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A SIMPLE REGULARIZATION METHOD FOR THE ILL-POSED EVOLUTION EQUATION

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Abstract. The nonhomogeneous backward Cauchy problem

$$u_t + Au(t) = f(t), \quad u(T) = \varphi,$$

where A is a positive self-adjoint unbounded operator which has continuous spectrum and f is a given function being given is regularized by the well-posed problem. New error estimates of the regularized solution are obtained. This work extends earlier results by N. Boussetila and by M. Denche and S. Djezzar.

Keywords: nonlinear parabolic problem, backward problem, semigroup of operators, ill-posed problem, contraction principle

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1. Introduction

Let H be a Hilbert space. For a positive number T we shall consider the problem of finding the function $u\colon [0,T] \to H$ from the system

$$u_t + Au = f(t), \quad 0 < t < T, \tag{1}$$

$$u(T) = \varphi, \tag{2}$$

for a prescribed final value φ in H and a given $f\colon [0,T] \to H$. The operator A is a positive self-adjoint operator such that $0 \in \varrho(A)$. This problem is well known to be severely ill-posed and regularization methods for it are required.

The case $f = 0$ and A a self-adjoint operator having the discrete spectrum on H has been considered by many authors using different approaches. Such authors as Lattès and Lions [13], Miller [16], Payne [17] have approximated (1)–(2) by perturbing the operator A. This method is called the quasi-reversibility method (QR). The main idea of the method is adding a "corrector" to the main equation. In fact, they considered the problem

$$u_t + Au - \varepsilon A^* Au = 0, \quad t \in [0,T], \quad u(T) = \varphi.$$

The stability magnitude of the method is of order $\mathrm{e}^{c\varepsilon^{-1}}$. In [10], the problem is approximated

$$u_t + Au + \varepsilon Au_t = 0, \quad t \in [0,T], \quad u(T) = \varphi. \tag{3}$$

Ames and Hughes [1] gave a survey on the relation between the operator-theoretic methods and the QR method treating the abstract Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \quad u(T) = \chi, \quad 0 < t < T.$$

The authors considered the problem both in the Hilbert space and in the Banach space. They also gave many structural stability results. Very recently, using the QR method, Yongzhong Huang and Quan Zheng [12] considered the problem in an abstract setting, i.e., $-A$ is the generator of an analytic semigroup in a Banach space.

In [19], Showalter presented a different method called the quasiboundary value (QBV) method to regularize the linear homogeneous problem which gave a stability estimate better than the one of the methods discussed. The main idea of the method is adding an appropriate "corrector" to the final data. Using the method, Clark-Oppenheimer in [3], and Denche-Bessila very recently in [4], regularized the backward problem by replacing the final condition by

$$u(T) + \varepsilon u(0) = \varphi$$

and

$$u(T) - \varepsilon u'(0) = \varphi,$$

respectively.

Very recently, an improved version for the homogeneous ill-posed problem has been also given in [11] by Dinh Nho Hao and his group.

To our knowledge, the case when A has discrete spectrum has been treated in many recent papers, such as [11], [4]. However, the literature on the homogeneous case of the problem in the case A has continuous spectrum are quite scarce. For some related works on this problem, we refer the reader to N. Boussetila and F. Rebbani [2], Denche and S. Djezzar [5].

In the present paper we shall use a new truncated method to extend the continuous dependence results of [2], [5] to more general nonhomogeneous problems. Recently, the truncated regularization method has been effectively applied to solve the sideways heat equation [6], [7], a more general sideways parabolic equation [8] and backward heat [9]. This regularization method is rather simple and convenient for dealing with some ill-posed problems. However, as far as we know, there have not any results of the truncated method for treating the problem (1)–(2) until now. Our article is the first work in the nonhomogeneous backward Cauchy problem when the operator A has continuous spectrum. Moreover, we establish some new error estimates including the order of Hölder type. Especially, the convergence of the approximate solution at $t = 0$ is also proved.

This paper is organized as follows. In the next section, for easy reading, we summarize some well-known facts concerning the semigroup of operators. Stability estimates of the regularized solution will be presented in Section 3.

2. Semigroup of operators

In this section we present the notation and the functional setting which will be used in this paper and prepare some material which will be used in our analysis.

