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ASYMPTOTICS FOR LARGE TIME OF SOLUTIONS TO NONLINEAR SYSTEM ASSOCIATED WITH THE PENETRATION OF A MAGNETIC FIELD INTO A SUBSTANCE

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Abstract. The nonlinear integro-differential system associated with the penetration of a magnetic field into a substance is considered. The asymptotic behavior as $t \to \infty$ of solutions for two initial-boundary value problems are studied. The problem with non-zero conditions on one side of the lateral boundary is discussed. The problem with homogeneous boundary conditions is studied too. The rates of convergence are given. Results presented show the difference between stabilization characters of solutions of these two cases.

Keywords: system of nonlinear integro-differential equations, magnetic field, asymptotics for large time

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1. Introduction

Integro-differential equations arise in the study of various problems in physics, chemistry, technology, economics etc. (see, for example, [1]–[4], [6]–[8], [10]–[12], [19], [26], [27], [29]). One kind of integro-differential systems arises in mathematical modelling of the process of penetrating of a magnetic field into a substance. Penetrating into a material a variable magnetic field induces in it a variable electric field which causes the appearance of currents. The currents lead to the heating of the material and increasing of its temperature. For quasistationary approximation the corresponding system of Maxwell's equations has the form [20]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is the vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the heat capacity and the electroconductivity of the substance.

If c_ν and ν_m have the form $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$, then the system (1.1)–(1.2) can be rewritten in the following form [9]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\Big[a\Big(\int_0^t |\operatorname{rot} H|^2 \, \mathrm{d}\tau\Big) \operatorname{rot} H\Big], \tag{1.3}$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$.

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x, t)$, $V = V(x, t)$, then we have

$$\operatorname{rot}(a(S) \operatorname{rot} H) = \Big(0, -\frac{\partial}{\partial x}\Big(a(S)\frac{\partial U}{\partial x}\Big), -\frac{\partial}{\partial x}\Big(a(S)\frac{\partial V}{\partial x}\Big)\Big),$$

where

$$S(x, t) = \int_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] \, \mathrm{d}\tau. \tag{1.4}$$

Therefore, we obtain the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial V}{\partial x}\Big], \tag{1.5}$$

where S is defined by relation (1.4).

The model of (1.3) type is complex and has been intensively studied by many authors. The existence and uniqueness of global solutions of initial-boundary value problems for equations and systems of (1.3) type were studied in [9], [13], [21], [22], [24], [25] and in a number of other works as well. The existence theorems that are proved in [9] and [13] are based on a priori estimates, Galerkin's method and compactness arguments as in [23] and [28] for nonlinear parabolic equations. The asymptotic behavior as $t \to \infty$ of the solutions of such type models have been object of intensive research in recent years [14]–[17]. Note that in [14] and [17] the corresponding scalar equation of the (1.3) type was considered. System (1.4)–(1.5) for the case $a(S) = (1 + S)^p$, $-1/2 \leqslant p < 0$ was investigated in [16].

In [15] the asymptotic behavior of solutions of the initial-boundary value problem for the system (1.4)–(1.5) with homogeneous boundary data is studied. In the present work the study of asymptotics for large time of solutions of the first boundary value problems for the system (1.4)–(1.5) is continued. The attention is paid to the case $a(S) = (1 + S)^p$, $0 < p \leqslant 1$.

We organize our paper as follows. Section two is devoted to the asymptotic behavior of the solutions as $t \to \infty$ of the initial-boundary value problem with non-zero boundary data on one side of the lateral boundary. In the third section the same problem with zero lateral boundary data on the whole boundary is studied.

2. The problem with non-zero data on one side of lateral boundary

In this section we study asymptotic behavior of the solution to the following nonlinear system of integro-differential equations under nonhomogeneous Dirichlet boundary conditions on one side of lateral boundary:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial V}{\partial x}\Big], \quad (x,t) \in Q, \tag{2.1}$$

$$U(0,t) = V(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(1,t) = \psi_2, \quad t \geqslant 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{2.3}$$

where

$$S(x,t) = \int_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}\tau, \tag{2.4}$$

$a(S) = (1+S)^p$, $0 < p \leqslant 1$; $\psi_1 = \mathrm{const} \geqslant 0$, $\psi_2 = \mathrm{const} \geqslant 0$, $\psi_1^2+\psi_2^2 \neq 0$; $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions; $Q = (0,1) \times (0,\infty)$.

Now we are going to investigate the asymptotic behavior of the solutions of the problem (2.1)–(2.4) as $t \to \infty$.

We use the usual L_2-inner product and norm:

$$(u,v) = \int_0^1 u(x)v(x)\,\mathrm{d}x, \quad \|u\| = (u,u)^{1/2}.$$

Denote by H^k and H_0^k the usual Sobolev spaces of real functions.

In this section we use the scheme of [5] in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied.

Let us mention that boundary conditions (2.2) are used here taking into account the physical problem considered in [18].

Note that for problem (2.1)–(2.4) the following statement is valid.

Theorem 2.1. *Let $a(S) = (1+S)^p$, $0 < p \leqslant 1$. Suppose also that $U_0, V_0 \in H^2(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$. Then for the* *solution of problem* (2.1)–(2.4) *the following asymptotic relations hold as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial x} - \psi_1 = O\Big(\frac{1}{t^{1+p}}\Big), \quad \frac{\partial V(x,t)}{\partial x} - \psi_2 = O\Big(\frac{1}{t^{1+p}}\Big)$$

uniformly in x *on* $[0,1]$.

A series of lemmas is necessary in order to prove Theorem 2.1.

Lemma 2.1. *For the solution of problem* (2.1)–(2.4) *the following estimates are true:*

$$\int_0^t\int_0^1\Big(\frac{\partial U}{\partial \tau}\Big)^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqslant C, \quad \int_0^t\int_0^1\Big(\frac{\partial V}{\partial \tau}\Big)^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqslant C.$$

Note that in this work C, C_i, and c denote positive constants independent of t.

P r o o f. Let us differentiate the first equation of the system (2.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\Big[\frac{\partial (1+S)^p}{\partial t}\frac{\partial U}{\partial x} + (1+S)^p\frac{\partial^2 U}{\partial t\partial x}\Big] = 0. \tag{2.5}$$

After multiplying (2.5) by $\partial U/\partial t$, carrying out integration by parts gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x &+ \int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\\ &+ p\int_0^1(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\\ &+ p\int_0^1(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x = 0.\end{aligned} \tag{2.6}$$

In an analogous way we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\,\mathrm{d}x &+ \int_0^1(1+S)^p\Big(\frac{\partial^2 V}{\partial t\partial x}\Big)^2\,\mathrm{d}x\\ &+ p\int_0^1(1+S)^{p-1}\Big(\frac{\partial V}{\partial x}\Big)^3\frac{\partial^2 V}{\partial t\partial x}\,\mathrm{d}x\\ &+ p\int_0^1(1+S)^{p-1}\frac{\partial V}{\partial x}\Big(\frac{\partial U}{\partial x}\Big)^2\frac{\partial^2 V}{\partial t\partial x}\,\mathrm{d}x = 0.\end{aligned} \tag{2.7}$$

