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SOME NOTES ON EMBEDDING FOR ANISOTROPIC
SOBOLEV SPACES

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Abstract. In this paper, we prove new embedding theorems for generalized anisotropic Sobolev spaces, $W_{\Lambda^{p,q}(w)}^{r_1, \dots, r_n}$ and $W_X^{r_1, \dots, r_n}$, where $\Lambda^{p,q}(w)$ is the weighted Lorentz space and X is a rearrangement invariant space in \mathbb{R}^n . The main methods used in the paper are based on some estimates of nonincreasing rearrangements and the applications of B_p weights.

Keywords: Lorentz spaces, Sobolev spaces, Besov spaces, Sobolev embedding, rearrangement invariant spaces

MSC 2010: 46E35, 42B35

1. INTRODUCTION

Let $\mathcal{M}(X, \mu)$ be the class of all measurable and almost everywhere finite functions on X . For $f \in \mathcal{M}(X, \mu)$, a non-increasing rearrangement of f is the non-increasing function f^* on $\mathbb{R}_+ = (0, +\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \inf\{\lambda: \mu_f(\lambda) \leq t\}, \quad 0 < t < \infty,$$

where

$$\mu_f(\lambda) = \mu\{x \in X: |f(x)| > \lambda\}, \quad \lambda \geq 0.$$

Assume that $0 < q, p < \infty$. A function $f \in \mathcal{M}(X, \mu)$ belongs to the Lorentz space $L^{q,p}(X)$ if

$$\|f\|_{q,p} = \left(\int_0^\infty (t^{1/q} f^*(t))^p \frac{dt}{t} \right)^{1/p} < \infty.$$

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For $0 < p < \infty$, the space $L^{p,\infty}(X)$ is defined as the class of all $f \in \mathcal{M}(X, \mu)$ such that

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty.$$

We also let $L^{\infty,\infty}(X) = L^\infty(X)$.

Let w be a weight on \mathbb{R}_+ (a nonnegative locally integrable function on \mathbb{R}_+). If $(X, \mu) = (\mathbb{R}_+, w(t) dt)$, we replace $L^{q,p}(X)$ with $L^{q,p}(w)$. For $0 < p, q < \infty$, or $0 < p \leq \infty$ and $q = \infty$, the weighted Lorentz space $\Lambda_{\mathbb{R}^n}^{p,q}(w) = \Lambda^{p,q}(w)$ is defined in [5, Ch. 2] by

$$\Lambda^{p,q}(w) = \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{\Lambda^{p,q}(w)} = \|f^*\|_{L^{p,q}(w)} < \infty\}.$$

If $p = q$, denote $\Lambda^p(w) = \Lambda^{p,p}(w)$. It is well known [5, Ch. 2] that

$$\Lambda^{p,q}(1) = L^{p,q}(\mathbb{R}^n)$$

and if $0 < p, q < \infty$, then

$$\Lambda^{p,q}(w) = \Lambda^q(\tilde{w}),$$

where

$$\tilde{w}(t) = W^{q/p-1}(t)w(t), \quad W(t) = \int_0^t w(s) ds.$$

In the following part of this paper, we will always denote $W(t) = \int_0^t w(s) ds$.

The weighted Lorentz spaces have close connection with weights of $B_p, B_{p,\infty}^\infty$ for $0 < p < \infty$ (see [5, Ch. 1]). Let A be the Hardy operator defined as follows:

$$Af(t) = \frac{1}{t} \int_0^t f(s) ds, \quad t > 0.$$

The space $L_{\text{dec}}^p (L_{\text{dec}}^{p,\infty})$ is the cone of all nonnegative non-increasing functions in $L^p (L^{p,\infty})$. We denote $w \in B_p$ if

$$A: L_{\text{dec}}^p(w) \rightarrow L^p(w)$$

is bounded and denote $w \in B_{p,\infty}^\infty$ if

$$A: L_{\text{dec}}^{p,\infty}(w) \rightarrow L^{p,\infty}(w)$$

is bounded.

Let $1 \leq p < \infty$ and $r \in \mathbb{N}$. We denote by W_p^r the isotropic Sobolev spaces for functions $f \in L^p(\mathbb{R}^n)$ which have all generalized derivatives $D^s f$ ($s = (s_1, \dots, s_n)$) of order

$$|s| = s_1 + \dots + s_n \leq r,$$

which belong to $L^p(\mathbb{R}^n)$. It is well known that for

$$1 \leq p < n/r \quad \text{and} \quad q^* = np/(n - rp)$$

we get $W_p^r \subset L^{q^*}(\mathbb{R}^n)$ (see [15], [3]). This embedding has been generalized and developed in different directions (see [7], [9], [12], [13], [14], [17], [18] for details and references).