We denote by $\{E_\lambda,\ \lambda \geqslant 0\}$ the spectral resolution of the identity associated to A.

We also denote by $S(t) = \mathrm{e}^{-tA} = \int_0^\infty \mathrm{e}^{-t\lambda}\,\mathrm{d}E_\lambda \in \mathcal{L}(\mathcal{H})$, $t \geqslant 0$, the C_0-semigroup generated by $-A$.

The first main theorem of spectral theory (Hilbert (1906), Neumann (1929))**.** *Let $A\colon D(A) \subset H \to H$ be a self-adjoint operator on the Hilbert space H over K. Then there exists exactly one spectral family $\{E_\lambda\}$ such that*

$$Au = \int_0^{+\infty} \lambda\,\mathrm{d}E_\lambda u$$

for all $u \in D(A)$.

In this connection, $u \in D(A)$ iff the integral (4) exists, i.e.,

$$\int_0^{+\infty} \lambda^2\,\mathrm{d}\|E_\lambda u\|^2 < \infty.$$

Definition. Let $A\colon D(A) \subset H \to H$ be a self-adjoint operator on the Hilbert space H over K and let $f, g\colon \mathbb{R} \to K$ be piecewise continuous functions. We set

$$D(f(A)) = \left\{u \in H\colon \int_0^{+\infty} |f(\lambda)|^2\,\mathrm{d}\|E_\lambda u\|^2 < \infty\right\}$$

and define the linear operator $f(A)\colon\ D(A) \subset H \to H$ by the formula

$$f(A)u = \int_0^{+\infty} f(\lambda)\,\mathrm{d}E_\lambda u$$

for all $u \in D(f(A))$.

3. Regularization and error estimates

Now we are ready to state and prove the main results of this paper.

If the problem (1)–(2) admits a solution u then it can be represented by

$$u(t) = \int_0^\infty \mathrm{e}^{\lambda(T-t)}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \mathrm{e}^{\lambda(s-t)}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s. \tag{4}$$

Since $t < T$, we know from (4) that the terms $\mathrm{e}^{-(t-T)\lambda}$ and $\mathrm{e}^{-(t-s)\lambda}$ are the cause of unstability. So, to regularize problem (1)–(2), we hope to recover the stability of problem (4) by filtering the high frequencies using a suitable method. The essence of our regularization method is just to eliminate all high frequencies from the solution, and instead consider (4) only for $\lambda \leqslant A_\varepsilon$, where A_ε is an appropriate positive constant satisfying $\lim\limits_{\varepsilon\to 0} A_\varepsilon = \infty$. We note that A_ε is a constant which will be selected appropriately as the regularization parameter. Let φ and φ_ε denote the exact and the measured data at $t = T$, respectively, which satisfy

$$\|\varphi - \varphi_\varepsilon\| \leqslant \varepsilon.$$

Hence, the ill-posed problem (1)–(2) can be approximated by the problem

$$u_\varepsilon(t) = \int_0^\infty \mathrm{e}^{\lambda(T-t)}\chi_{[0,A_\varepsilon]}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \mathrm{e}^{\lambda(s-t)}\chi_{[0,A_\varepsilon]}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \tag{5}$$

where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$ for $a < b$.

Our first main theorem is

Theorem 1. *The solution defined in* (5) *depends continuously (in* $C([0,T];H)$*) on* φ*, which means that, if* u *and* v *are two solutions of problem* (5) *corresponding to the final value* φ *and* ω*, respectively, then*

$$\|u(t) - v(t)\| \leqslant \mathrm{e}^{(T-t)A_\varepsilon}\|\varphi - \omega\|.$$

P r o o f. It is well known that for all $t \in [0,T]$,

$$(6) \qquad u(t) - v(t) = \int_0^{A_\varepsilon} \mathrm{e}^{\lambda(T-t)} \, \mathrm{d}E_\lambda(\varphi - \omega).$$

Using (6), we obtain

$$\begin{aligned} \|u(t) - v(t)\|^2 &\leqslant \mathrm{e}^{2(T-t)A_\varepsilon} \int_0^\infty \mathrm{d}\|E_\lambda(\varphi - \omega)\|^2 \\ &\leqslant \mathrm{e}^{2(T-t)A_\varepsilon} \|\varphi - \omega\|^2. \end{aligned}$$