Combining (2.6), (2.7), and taking into account the relation $S(x,t) \geqslant 0$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big\|\frac{\partial U}{\partial t}\Big\|^2 + \Big\|\frac{\partial V}{\partial t}\Big\|^2\Big) &+ 2\Big(\Big\|\frac{\partial^2 U}{\partial x\partial t}\Big\|^2 + \Big\|\frac{\partial^2 V}{\partial x\partial t}\Big\|^2\Big)\\ &+ \frac{p}{2}\int_0^1(1+S)^{p-1}\frac{\partial}{\partial t}\Big[\Big(\frac{\partial U}{\partial x}\Big)^4 + \Big(\frac{\partial V}{\partial x}\Big)^4\Big]\,\mathrm{d}x\\ &+ p\int_0^1(1+S)^{p-1}\frac{\partial}{\partial t}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x \leqslant 0,\end{aligned}$$

or

$$\Big\|\frac{\partial U}{\partial t}\Big\|^2+\Big\|\frac{\partial V}{\partial t}\Big\|^2+2\int_0^t\Big(\Big\|\frac{\partial^2 U}{\partial x\partial \tau}\Big\|^2+\Big\|\frac{\partial^2 V}{\partial x\partial \tau}\Big\|^2\Big)\,\mathrm{d}\tau$$
$$+\frac{p}{2}\int_0^t\int_0^1(1+S)^{p-1}\frac{\partial}{\partial \tau}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]^2\,\mathrm{d}x\,\mathrm{d}\tau\leqslant C.$$

Using Poincaré's inequality and taking into account the restriction on p, after simple transformations we have

$$\int_0^1\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x+2\int_0^t\int_0^1\Big[\Big(\frac{\partial U}{\partial \tau}\Big)^2+\Big(\frac{\partial V}{\partial \tau}\Big)^2\Big]\,\mathrm{d}x\,\mathrm{d}\tau\leqslant C. \tag{2.8}$$

Thanks to (2.8) we deduce the desired results of Lemma 2.1. □

Lemma 2.2. *For the function S the following estimates hold:*

$$c\varphi^{1/(1+2p)}(t)\leqslant 1+S(x,t)\leqslant C\varphi^{1/(1+2p)}(t),$$

where

$$\varphi(t)=1+\int_0^t\int_0^1(\sigma_1^2+\sigma_2^2)\,\mathrm{d}x\,\mathrm{d}\tau \tag{2.9}$$

and $\sigma_1=(1+S)^p(\partial U/\partial x)$, $\sigma_2=(1+S)^p(\partial V/\partial x)$.

P r o o f. From (2.4) it follows that

$$\frac{\partial S}{\partial t}=\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2,\quad S(x,0)=0. \tag{2.10}$$

Let us multiply the first relation of (2.10) by $(1+S)^{2p}$:

$$\frac{1}{1+2p}\frac{\partial(1+S)^{1+2p}}{\partial t}=\Big(\frac{\partial U}{\partial x}\Big)^2(1+S)^{2p}+\Big(\frac{\partial V}{\partial x}\Big)^2(1+S)^{2p}.$$

The system (2.1) can be rewritten as

$$\frac{\partial U}{\partial t}=\frac{\partial \sigma_1}{\partial x},\quad \frac{\partial V}{\partial t}=\frac{\partial \sigma_2}{\partial x}. \tag{2.11}$$

We have

$$\frac{1}{1+2p}\frac{\partial(1+S)^{1+2p}}{\partial t}=\sigma_1^2+\sigma_2^2, \tag{2.12}$$

$$
(2.13)\qquad \begin{aligned}
\sigma_1^2(x,t) &= \int_0^1 \sigma_1^2(y,t)\,\mathrm{d}y + \int_0^1\int_y^x \frac{\partial\sigma_1^2(\xi,t)}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}y\\
&= \int_0^1 \sigma_1^2(y,t)\,\mathrm{d}y + 2\int_0^1\int_y^x \sigma_1(\xi,t)\frac{\partial U(\xi,t)}{\partial t}\,\mathrm{d}\xi\,\mathrm{d}y,\\
\sigma_2^2(x,t) &= \int_0^1 \sigma_2^2(y,t)\,\mathrm{d}y + \int_0^1\int_y^x \frac{\partial\sigma_2^2(\xi,t)}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}y\\
&= \int_0^1 \sigma_2^2(y,t)\,\mathrm{d}y + 2\int_0^1\int_y^x \sigma_2(\xi,t)\frac{\partial V(\xi,t)}{\partial t}\,\mathrm{d}\xi\,\mathrm{d}y.
\end{aligned}
$$

In view of (2.9), (2.13), and Lemma 2.1, from (2.12) we obtain

$$
\begin{aligned}
&\frac{1}{1+2p}(1+S)^{1+2p}\\
&\quad= \int_0^t(\sigma_1^2+\sigma_2^2)\,\mathrm{d}\tau + \frac{1}{1+2p}\\
&\quad= \int_0^t\int_0^1(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau\\
&\qquad+2\int_0^t\int_0^1\int_y^x\Big(\sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial\tau}+\sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial\tau}\Big)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\tau+\frac{1}{1+2p}\\
&\quad\leqslant 2\int_0^t\int_0^1(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau\\
&\qquad+\int_0^t\int_0^1\Big[\Big(\frac{\partial U(x,\tau)}{\partial\tau}\Big)^2+\Big(\frac{\partial V(x,\tau)}{\partial\tau}\Big)^2\Big]\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{1+2p}\\
&\quad\leqslant 2\int_0^t\int_0^1(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau+C_1\leqslant C_2\varphi(t),
\end{aligned}
$$

i.e.,

$$
(2.14)\qquad 1+S(x,t)\leqslant C\varphi^{1/(1+2p)}(t).
$$

In an analogous way we deduce

$$
(2.15)\qquad \begin{aligned}
&\frac{1}{1+2p}(1+S)^{1+2p}\\
&\quad= \int_0^t\int_0^1(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau\\
&\qquad+2\int_0^t\int_0^1\int_y^x\Big(\sigma_1(\xi,\tau)\frac{\partial U(\xi,\tau)}{\partial\tau}+\sigma_2(\xi,\tau)\frac{\partial V(\xi,\tau)}{\partial\tau}\Big)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\tau\\
&\qquad+\frac{1}{1+2p}\\
&\quad\geqslant \frac{1}{2}\int_0^t\int_0^1(\sigma_1^2(y,\tau)+\sigma_2^2(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau-C_1=\frac{1}{2}\varphi(t)-C_2.
\end{aligned}
$$