We denote by $W_{p,s}^{r_1, \dots, r_n}$ ($1 \leq p < \infty$, $0 < s < \infty$, $r_1, \dots, r_n \in \mathbb{N}$) the anisotropic space of functions $f \in L^{p,s}$ that have generalized partial derivatives

$$D_i^{r_i} f = \frac{\partial^{r_i} f}{\partial x_i^{r_i}} \in L^{p,s}.$$

We write $r = n \left(\sum_{i=1}^n r_i^{-1} \right)^{-1}$. In [6], V.I. Kolyada got that if $1 < p < n/r$, $q^* = np/(n - rp)$, then

$$W_{p,s}^{r_1, \dots, r_n} \subset L^{q^*,s},$$

which of course implies

$$W_p^{r_1, \dots, r_n} \subset L^{q^*,p} \quad \text{and} \quad W_p^{r_1, \dots, r_n} \subset L^{q^*}.$$

Moreover, in the same paper it is also proved that if $p < q < \infty$ and $1/p - 1/q < r/n$, then for any $\theta > 0$

$$(*) \quad W_{p,s}^{r_1, \dots, r_n} \subset L^{q,\theta}.$$

Let w be a weight in \mathbb{R}_+ . We denote $W_{\Lambda^{p,s}(w)}^{r_1, \dots, r_n}$ ($1 \leq p < \infty$, $0 < s < \infty$, $r_1, \dots, r_n \in \mathbb{N}$) the space of functions $f \in \Lambda^{p,s}(w)$ which have generalized partial derivatives $D_i^{r_i} f \in \Lambda^{p,s}(w)$ and denote

$$\|f\|_{W_{\Lambda^{p,s}(w)}^{r_1, \dots, r_n}} = \|f\|_{\Lambda^{p,s}(w)} + \sum_{i=1}^n \|D_i^{r_i} f\|_{\Lambda^{p,s}(w)}.$$

If a function f is defined on \mathbb{R}^n , $k \in \mathbb{N}$, e_i is the unit coordinate vector, then we set

$$\Delta_i^k(h)f(x) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x + jhe_i) \quad (h \in \mathbb{R})$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}$. Let $1 \leq p, q < \infty$, $1 \leq s < \infty$, $\alpha_1, \dots, \alpha_n$ be positive numbers and w be a weight in \mathbb{R}_+ . Suppose $k_i > \alpha_i$, $k_i \in \mathbb{N}$. We define Besov space $B_{\Lambda^{q,p}(w),s}^{\alpha_1, \dots, \alpha_n}$ as the space of functions $f \in \Lambda^{q,p}(w)$ for which

$$\|f\|_{B_{\Lambda^{q,p}(w),s}^{\alpha_1, \dots, \alpha_n}} = \|f\|_{\Lambda^{q,p}(w)} + \sum_{j=1}^n \left(\int_0^\infty \left[h^{-\alpha_j} \|\Delta_j^{k_j}(h)f\|_{\Lambda^{q,p}(w)} \right]^s \frac{dh}{h} \right)^{1/s} < \infty.$$

We note that the definition of $B_{\Lambda^{q,p}(w),s}^{\alpha_1,\dots,\alpha_n}$ does not depend on the choice of $k_i > \alpha_i$ (see [3, Sec. 18], [10]).

In this paper, we prove in Theorem 3.1 that if w satisfies some special conditions, and $1/q^* = 1/p - r/n$, then if $1 \leq p < n/r$, $0 < s < \infty$ we get

$$W_{\Lambda^{p,s}(w)}^{r_1,\dots,r_n} \subset \Lambda^{q^*,s}(w),$$

and if $1 \leq p < q < q^*$, $0 < s < \infty$, then for any $qs/q^* < \theta < \infty$, we get

$$W_{\Lambda^{p,s}(w)}^{r_1,\dots,r_n} \subset \Lambda^{q,\theta}(w).$$

If we put $p > 1$, $w = 1$, then the range of θ may be enlarged to $0 < \theta < \infty$, and the above embedding is the result in [6]. The other result, Theorem 3.4, extends Theorem 4 in [6] by replacing the Lorentz space $L^{p,q}$ by the weighted Lorentz space $\Lambda^{p,q}(w)$ where w satisfies some mild conditions, which assumes as

$$W_{\Lambda^p(w)}^{r_1,\dots,r_n} \subset B_{\Lambda^{q,p}(w),p}^{\alpha_1,\dots,\alpha_n},$$

where $1 < p < q < \infty$, $1/p - 1/q < r/n$, $r = n \left(\sum_{i=1}^n r_i^{-1} \right)^{-1}$, and $\alpha_i = r_i [1 - nr^{-1}(1/p - 1/q)]$, $i = 1, \dots, n$.