This inequality implies that the solution of the problem (5) depends continuously on φ and Theorem 1 is proved. $\square$

Theorem 2. *Let $u \in C([0,T];H)$ be a solution of (1)–(2). Assume that f has the eigenfunction expansion $f(t) = \int_0^\infty \mathrm{d}E_\lambda f(t)$ satisfying*

$$(7) \qquad \int_0^T \mathrm{e}^{2\lambda s} \, \mathrm{d}\|E_\lambda f(s)\|^2 \, \mathrm{d}s < \infty.$$

Then the following estimate is true

$$(8) \qquad \|u(t) - u_\varepsilon(t)\| \leqslant \mathrm{e}^{-tA_\varepsilon} N, \quad \forall\, t \in (0,T],$$

where

$$N = \sqrt{2\Big(\|u(0)\|^2 + T\int_0^T \mathrm{e}^{2\lambda s} \, \mathrm{d}\|E_\lambda f(s)\|^2 \, \mathrm{d}s\Big)},$$

and u_ε is the unique solution of problem (5).

P r o o f. The functions $u(t)$, $u_\varepsilon(t)$ have the expansion

$$(9) \qquad u(t) = \int_0^\infty \mathrm{e}^{\lambda(T-t)} \, \mathrm{d}E_\lambda \varphi - \int_t^T \int_0^\infty \mathrm{e}^{\lambda(s-t)} \, \mathrm{d}E_\lambda f(s) \, \mathrm{d}s,$$

$$(10) \qquad u_\varepsilon(t) = \int_0^\infty \mathrm{e}^{\lambda(T-t)} \chi_{[0,A_\varepsilon]} \, \mathrm{d}E_\lambda \varphi - \int_t^T \int_0^\infty \mathrm{e}^{\lambda(s-t)} \chi_{[0,A_\varepsilon]} \, \mathrm{d}E_\lambda f(s) \, \mathrm{d}s.$$

Hence, we get

$$\begin{aligned}
u(t)-u_\varepsilon(t) &= \int_{A_\varepsilon}^{\infty} \mathrm{e}^{\lambda(T-t)}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_{A_\varepsilon}^{\infty} \mathrm{e}^{\lambda(s-t)}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&= \int_{A_\varepsilon}^{\infty} \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda T}\,\mathrm{d}E_\lambda\varphi - \int_0^T\int_{A_\varepsilon}^{\infty} \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&\quad + \int_0^t\int_{A_\varepsilon}^{\infty} \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&= \int_0^{\infty} \mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]}\mathrm{e}^{\lambda T}\,\mathrm{d}E_\lambda\varphi - \int_0^T\int_0^{\infty} \mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]}\mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&\quad + \int_0^t\int_0^{\infty} \mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]}\mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned}
\|u(t)-u_\varepsilon(t)\|^2 &\leqslant 2\int_0^{\infty} (\mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]})^2\Big(\mathrm{e}^{\lambda T}\,\mathrm{d}E_\lambda\varphi - \int_0^T \mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s\Big)^2 \\
&\quad + 2\int_0^{\infty} (\mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]})^2\Big(\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s\Big)^2.
\end{aligned}$$

Using

$$(\mathrm{e}^{-\lambda t}\chi_{[A_\varepsilon,\infty]})^2 \leqslant \mathrm{e}^{-2tA_\varepsilon}$$

and

$$\|u(0)\|^2 = \int_0^{\infty}\Big(\mathrm{e}^{\lambda T}\,\mathrm{d}E_\lambda\varphi - \int_0^T \mathrm{e}^{\lambda s}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s\Big)^2$$

we obtain

$$\begin{aligned}
\|u(t)-u_\varepsilon(t)\|^2 &\leqslant 2\mathrm{e}^{-2tA_\varepsilon}\Big(\|u(0)\|^2 + T\int_0^T\int_0^{\infty} \mathrm{e}^{2\lambda s}(\mathrm{d}E_\lambda f(s)\,\mathrm{d}s)^2\Big) \\
&= 2\mathrm{e}^{-2tA_\varepsilon}\Big(\|u(0)\|^2 + T\int_0^T \mathrm{e}^{2\lambda s}\,\mathrm{d}\|E_\lambda f(s)\|^2\,\mathrm{d}s\Big).
\end{aligned}$$