We have

$$C_2(1+S)^{1+2p} \geqslant C_2. \tag{2.16}$$

Thus, via relations (2.15) and (2.16) we obtain

$$\Big(\frac{1}{1+2p}+C_2\Big)(1+S)^{1+2p} \geqslant \frac{1}{2}\varphi(t),$$

or

$$1+S(x,t) \geqslant c\varphi^{1/(1+2p)}(t). \tag{2.17}$$

Estimates (2.14) and (2.17) yield that Lemma 2.2 is true. □

Lemma 2.3. *The following inequalities are true:*

$$c\varphi^{2p/(1+2p)}(t) \leqslant \int_0^1 (\sigma_1^2(x,t)+\sigma_2^2(x,t))\,\mathrm{d}x \leqslant C\varphi^{2p/(1+2p)}(t).$$

Proof. Taking into account Lemma 2.2, we get

$$\begin{aligned}\int_0^1 (\sigma_1^2+\sigma_2^2)\,\mathrm{d}x &= \int_0^1 (1+S)^{2p}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x\\ &\geqslant c\varphi^{2p/(1+2p)}(t)\int_0^1\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x\\ &\geqslant c\varphi^{2p/(1+2p)}(t)\Big[\Big(\int_0^1\frac{\partial U}{\partial x}\,\mathrm{d}x\Big)^2+\Big(\int_0^1\frac{\partial V}{\partial x}\,\mathrm{d}x\Big)^2\Big]\\ &= (\psi_1^2+\psi_2^2)c\varphi^{2p/(1+2p)}(t),\end{aligned}$$

or

$$\int_0^1 (\sigma_1^2(x,t)+\sigma_2^2(x,t))\,\mathrm{d}x \geqslant c\varphi^{2p/(1+2p)}(t). \tag{2.18}$$

From (2.8) it follows that

$$\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x \leqslant C, \quad \int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\,\mathrm{d}x \leqslant C. \tag{2.19}$$

Let us multiply the first and second equations of the system (2.1) scalarly by U and V, respectively. Using the boundary conditions (2.2), we have

$$\begin{aligned}\int_0^1 U\frac{\partial U}{\partial t}\,\mathrm{d}x+\int_0^1(1+S)^p\Big(\frac{\partial U}{\partial x}\Big)^2\,\mathrm{d}x &= \psi_1\sigma_1(1,t),\\ \int_0^1 V\frac{\partial V}{\partial t}\,\mathrm{d}x+\int_0^1(1+S)^p\Big(\frac{\partial V}{\partial x}\Big)^2\,\mathrm{d}x &= \psi_2\sigma_2(1,t).\end{aligned}$$

Using these equalities, Lemma 2.2, relations (2.11), (2.13), (2.18), (2.19), and the maximum principle

$$|U(x,t)| \leqslant \max_{0\leqslant y\leqslant 1} |U_0(y)|, \quad |V(x,t)| \leqslant \max_{0\leqslant y\leqslant 1} |V_0(y)|, \quad 0\leqslant x\leqslant 1,\ t\geqslant 0,$$

we get

$$\begin{aligned}
&\left\{\int_0^1 [\sigma_1^2(x,t)+\sigma_2^2(x,t)]\,\mathrm{d}x\right\}^2 \\
&\quad\leqslant 2\left[\int_0^1 \sigma_1^2(x,t)\,\mathrm{d}x\right]^2 + 2\left[\int_0^1 \sigma_2^2(x,t)\,\mathrm{d}x\right]^2 \\
&\quad\leqslant 2C_1\varphi^{2p/(1+2p)}(t)\left\{\left[\int_0^1 (1+S)^p\Big(\frac{\partial U}{\partial x}\Big)^2\mathrm{d}x\right]^2 + \left[\int_0^1 (1+S)^p\Big(\frac{\partial V}{\partial x}\Big)^2\mathrm{d}x\right]^2\right\} \\
&\quad\leqslant 4C_1\varphi^{2p/(1+2p)}(t) \\
&\qquad\times\left[(\psi_1\sigma_1(1,t))^2 + \Big(\int_0^1 U\frac{\partial U}{\partial t}\,\mathrm{d}x\Big)^2 + (\psi_2\sigma_2(1,t))^2 + \Big(\int_0^1 V\frac{\partial V}{\partial t}\,\mathrm{d}x\Big)^2\right] \\
&\quad\leqslant 4C_1\varphi^{2p/(1+2p)}(t) \\
&\qquad\times\left[(\psi_1^2+\psi_2^2)\Big(2\int_0^1\sigma_1^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial\sigma_1}{\partial x}\Big)^2\mathrm{d}x + 2\int_0^1\sigma_2^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial\sigma_2}{\partial x}\Big)^2\mathrm{d}x\Big)\right. \\
&\qquad\left.+\int_0^1 U^2\,\mathrm{d}x\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\mathrm{d}x + \int_0^1 V^2\,\mathrm{d}x\int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\mathrm{d}x\right] \\
&\quad\leqslant 4C_1\varphi^{2p/(1+2p)}(t) \\
&\qquad\times\left[(\psi_1^2+\psi_2^2)\Big(2\int_0^1\sigma_1^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\mathrm{d}x + 2\int_0^1\sigma_2^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\mathrm{d}x\Big)\right. \\
&\qquad\left.+\big(\max_{0\leqslant y\leqslant 1}|U_0(y)|\big)^2\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\mathrm{d}x + \big(\max_{0\leqslant y\leqslant 1}|V_0(y)|\big)^2\int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\mathrm{d}x\right] \\
&\quad\leqslant 4C_1\varphi^{2p/(1+2p)}(t)\left[C_2\int_0^1(\sigma_1^2+\sigma_2^2)\,\mathrm{d}x + \frac{C_3}{\varphi^{2p/(1+2p)}(t)}\int_0^1(\sigma_1^2+\sigma_2^2)\,\mathrm{d}x\right].
\end{aligned}$$

Now, taking into account relation $\varphi(t)\geqslant 1$, we get

$$\int_0^1(\sigma_1^2(x,t)+\sigma_2^2(x,t))\,\mathrm{d}x \leqslant C\varphi^{2p/(1+2p)}(t).$$

So, Lemma 2.3 is proved. □

From Lemma 2.3 and relation (2.9) we deduce

$$c\varphi^{2p/(1+2p)}(t) \leqslant \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \leqslant C\varphi^{2p/(1+2p)}(t). \tag{2.20}$$

Lemma 2.4. *The derivatives $\partial U/\partial t$ and $\partial V/\partial t$ satisfy the inequality*

$$\int_0^1 \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x \leqslant C\varphi^{-2/(1+2p)}(t).$$

P r o o f. Identity (2.6) yields

$$\text{(2.21)}\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x + \int_0^1 (1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x \leqslant 2p^2\int_0^1 (1+S)^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^6\,\mathrm{d}x + 2p^2\int_0^1 (1+S)^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial V}{\partial x}\Big)^4\,\mathrm{d}x.$$