On the other hand, we think it is meaningful if we can generalize (*) into the rearrangement invariant spaces. Indeed, there are many mathematicians, e.g., Bastero, Milman, Ruiz, Martín, Pustylnik, who have researched this kind of question and found many meaningful and important results. The first result we get, Theorem 3.5, can be regarded as an extension of Theorem 1.2 and Corollary 1.3 in [11]. The second result is Theorem 3.7, which can be regarded as a generalization of Theorem 3 in [6] in the background of the rearrangement invariant spaces.

Incidentally, in this paper, we do not deal with the situation of the limit indices. For example, if we let $p = n/r$ in Theorem 3.1, then $q^* = \infty$. We prefer to leave this kind of questions related to the weighted Lorentz spaces for further research.

Throughout this paper, we let $r_1, \dots, r_n \in \mathbb{N}$ and $r = n \left(\sum_{i=1}^n r_i^{-1} \right)^{-1}$. As usual, $f \approx g$ will indicate the existence of a universal constant $C > 0$ (independent of all parameters involved) so that $(1/C)f \leq g \leq Cf$.

This paper is organized as follows. Section 2 gives some necessary lemmas. In Section 3 we state and prove the main results of this paper.

2. SOME LEMMAS

Set $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$. Then we have

Lemma 2.1 (see [6, Lemma 8]). *If $f \in W_p^{r_1, \dots, r_n}$ ($1 \leq p < \infty$), then for any $0 < t < \tau < \infty$, there holds that*

$$(2.1) \quad f^*(t) \leq B \left[f^*(\tau) + \left(\frac{\tau}{t} \right)^r t^{r/n} \sum_{i=1}^n (D_i^{r_i} f)^{**}(\tau) \right],$$

where B depends only on r_1, \dots, r_n .

The following two lemmas disclose the relation between the weights in the weighted Lorentz spaces.

Lemma 2.2. *Let $0 < p, s < \infty$. If $w \in B_p$, then*

$$W(t)^{s/p-1} w(t) \in B_s.$$

Proof. Let $v(t) = W(t)^{s/p-1} w(t)$. Since $w \in B_p$, by [5, Ch. 1] we get

$$\int_0^r \frac{1}{W(t)^{1/p}} dt \leq C \frac{r}{W(r)^{1/p}}, \quad \forall r > 0.$$

Then

$$\int_0^r \frac{1}{V(t)^{1/s}} dt \leq C \frac{r}{V(r)^{1/s}}, \quad \forall r > 0,$$

where

$$V(t) = \int_0^t v(t) dt.$$

So $v \in B_s$. □

The next result is an important discovery of J. Soria in 1998.

Lemma 2.3 (see [16], [5]). $B_p = B_{p, \infty}^\infty$ ($0 < p < \infty$).

3. MAIN RESULTS

Note that in the following the constants may differ from one occurrence to another.

Theorem 3.1. *Let $1 \leq p < \infty$, $0 < s < \infty$, and w be a weight on \mathbb{R}_+ . Suppose $1/q^* = 1/p - r/n$. Let w satisfy the following conditions:*

- (i) $w \in B_p$,
- (ii) *there exists a number $a > 0$ such that*

$$W(t) \geq at, \quad \forall t > 0,$$

- (iii) *there exists a constant β with $\beta < 1$ such that*

$$(3.1) \quad W\left(\frac{t}{\xi}\right)^{s/q^* - 1} w\left(\frac{t}{\xi}\right) \leq C\xi^\beta W(t)^{s/q^* - 1} w(t), \quad \forall t > 0, \quad \forall \xi > 1.$$

Then for every $f \in W_{\Lambda^{p,s}(w)}^{r_1, \dots, r_n}$, there holds that if $p < n/r$, then

$$(3.2) \quad \|f\|_{\Lambda^{q^*,s}(w)} \leq C \sum_{i=1}^n \|D_i^{r_i} f\|_{\Lambda^{p,s}(w)},$$

and if $p < q < q^*$, $qs/q^* < \theta < \infty$, and the condition (ii) is substituted by the condition

- (ii') *there exists a number $a > 0$ such that*

$$W(t) \geq at, \quad \forall 0 < t < 1,$$

then

$$(3.3) \quad \|f\|_{\Lambda^{q,\theta}(w)} \leq C \left(\|f\|_{\Lambda^{p,s}(w)} + \sum_{i=1}^n \|D_i^{r_i} f\|_{\Lambda^{p,s}(w)} \right).$$

Proof. We first prove (3.2). Since $f \in \Lambda^{p,s}(w)$ and $q^* > p$, for any $\delta > 0$ we obtain

$$I_\delta = \int_\delta^\infty (W(t)^{1/q^*} f^*(t))^s \frac{w(t)}{W(t)} dt < \infty.$$