This completes the proof of Theorem 2. □

Remark 1. If $f(t)=0$ and $A_\varepsilon = T^{-1}\ln\varepsilon^{-1}$, the estimate (8) becomes

$$\|u(t)-u_\varepsilon(t)\| \leqslant \sqrt{2}\varepsilon^{t/T}\|u(0)\|. \tag{11}$$

This error is also given by Clark and Oppenheimer [3], Tautenhahn [21].

Theorem 3. *Let $u \in C([0,T];H)$ be a solution of* (1)–(2). *Assume that u has the eigenfunction expansion $u(t) = \int_0^\infty \mathrm{d}E_\lambda u(t)$.*

a) *If there exists a positive β such that*

$$\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2 < \infty, \tag{12}$$

then for every $t \in [0,T]$

$$\|u(t) - u_\varepsilon(t)\| \leqslant A_\varepsilon^{-\beta}\sqrt{\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2}. \tag{13}$$

b) *If there exists a positive m such that*

$$\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2 < \infty, \tag{14}$$

then for every $t \in [0,T]$

$$\|u(t) - u_\varepsilon(t)\| \leqslant \mathrm{e}^{-mA_\varepsilon}\sqrt{\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2} \tag{15}$$

where u_ε is the unique solution of problem (5).

P r o o f. a) Due to (9) and (10), we have

$$\begin{aligned}
u(t) - u_\varepsilon(t) &= \int_0^\infty \mathrm{e}^{\lambda(T-t)}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \mathrm{e}^{\lambda(s-t)}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&= \int_0^\infty \lambda^{-\beta}\mathrm{e}^{\lambda(T-t)}\lambda^{\beta}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda\varphi \\
&\quad - \int_t^T\int_0^\infty \lambda^{\beta}\mathrm{e}^{\lambda(s-t)}\lambda^{-\beta}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned}
\|u(t) - u_\varepsilon(t)\|^2 &= \int_0^\infty (\lambda^{-\beta}\chi_{[A_\varepsilon,\infty]})^2\Big(\mathrm{e}^{\lambda(T-t)}\lambda^{\beta}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \lambda^{\beta}\mathrm{e}^{\lambda(s-t)}f(s)\,\mathrm{d}s\Big)^2 \\
&\leqslant A_\varepsilon^{-2\beta}\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2.
\end{aligned}$$

b) Due to (9) and (10), we also have

$$\begin{aligned}
u(t) - u_\varepsilon(t) &= \int_0^\infty \mathrm{e}^{\lambda(T-t)}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \mathrm{e}^{\lambda(s-t)}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s \\
&= \int_0^\infty \mathrm{e}^{-m\lambda}\mathrm{e}^{\lambda(T-t)}\mathrm{e}^{m\lambda}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda\varphi \\
&\quad - \int_t^T\int_0^\infty \mathrm{e}^{m\lambda}\mathrm{e}^{\lambda(s-t)}\mathrm{e}^{-m\lambda}\chi_{[A_\varepsilon,\infty]}\,\mathrm{d}E_\lambda f(s)\,\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned}\|u(t)-u_\varepsilon(t)\|^2 &= \int_0^\infty (\mathrm{e}^{-m\lambda}\chi_{[A_\varepsilon,\infty]})^2 \\ &\quad\times\left(\mathrm{e}^{\lambda(T-t)}\mathrm{e}^{m\lambda}\,\mathrm{d}E_\lambda\varphi - \int_t^T\int_0^\infty \mathrm{e}^{m\lambda}\mathrm{e}^{\lambda(s-t)}f(s)\,\mathrm{d}s\right)^2 \\ &\leqslant \mathrm{e}^{-2mA_\varepsilon}\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2.\end{aligned}$$