Now using Lemmas 2.2, 2.3, keeping in mind the definitions of σ_1, σ_2 the relations

$$\int_0^1 \Big(\frac{\partial \sigma_1}{\partial x}\Big)^2\,\mathrm{d}x = -\int_0^1 \sigma_1\frac{\partial^2\sigma_1}{\partial x^2}\,\mathrm{d}x, \quad \int_0^1 \Big(\frac{\partial \sigma_2}{\partial x}\Big)^2\,\mathrm{d}x = -\int_0^1 \sigma_2\frac{\partial^2\sigma_2}{\partial x^2}\,\mathrm{d}x,$$

and (2.13), from (2.21) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x + c\varphi^{p/(1+2p)}(t)\int_0^1 \Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\\
&\quad\leqslant C_1\varphi^{-(5p+2)/(1+2p)}(t)\int_0^1 (\sigma_1^6+\sigma_1^2\sigma_2^4)\,\mathrm{d}x\\
&\quad\leqslant C_1\varphi^{-(5p+2)/(1+2p)}(t)\int_0^1 \sigma_1^2(x,t)\,\mathrm{d}x\Big\{\big[\max_{0\leqslant x\leqslant 1}\sigma_1^2(x,t)\big]^2 + \big[\max_{0\leqslant x\leqslant 1}\sigma_2^2(x,t)\big]^2\Big\}\\
&\quad\leqslant C_2\varphi^{-(3p+2)/(1+2p)}(t)\Bigg(\Big\{\int_0^1 \sigma_1^2\,\mathrm{d}x + 2\Big[\int_0^1\sigma_1^2\,\mathrm{d}x\Big]^{1/2}\Big[\int_0^1\Big(\frac{\partial\sigma_1}{\partial x}\Big)^2\,\mathrm{d}x\Big]^{1/2}\Big\}^2\\
&\qquad + \Big\{\int_0^1 \sigma_2^2\,\mathrm{d}x + 2\Big[\int_0^1\sigma_2^2\,\mathrm{d}x\Big]^{1/2}\Big[\int_0^1\Big(\frac{\partial\sigma_2}{\partial x}\Big)^2\,\mathrm{d}x\Big]^{1/2}\Big\}^2\Bigg)\\
&\quad\leqslant C_2\varphi^{-(3p+2)/(1+2p)}(t)\Bigg(\Big\{\int_0^1 \sigma_1^2\,\mathrm{d}x + 2\Big[\int_0^1\sigma_1^2\,\mathrm{d}x\Big]^{3/4}\Big[\int_0^1\Big(\frac{\partial^2\sigma_1}{\partial x^2}\Big)^2\,\mathrm{d}x\Big]^{1/4}\Big\}^2\\
&\qquad + \Big\{\int_0^1 \sigma_2^2\,\mathrm{d}x + 2\Big[\int_0^1\sigma_2^2\,\mathrm{d}x\Big]^{3/4}\Big[\int_0^1\Big(\frac{\partial^2\sigma_2}{\partial x^2}\Big)^2\,\mathrm{d}x\Big]^{1/4}\Big\}^2\Bigg)\\
&\quad\leqslant C_3\varphi^{(p-2)/(1+2p)}(t) + C_4\varphi^{-(3p+2)/(1+2p)}(t)\varphi^{3p/(1+2p)}(t)\\
&\qquad\times\Big\{\Big[\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\Big]^{1/2} + \Big[\int_0^1\Big(\frac{\partial^2 V}{\partial t\partial x}\Big)^2\,\mathrm{d}x\Big]^{1/2}\Big\}\\
&\quad\leqslant C_3\varphi^{(p-2)/(1+2p)}(t) + C_5\varphi^{-(p+4)/(1+2p)}(t)\\
&\qquad+\frac{c}{4}\varphi^{p/(1+2p)}(t)\Big[\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial^2 V}{\partial t\partial x}\Big)^2\,\mathrm{d}x\Big].
\end{aligned}$$

So, taking into account the restrictions on p, the last inequality gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x+\frac{c}{4}\varphi^{p/(1+2p)}(t)\int_0^1\Big[3\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2-\Big(\frac{\partial^2 V}{\partial t\partial x}\Big)^2\Big]\,\mathrm{d}x\leqslant C\varphi^{(p-2)/(1+2p)}(t).$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{\partial V}{\partial t}\Big)^2\,\mathrm{d}x+\frac{c}{4}\varphi^{p/(1+2p)}(t)\int_0^1\Big[3\Big(\frac{\partial^2 V}{\partial t\partial x}\Big)^2-\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\Big]\,\mathrm{d}x\leqslant C\varphi^{(p-2)/(1+2p)}(t).$$

Thanks to Poincaré's inequality we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x+\frac{c}{2}\varphi^{p/(1+2p)}(t)\int_0^1\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x \leqslant C\varphi^{(p-2)/(1+2p)}(t). \tag{2.22}$$

From (2.22), using Gronwall's inequality, we get

$$\begin{aligned}\int_0^1\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x &\leqslant \exp\Big(-\frac{c}{2}\int_0^t\varphi^{p/(1+2p)}(\tau)\,\mathrm{d}\tau\Big)\Big\{\int_0^1\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x\Big|_{t=0}\\ &\quad +C\int_0^t\exp\Big(\frac{c}{2}\int_0^\tau\varphi^{p/(1+2p)}(\xi)\,\mathrm{d}\xi\Big)\varphi^{(p-2)/(1+2p)}(\tau)\,\mathrm{d}\tau\Big\}.\end{aligned} \tag{2.23}$$

Noting that $\varphi(t)\geqslant 1$, applying L'Hopital's rule and estimate (2.20), we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{\int_0^t\exp\big(\frac{c}{2}\int_0^\tau\varphi^{\frac{p}{1+2p}}(\xi)\,\mathrm{d}\xi\big)\varphi^{\frac{p-2}{1+2p}}(\tau)\,\mathrm{d}\tau}{\exp\big(\frac{c}{2}\int_0^t\varphi^{\frac{p}{1+2p}}(\tau)\,\mathrm{d}\tau\big)\varphi^{-\frac{2}{1+2p}}(t)}\\ &\quad=\lim_{t\to\infty}\frac{\exp\big(\frac{c}{2}\int_0^t\varphi^{\frac{p}{1+2p}}(\tau)\,\mathrm{d}\tau\big)\varphi^{\frac{p-2}{1+2p}}(t)}{\exp\big(\frac{c}{2}\int_0^t\varphi^{\frac{p}{1+2p}}(\tau)\,\mathrm{d}\tau\big)\big(\frac{c}{2}\varphi^{\frac{p-2}{1+2p}}(t)-\frac{2}{1+2p}\varphi^{\frac{-3-2p}{1+2p}}(t)\frac{\mathrm{d}\varphi}{\mathrm{d}t}\big)}\\ &\quad\leqslant\lim_{t\to\infty}\frac{1}{\frac{c}{2}-\frac{2C}{1+2p}\varphi^{-\frac{p+1}{1+2p}}(t)}\leqslant C.\end{aligned} \tag{2.24}$$

The inequalities (2.23) and (2.24) ensure validity of Lemma 2.4. □

Let us now estimate $\partial S/\partial x$ in $L_1(0,1)$.

Lemma 2.5. *For $\partial S/\partial x$ the following inequality is true:*

$$\int_0^1\Big|\frac{\partial S}{\partial x}\Big|\,\mathrm{d}x\leqslant C\varphi^{-p/(1+2p)}(t).$$

P r o o f. Differentiating (2.12) with respect to x, we get

$$\frac{\partial}{\partial t}\Big[(1+S)^{2p}\frac{\partial S}{\partial x}\Big] = 2\sigma_1\frac{\partial \sigma_1}{\partial x} + 2\sigma_2\frac{\partial \sigma_2}{\partial x}. \tag{2.25}$$

From Lemmas 2.3 and 2.4 one can easily see that the following estimates are true:

$$\begin{aligned} &\int_0^1 \Big|\sigma_1\frac{\partial U}{\partial t}\Big|\,\mathrm{d}x \leqslant C\varphi^{p/(1+2p)}(t)\varphi^{-1/(1+2p)}(t) = C\varphi^{(p-1)/(1+2p)}(t),\\ &\int_0^1 \Big|\sigma_2\frac{\partial V}{\partial t}\Big|\,\mathrm{d}x \leqslant C\varphi^{p/(1+2p)}(t)\varphi^{-1/(1+2p)}(t) = C\varphi^{(p-1)/(1+2p)}(t). \end{aligned} \tag{2.26}$$

Finally, using Lemma 2.2, relations (2.11), (2.20), (2.25), and (2.26), we have

$$\begin{aligned} (1+S)^{2p}\frac{\partial S}{\partial x} &= \int_0^t \Big(2\sigma_1\frac{\partial U}{\partial \tau} + 2\sigma_2\frac{\partial V}{\partial \tau}\Big)\,\mathrm{d}\tau,\\ \int_0^1\Big|\frac{\partial S}{\partial x}\Big|\,\mathrm{d}x &\leqslant C_1\varphi^{-2p/(1+2p)}(t)\int_0^t \varphi^{(p-1)/(1+2p)}(\tau)\,\mathrm{d}\tau\\ &\leqslant C_2\varphi^{-2p/(1+2p)}(t)\int_1^\varphi \varphi^{-(p+1)/(1+2p)}\,\mathrm{d}\varphi\\ &= C_3\varphi^{-2p/(1+2p)}(t)\int_1^\varphi \mathrm{d}\varphi^{p/(1+2p)}\\ &= C_3\varphi^{-2p/(1+2p)}(t)(\varphi^{p/(1+2p)}(t)-1) \leqslant C\varphi^{-p/(1+2p)}(t). \end{aligned}$$

Thus, Lemma 2.5 is proved. □

Now we are ready to prove Theorem 2.1.

P r o o f of Theorem 2.1. From (2.13), keeping in mind Lemma 2.3 and relation (2.26), we get

$$\begin{aligned} \sigma_1^2(x,t) &\leqslant \int_0^1 \sigma_1^2(y,t)\,\mathrm{d}y + 2\int_0^1\Big|\sigma_1(y,t)\frac{\partial U(y,t)}{\partial t}\Big|\,\mathrm{d}y\\ &\leqslant C_1\varphi^{2p/(1+2p)}(t) + C_2\varphi^{(p-1)/(1+2p)}(t) \leqslant C\varphi^{2p/(1+2p)}(t), \end{aligned}$$

or

$$|\sigma_1(x,t)| \leqslant C\varphi^{p/(1+2p)}(t). \tag{2.27}$$

Taking into account (2.11), (2.27), Lemmas 2.2, 2.4, 2.5, and the relation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial \sigma_1}{\partial x}(1+S)^{-p} - p\sigma_1(1+S)^{-p-1}\frac{\partial S}{\partial x},$$

we derive

$$\begin{aligned}\int_0^1 \Big|\frac{\partial^2 U(x,t)}{\partial x^2}\Big|\,\mathrm{d}x &\leqslant \int_0^1 \Big|\frac{\partial U}{\partial t}(1+S)^{-p}\Big|\,\mathrm{d}x + p\int_0^1 \Big|\sigma_1(1+S)^{-p-1}\frac{\partial S}{\partial x}\Big|\,\mathrm{d}x\\
&\leqslant \Big[\int_0^1 \Big|\frac{\partial U}{\partial t}\Big|^2\,\mathrm{d}x\Big]^{1/2}\Big[\int_0^1 (1+S)^{-2p}\,\mathrm{d}x\Big]^{1/2}\\
&\quad + p\int_0^1 \Big|\sigma_1(1+S)^{-p-1}\frac{\partial S}{\partial x}\Big|\,\mathrm{d}x\\
&\leqslant C_1\varphi^{-1/(1+2p)}(t)\varphi^{-p/(1+2p)}(t)\\
&\quad + C_2\varphi^{p/(1+2p)}(t)\varphi^{-(p+1)/(1+2p)}(t)\int_0^1 \Big|\frac{\partial S}{\partial x}\Big|\,\mathrm{d}x\\
&\leqslant C\varphi^{-(p+1)/(1+2p)}(t).\end{aligned}$$

Hence, we have

$$\int_0^1 \Big|\frac{\partial^2 U(x,t)}{\partial x^2}\Big|\,\mathrm{d}x \leqslant C\varphi^{-(p+1)/(1+2p)}(t).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}\,\mathrm{d}y + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,\mathrm{d}\xi\,\mathrm{d}y,$$

we obtain that

$$\begin{aligned}\Big|\frac{\partial U(x,t)}{\partial x} - \psi_1\Big| &= \Big|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}\,\mathrm{d}\xi\,\mathrm{d}y\Big|\\
&\leqslant \int_0^1 \Big|\frac{\partial^2 U(y,t)}{\partial y^2}\Big|\,\mathrm{d}y \leqslant C\varphi^{-(p+1)/(1+2p)}(t).\end{aligned} \tag{2.28}$$

The same estimate is valid for $\partial V/\partial x$:

$$\Big|\frac{\partial V(x,t)}{\partial x} - \psi_2\Big| \leqslant C\varphi^{-(p+1)/(1+2p)}(t). \tag{2.29}$$

After integrating, from (2.20) it is easy to show that the following estimates are true:

$$ct^{1+2p} \leqslant \varphi(t) \leqslant Ct^{1+2p}, \quad t \geqslant 1. \tag{2.30}$$

Once (2.30) is checked, one derives from (2.28) and (2.29) the validity of Theorem 2.1. □

Now, let us prove the second main result of this section.

Theorem 2.2. *Let* $a(S) = (1+S)^p$, $0 < p \leqslant 1$. *Suppose also that* $U_0, V_0 \in H^3(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$. *Then for the solution of problem* (2.1)–(2.4) *the following asymptotic relations hold as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial t} = O\Big(\frac{1}{t}\Big), \quad \frac{\partial V(x,t)}{\partial t} = O\Big(\frac{1}{t}\Big), \quad t \geqslant 1,$$

uniformly in x *on* $[0,1]$.