□

Theorem 4. *Let $\varphi_\varepsilon \in H$ be a measured data such that*

$$\|\varphi_\varepsilon - \varphi\| \leqslant \varepsilon.$$

Suppose problem (1)–(2) *has a unique solution $u \in C([0,T];H)$ and let $w_\varepsilon \in C([0,T];H)$ be the unique solution of problem* (5) *corresponding to φ_ε.*

a) *If* (12) *holds then for $A_\varepsilon = (p/T)\ln(1/\varepsilon)$, $(0<p<1)$ we get*

$$\|w_\varepsilon(t)-u(t)\| \leqslant \varepsilon^{(pt/T)+1-p} + \left(\frac{T}{p}\right)^\beta \left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\beta}\sqrt{\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2}.$$

b) *If* (14) *holds then for $A_\varepsilon = (T+m)^{-1}\ln(1/\varepsilon)$ we get*

$$\|w_\varepsilon(t)-u(t)\| \leqslant \varepsilon^{(m/(T+m))}\left(\varepsilon^{(t/(T+m))} + \sqrt{\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2}\right).$$

Proof. Using Theorem 2, we get

$$\|w_\varepsilon(t)-u_\varepsilon(t)\| \leqslant \mathrm{e}^{(T-t)A_\varepsilon}\|\varphi_\varepsilon-\varphi\| \leqslant \varepsilon\mathrm{e}^{(T-t)A_\varepsilon}.$$

If (12) holds then

$$\begin{aligned}\|w_\varepsilon(t)-u(t)\| &\leqslant \|w_\varepsilon(t)-u_\varepsilon(t)\| + \|u_\varepsilon(t)-u(t)\| \\ &\leqslant \varepsilon\mathrm{e}^{(T-t)A_\varepsilon} + A_\varepsilon^{-\beta}\sqrt{\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2} \\ &= \varepsilon^{(pt/T)+1-p} + \left(\frac{T}{p}\right)^\beta\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{-\beta}\sqrt{\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2}.\end{aligned}$$

If (14) holds then

$$\begin{aligned}
\|w_\varepsilon(t)-u(t)\| &\leqslant \|w_\varepsilon(t)-u_\varepsilon(t)\|+\|u_\varepsilon(t)-u(t)\| \\
&\leqslant \varepsilon \mathrm{e}^{(T-t)A_\varepsilon}+\mathrm{e}^{-mA_\varepsilon}\sqrt{\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2} \\
&= \varepsilon^{m/(T+m)}\biggl(\varepsilon^{t/(T+m)}+\sqrt{\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2}\biggr).
\end{aligned}$$

□

Remark 2. 1. One superficial advantage of this method is that there is an error estimation at the original time $t=0$, which is not given in [14], [23]. We have the following estimates at $t=0$:

$$\|w_\varepsilon(0)-u(0)\| \leqslant \varepsilon^{1-p}+\Bigl(\frac{T}{p}\Bigr)^\beta\Bigl(\ln\Bigl(\frac{1}{\varepsilon}\Bigr)\Bigr)^{-\beta}\sqrt{\int_0^\infty \lambda^{2\beta}\,\mathrm{d}\|E_\lambda u(t)\|^2} \tag{16}$$

and

$$\|w_\varepsilon(0)-u(0)\| \leqslant \varepsilon^{m/(T+m)}\biggl(1+\sqrt{\int_0^\infty \mathrm{e}^{2m\lambda}\,\mathrm{d}\|E_\lambda u(t)\|^2}\biggr). \tag{17}$$

2. It follows from (16) that if $\varepsilon \to 0$ then the second term on the right-hand side of the inequality approaches zero with logarithmic speed, and the first as a power. So, the right-hand side of (16) is logarithmic stability estimate. This logarithmic order is also given in [2], [9], [14], [11], [21], [22].

It follows from (17) that we obtain the Hölder stability. As we know, the error of Hölder form is the optimal error. We note again that such order is not considered in [11].

3. Notice that the error in [5] (see Theorem 2.6, page 5) is

$$\|u(0)-u_\varepsilon(0)\| \leqslant NT\mathrm{e}^{kT}\Bigl(1+\ln\Bigl(\frac{T}{\varepsilon}\Bigr)\Bigr)^{-1}. \tag{18}$$

Comparing (16) and (17) with (18) and the results obtained in [11], [4], [5], [23], we realize (17) is sharp and the best known estimate. This is a generalization of many results discussed earlier.

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