P r o o f. Let us multiply (2.21) on $\varphi^{2/(1+2p)}(t)$. Keeping in mind Lemma 2.2 and estimates (2.28), (2.29), we arrive at

$$\begin{aligned}\varphi^{2/(1+2p)}(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x + c\varphi^{2/(1+2p)}(t)\varphi^{p/(1+2p)}(t)\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x \\ \leqslant C\varphi^{p/(1+2p)}(t).\end{aligned}$$

Integrating the last inequality on $(0,t)$, using the formula of integrating by parts, relation (2.20), and Lemma 2.4 we get

$$\begin{aligned}&c\int_0^t \varphi^{(p+2)/(1+2p)}(\tau)\int_0^1\Big(\frac{\partial^2 U}{\partial \tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant -\varphi^{2/(1+2p)}(t)\int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x + \int_0^1\Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x\Big|_{t=0} \\ &\quad + \frac{2}{1+2p}\int_0^t \varphi^{(1-2p)/(1+2p)}(\tau)\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\int_0^1\Big(\frac{\partial U}{\partial \tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau + C\int_0^t \varphi^{p/(1+2p)}(\tau)\,\mathrm{d}\tau \\ &\leqslant C_1\int_1^{\varphi}\varphi^{-1}\,\mathrm{d}\varphi + C_2\int_1^{\varphi}\varphi^{-p/(1+2p)}\,\mathrm{d}\varphi + C_3 \\ &\leqslant C\varphi^{(p+1)/(1+2p)}(t),\end{aligned}$$

or

$$\int_0^t \varphi^{(p+2)/(1+2p)}(\tau)\int_0^1\Big(\frac{\partial^2 U}{\partial \tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C\varphi^{(p+1)/(1+2p)}(t). \tag{2.31}$$

Analogously, we can show that

$$\int_0^t \varphi^{(p+2)/(1+2p)}(\tau)\int_0^1\Big(\frac{\partial^2 V}{\partial \tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C\varphi^{(p+1)/(1+2p)}(t). \tag{2.32}$$

Multiply equation (2.5) scalarly on $\varphi^{3/(1+2p)}(t)(\partial^2 U/\partial t^2)$. Integration by parts gives

$$\int_0^1 \varphi^{3/(1+2p)}(t)\Big(\frac{\partial^2 U}{\partial t^2}\Big)^2 \,\mathrm{d}x + \int_0^1 \varphi^{3/(1+2p)}(t)(1+S)^p \frac{\partial^2 U}{\partial t\partial x}\frac{\partial^3 U}{\partial t^2 \partial x}\,\mathrm{d}x \\ + p\int_0^1 \varphi^{3/(1+2p)}(t)(1+S)^{p-1}\frac{\partial U}{\partial x}\frac{\partial^3 U}{\partial t^2\partial x}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x = 0.$$

After integrating the last equality on $(0, t)$ we get

$$\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)\Big(\frac{\partial^2 U}{\partial \tau^2}\Big)^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{1}{2}\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^p \frac{\partial}{\partial\tau}\Big(\frac{\partial^2 U}{\partial \tau\partial x}\Big)^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ + p\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial}{\partial\tau}\Big(\frac{\partial^2 U}{\partial \tau\partial x}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ + p\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial}{\partial\tau}\Big(\frac{\partial^2 U}{\partial \tau\partial x}\Big)\,\mathrm{d}x\,\mathrm{d}\tau = 0.$$

Applying again the formula of integrating by parts and relation (2.4) we obtain

$$\frac{1}{2}\varphi^{3/(1+2p)}(t)\int_0^1 (1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x - \frac{1}{2}\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\Big|_{t=0} \\ \leqslant \frac{3}{2+4p}\int_0^t\int_0^1 \varphi^{(2-2p)/(1+2p)}(\tau)\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{p}{2}\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ - p\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x + p\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\Big|_{t=0} \\ + \frac{3p}{1+2p}\int_0^t\int_0^1\varphi^{(2-2p)/(1+2p)}(\tau)\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau \\ + p(p-1)\int_0^t\int_0^1\varphi^{3/(1+2p)}(\tau)(1+S)^{p-2}\Big[\Big(\frac{\partial U}{\partial x}\Big)^5 + \Big(\frac{\partial U}{\partial x}\Big)^3\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau \\ + 3p\int_0^t\int_0^1\varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ - p\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x + p\int_0^1\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\Big|_{t=0} \\ + \frac{3p}{1+2p}\int_0^t\int_0^1\varphi^{(2-2p)/(1+2p)}(\tau)\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau$$

$$+ p(p-1)\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-2} \times \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ p\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\Big(\frac{\partial V}{\partial x}\Big)^2\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ 2p\int_0^t\int_0^1 \varphi^{3/(1+2p)}(\tau)(1+S)^{p-1}\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial\tau\partial x}\frac{\partial^2 V}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau.$$

Using Lemma 2.2, a priori estimates (2.20), (2.28), (2.29), (2.31), (2.32), and Schwarz's inequality, we get

$$\begin{aligned}
\frac{c\varphi^{(3+p)/(1+2p)}(t)}{2}&\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\mathrm{d}x \\
&\leqslant \frac{1}{2}\Big\|\frac{\partial^2 U}{\partial t\partial x}\Big\|^2\Big|_{t=0} + C_1\varphi^{(p+1)/(1+2p)}(t) + C_2\varphi^{(p+1)/(1+2p)}(t) \\
&\quad + \frac{1}{8}\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\mathrm{d}x \\
&\quad + C_3\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^{p-2}\,\mathrm{d}x + C_4 \\
&\quad + \int_0^t\varphi^{2/(1+2p)}(\tau)\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C_5\int_0^t\varphi^{2/(1+2p)}(\tau)\int_0^1(1+S)^{p-2}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t\varphi^{2/(1+2p)}(\tau)\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C_6\int_0^t\varphi^{4/(1+2p)}(\tau)\int_0^1(1+S)^{p-4}\,\mathrm{d}x\,\mathrm{d}\tau + C_7\varphi^{(p+1)/(1+2p)}(t) \\
&\quad + \frac{1}{8}\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\mathrm{d}x \\
&\quad + C_8\varphi^{3/(1+2p)}(t)\int_0^1(1+S)^{p-2}\,\mathrm{d}x \\
&\quad + \int_0^t\int_0^1\varphi^{2/(1+2p)}(\tau)(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + C_9\int_0^t\int_0^1\varphi^{2/(1+2p)}(\tau)(1+S)^{p-2}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t\int_0^1\varphi^{2/(1+2p)}(\tau)(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+ C_{10} \int_0^t \int_0^1 \varphi^{4/(1+2p)}(\tau)(1+S)^{p-4}\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ C_{11} \int_0^t \varphi^{(p+2)/(1+2p)}(\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ C_{12} \int_0^t \varphi^{(p+2)/(1+2p)}(\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ C_{12} \int_0^t \varphi^{(p+2)/(1+2p)}(\tau) \int_0^1 \Big(\frac{\partial^2 V}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

Consequently, taking into account again Lemma 2.2 and estimates (2.20), (2.31), (2.32) we have

$$
\begin{aligned}
&\frac{c}{4}\varphi^{(p+3)/(1+2p)}(t) \int_0^1 \Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x \\
&\qquad\leqslant C_{13}\varphi^{(p+1)/(1+2p)}(t) + C_{14}\int_0^t \varphi^{p/(1+2p)}(\tau)\,\mathrm{d}\tau + C_4 \\
&\qquad\leqslant C_{15}\varphi^{(p+1)/(1+2p)}(t).
\end{aligned}
$$

Or, finally,

$$
\int_0^1 \Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x \leqslant C\varphi^{-2/(1+2p)}(t).
$$

From this, using the relation

$$
\frac{\partial U(x,t)}{\partial t} = \int_0^1 \frac{\partial U(y,t)}{\partial t}\,\mathrm{d}y + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial\xi\partial t}\,\mathrm{d}\xi\,\mathrm{d}y \tag{2.33}
$$

and Lemma 2.4, we obtain

$$
\Big|\frac{\partial U(x,t)}{\partial t}\Big| \leqslant C\varphi^{-1/(1+2p)}(t). \tag{2.34}
$$

Analogously,

$$
\Big|\frac{\partial V(x,t)}{\partial t}\Big| \leqslant C\varphi^{-1/(1+2p)}(t). \tag{2.35}
$$

Now taking into account (2.30), estimates (2.34) and (2.35) imply the validity of Theorem 2.2. □

3. The problem with zero boundary conditions

Now let us consider the following initial-boundary value problem under the homogeneous Dirichlet boundary conditions:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial U}{\partial x}\Big], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big[a(S)\frac{\partial V}{\partial x}\Big], \quad (x,t) \in Q, \tag{3.1}$$

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \geqslant 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{3.3}$$

where again

$$S(x,t) = \int_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}\tau, \tag{3.4}$$

$a(S) = (1+S)^p$, $0 < p \leqslant 1$; $U_0 = U_0(x)$ and $V_0 = V_0(x)$ are given functions.

It is easy to verify the following statement.

Lemma 3.1. *For the solution of problem* (3.1)–(3.4) *the following estimate is true:*

$$\|U\| + \|V\| \leqslant C\exp(-t).$$

Note that Lemma 3.1 gives exponential stabilization of the solution of problem (3.1)–(3.4) in the norm of the space $L_2(0,1)$. The stabilization is also achieved in the norm of the space $H^1(0,1)$. In particular, in [15] the following result is proved.

Theorem 3.1. *If* $a(S) = (1+S)^p$, $0 < p \leqslant 1$, $U_0, V_0 \in H^2(0,1) \cap H_0^1(0,1)$, *then for the solution of problem* (3.1)–(3.4) *the following estimate is true as* $t \to \infty$*:*

$$\Big\|\frac{\partial U}{\partial x}\Big\| + \Big\|\frac{\partial U}{\partial t}\Big\| + \Big\|\frac{\partial V}{\partial x}\Big\| + \Big\|\frac{\partial V}{\partial t}\Big\| \leqslant C\exp\Big(-\frac{t}{2}\Big).$$

Theorem 3.1 helps us to deduce that Lemma 2.2 holds also for the solution of problem (3.1)–(3.4). Therefore using this lemma, relation (2.9), and again Theorem 3.1 we obtain

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \int_0^1 (1+S)^{2p}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x \leqslant C\varphi^{2p/(1+2p)}(t)\exp(-t).$$

After integrating this inequality and taking into account definition (2.9), we arrive at

$$1 \leqslant \varphi(t) \leqslant C.$$

From this, keeping in mind Lemma 2.2, we get

$$1 \leqslant 1 + S(x,t) \leqslant C. \tag{3.5}$$

Identities (2.13) together with (3.5) and Theorem 3.1 give

$$\sigma_1^2(x,t) + \sigma_2^2(x,t) \leqslant 2\int_0^1 (1+S)^{2p}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,\mathrm{d}x + \int_0^1 \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial V}{\partial t}\Big)^2\Big]\,\mathrm{d}x \leqslant C\exp(-t).$$

Finally, if we recall the definition of σ_1 and σ_2, the validity of the following statement will be obvious.

Theorem 3.2. *If the conditions of Theorem* 3.1 *are satisfied, then for the solution of problem* (3.1)–(3.4) *the following estimates take place as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial x} = O\Big(\exp\Big(-\frac{t}{2}\Big)\Big), \quad \frac{\partial V(x,t)}{\partial x} = O\Big(\exp\Big(-\frac{t}{2}\Big)\Big)$$

uniformly in x *on* $[0,1]$.

Now let us prove the second main result of this section.

Theorem 3.3. *If* $a(S) = (1+S)^p$, $0 < p \leqslant 1$, $U_0, V_0 \in H^3(0,1) \cap H_0^1(0,1)$, *then for the solution of problem* (3.1)–(3.4) *the following estimates hold as* $t \to \infty$*:*

$$\frac{\partial U(x,t)}{\partial t} = O\Big(\exp\Big(-\frac{t}{2}\Big)\Big), \quad \frac{\partial V(x,t)}{\partial t} = O\Big(\exp\Big(-\frac{t}{2}\Big)\Big)$$

uniformly in x *on* $[0,1]$.

P r o o f. Note that (2.21) is valid for problem (3.1)–(3.4) as well. Let us multiply (2.21) scalarly by $\exp(2t)$ and integrate it on $(0,t)$. Using integration by parts, estimate (3.5) and Theorems 3.1, 3.2, we get

$$\int_0^t \exp(2\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \exp(2\tau)\int_0^1 (1+S)^p\Big(\frac{\partial^2 U}{\partial x\partial \tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 2p^2\int_0^t \exp(2\tau)\int_0^1 (1+S)^{p-2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big[\Big(\frac{\partial U}{\partial x}\Big)^4 + \Big(\frac{\partial V}{\partial x}\Big)^4\Big]\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$\int_0^t \exp(2\tau)\int_0^1 \Big(\frac{\partial^2 U}{\partial x\partial \tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant -\exp(2t)\int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x + \int_0^1 \Big(\frac{\partial U}{\partial t}\Big)^2\,\mathrm{d}x\Big)\Big|_{t=0} + 2\int_0^t \exp(2\tau)\int_0^1 \Big(\frac{\partial U}{\partial \tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau + C\int_0^t \exp(-\tau)\,\mathrm{d}\tau,$$

or

$$\int_0^t \exp(2\tau)\int_0^1 \Big(\frac{\partial^2 U}{\partial x\partial\tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C\exp(t). \tag{3.6}$$

Similarly,

$$\int_0^t \exp(2\tau)\int_0^1 \Big(\frac{\partial^2 V}{\partial x\partial\tau}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant C\exp(t). \tag{3.7}$$

Multiplying (2.5) scalarly by $\exp(2t)(\partial^2 U/\partial t^2)$, using integration by parts and boundary conditions (3.2), we get

$$\begin{aligned}
&\exp(2t)\int_0^1\Big(\frac{\partial^2 U}{\partial t^2}\Big)^2\,\mathrm{d}x+\frac12\int_0^1\exp(2t)(1+S)^p\frac{\partial}{\partial t}\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\\
&\quad= -p\int_0^1\exp(2t)(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial}{\partial t}\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)\,\mathrm{d}x\\
&\qquad -p\int_0^1\exp(2t)(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial}{\partial t}\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)\,\mathrm{d}x.
\end{aligned}$$

Let us integrate this equality from 0 to t. Using integration by parts, we obtain

$$\begin{aligned}
&\frac{\exp(2t)}{2}\int_0^1(1+S)^p\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\\
&\leqslant \frac12\int_0^1\Big(\frac{\partial^2 U}{\partial t\partial x}\Big)^2\,\mathrm{d}x\Big|_{t=0}+\int_0^t\int_0^1\exp(2\tau)(1+S)^p\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\frac{p}{2}\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\Big[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad-p\exp(2t)\int_0^1(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x+p\int_0^1\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\Big|_{t=0}\\
&\quad+2p\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^3\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+p(p-1)\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-2}\\
&\qquad\times\Big[\Big(\frac{\partial U}{\partial x}\Big)^5+\Big(\frac{\partial U}{\partial x}\Big)^3\Big(\frac{\partial V}{\partial x}\Big)^2\Big]\frac{\partial^2 U}{\partial\tau\partial x}\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+3p\int_0^t\int_0^1\exp(2\tau)(1+S)^{p-1}\Big(\frac{\partial U}{\partial x}\Big)^2\Big(\frac{\partial^2 U}{\partial\tau\partial x}\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad-p\exp(2t)\int_0^1(1+S)^{p-1}\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\\
&\quad+p\int_0^1\frac{\partial U}{\partial x}\Big(\frac{\partial V}{\partial x}\Big)^2\frac{\partial^2 U}{\partial t\partial x}\,\mathrm{d}x\Big|_{t=0}
\end{aligned}$$

$$
\begin{aligned}
&+ 2p \int_0^t \int_0^1 \exp(2\tau)(1+S)^{p-1} \frac{\partial U}{\partial x} \Big(\frac{\partial V}{\partial x}\Big)^2 \frac{\partial^2 U}{\partial \tau \partial x} \, \mathrm{d}x \, \mathrm{d}\tau \\
&+ p(p-1) \int_0^t \int_0^1 \exp(2\tau)(1+S)^{p-2} \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big] \frac{\partial U}{\partial x} \\
&\quad \times \Big(\frac{\partial V}{\partial x}\Big)^2 \frac{\partial^2 U}{\partial \tau \partial x} \, \mathrm{d}x \, \mathrm{d}\tau \\
&+ p \int_0^t \int_0^1 \exp(2\tau)(1+S)^{p-1} \Big(\frac{\partial V}{\partial x}\Big)^2 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \, \mathrm{d}x \, \mathrm{d}\tau \\
&+ 2p \int_0^t \int_0^1 \exp(2\tau)(1+S)^{p-1} \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} \frac{\partial^2 U}{\partial \tau \partial x} \frac{\partial^2 V}{\partial \tau \partial x} \, \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned}
$$

Now using Theorem 3.2, Schwarz's inequality, and a priori estimates (3.5)–(3.7), we deduce

$$
\begin{aligned}
&\frac{\exp(2t)}{2} \int_0^1 \Big(\frac{\partial^2 U}{\partial t \partial x}\Big)^2 \mathrm{d}x \\
&\quad \leqslant C_1 + C_2 \exp(t) + C_3 \int_0^t \exp(2\tau) \exp(-\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau \\
&\qquad + \frac{\exp(2t)}{8} \int_0^1 \Big(\frac{\partial^2 U}{\partial t \partial x}\Big)^2 \mathrm{d}x + C_4 \exp(-t) \\
&\qquad + \int_0^t \exp(2\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau + C_5 \int_0^t \exp(-\tau) \, \mathrm{d}\tau \\
&\qquad + \int_0^t \exp(2\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau + C_6 \int_0^t \exp(-3\tau) \, \mathrm{d}\tau \\
&\qquad + C_7 \int_0^t \exp(2\tau) \exp(-\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau + \frac{\exp(2t)}{8} \int_0^1 \Big(\frac{\partial^2 U}{\partial t \partial x}\Big)^2 \mathrm{d}x \\
&\qquad + C_8 \exp(-t) + \int_0^t \exp(2\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau + C_9 \int_0^t \exp(-\tau) \, \mathrm{d}\tau \\
&\qquad + \int_0^t \exp(2\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau + C_{10} \int_0^t \exp(-3\tau) \, \mathrm{d}\tau \\
&\qquad + C_{11} \int_0^t \exp(2\tau) \exp(-\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau \\
&\qquad + C_{12} \int_0^t \exp(\tau) \int_0^1 \Big(\frac{\partial^2 U}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau \\
&\qquad + C_{12} \int_0^t \exp(\tau) \int_0^1 \Big(\frac{\partial^2 V}{\partial \tau \partial x}\Big)^2 \mathrm{d}x \, \mathrm{d}\tau \\
&\quad \leqslant \frac{\exp(2t)}{4} \int_0^1 \Big(\frac{\partial^2 U}{\partial t \partial x}\Big)^2 \mathrm{d}x + C_{13} \exp(t),
\end{aligned}
$$

i.e.,

$$\int_0^1 \Big(\frac{\partial^2 U}{\partial t \partial x}\Big)^2 \,\mathrm{d}x \leqslant C \exp(-t). \tag{3.8}$$

Analogously,

$$\int_0^1 \Big(\frac{\partial^2 V}{\partial t \partial x}\Big)^2 \,\mathrm{d}x \leqslant C \exp(-t). \tag{3.9}$$

Finally, using Theorem 3.1, estimates (3.8), (3.9), and relation (2.33), we get the validity of the Theorem 3.3. □

Remarks.

(1) The existence of globally defined solutions of the problems (2.1)–(2.4) and (3.1)–(3.4) can be obtained by a routine procedure. One first establishes the existence of local solutions on a maximal time interval and then uses the derived a priori estimates to show that the solutions cannot escape in finite time (see, for example, [23], [28]). In particular, the following theorems hold:

Theorem 3.4. *If* $a(S) = (1+S)^p$, $0 < p \leqslant 1$, $U_0, V_0 \in H^3(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$, *then there is a unique solution* (U,V) *to* (2.1)–(2.4) *such that* $U, V \in H^2(Q)$.

Theorem 3.5. *If* $a(S) = (1+S)^p$, $0 < p \leqslant 1$, $U_0, V_0 \in H^3(0,1)$, $U_0(0) = V_0(0) = U_0(1) = V_0(1) = 0$, *then there is a unique solution* (U,V) *to* (3.1)–(3.4) *such that* $U, V \in H^2(Q) \cap L_2((0,\infty); H_0^1(0,1))$.

(2) Mathematical results given in the second and third sections show the difference between stabilization characters of solutions with nonhomogeneous and homogeneous boundary conditions.

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