Set $\tau = At$. Using Lemma 2.1 (inequality (2.1) is also applicable to the space $W_{\Lambda^{p,s}(w)}^{r_1, \dots, r_n}$) and the condition $W(t) \geq at$, we get

$$\begin{aligned}
I_\delta &\leq B^s \int_\delta^\infty \left[W(t)^{1/q^*} \left(f^{**}(At) + A^r W(t)^{r/n} \sum_{i=1}^n (D_i^{r_i} f)^{**}(At) \right) \right]^s \frac{w(t)}{W(t)} dt \\
&\leq B_1 \left(\int_\delta^\infty W(t)^{s/q^*} f^{**s}(At) \frac{w(t)}{W(t)} dt \right. \\
&\quad \left. + \sum_{i=1}^n \int_\delta^\infty W(t)^{(1/q^* + r/n)s-1} w(t) (D_i^{r_i} f)^{**s}(t) dt \right) \\
&\leq B_1 \left(\frac{1}{A} \int_\delta^\infty W(t/A)^{s/q^*} f^{**s}(t) \frac{w(t/A)}{W(t/A)} dt \right) \\
&\quad + B_1 \left(\sum_{i=1}^n \int_\delta^\infty W(t)^{(1/q^* + r/n)s-1} w(t) (D_i^{r_i} f)^{**s}(t) dt \right) \\
&= B_1(I_1 + I_2).
\end{aligned}$$

By (3.1), we get

$$I_1 \leq \frac{1}{A^{1-\beta}} \int_\delta^\infty W(t)^{s/q^*} f^{**s}(t) \frac{w(t)}{W(t)} dt = \frac{1}{A^{1-\beta}} I_\delta.$$

Setting $A = (2B_1)^{1/(1-\beta)}$, we have

$$I_\delta \leq B_2 I_2.$$

So

$$\begin{aligned}
I_\delta^{1/s} &\leq B_3 \sum_{i=1}^n \left(\int_0^\infty W(t)^{(1/q^* + r/n)s-1} w(t) (D_i^{r_i} f)^{**s}(t) dt \right)^{1/s} \\
&= B_3 \sum_{i=1}^n \left(\int_0^\infty W(t)^{s/p-1} w(t) (D_i^{r_i} f)^{**s}(t) dt \right)^{1/s}.
\end{aligned}$$

For $w \in B_p$, by Lemma 2.2 we get $W^{s/p-1}w \in B_s$. Hence

$$I_\delta^{1/s} \leq B_3 \sum_{i=1}^n \|D_i^{r_i} f\|_{\Lambda^{p,s}(w)},$$

i.e. (3.2) is proved.

In the sequel, we prove the inequality (3.3). Firstly

$$\begin{aligned} J_1 &= \int_1^\infty W(t)^{\theta/q-1} w(t) f^{*\theta}(t) dt \\ &\leq \sup_{t>0} (W(t)^{1/p} f^*(t))^\theta \int_1^\infty W(t)^{(1/q-1/p)\theta-1} w(t) dt \\ &\leq C \|f\|_{\Lambda^{p,\infty}(w)}^\theta. \end{aligned}$$

On the other hand,

$$J_\delta = \int_\delta^\infty W(t)^{\theta/q-1} w(t) f^{*\theta}(t) dt \leq J_1 + \int_\delta^1 W(t)^{\theta/q-1} w(t) f^{*\theta}(t) dt.$$

As above, noticing that (3.1) is true if $\theta < qs/q^*$ and applying it we get

$$\begin{aligned} J_\delta &\leq C \left(J_1 + \sum_{i=1}^n \int_0^1 W(t)^{\theta r/n+\theta/q-1} w(t) (D_i^{r_i} f)^{**\theta}(t) dt \right) \\ &\leq C \left(J_1 + \sum_{i=1}^n \left(\sup_{t>0} (W(t)^{1/p} (D_i^{r_i} f)^{**}(t)) \right)^\theta \right. \\ &\quad \left. \times \int_0^1 W(t)^{\theta(r/n+1/q-1/p)-1} w(t) dt \right) \\ &\leq C \left(J_1 + \sum_{i=1}^n \|(D_i^{r_i} f)^{**}\|_{L^{p,\infty}(w)}^\theta \right). \end{aligned}$$

Since $w \in B_p$, by Lemma 2.3, we get $w \in B_{p,\infty}^\infty$. Thus

$$J_\delta \leq C \left(J_1 + \sum_{i=1}^n \|(D_i^{r_i} f)^*\|_{L^{p,\infty}(w)}^\theta \right) \leq C \left(J_1 + \sum_{i=1}^n \|D_i^{r_i} f\|_{\Lambda^{p,s}(w)}^\theta \right).$$

Now (3.3) is proved. □

Remark 3.2. There are many weights satisfying conditions of Theorem 3.1. For example,

- (i) $w = t^{-\alpha} + a$, where $0 < \alpha < \min(s/q^*, 1)$, $0 < a < \infty$;
- (ii)

$$w = \begin{cases} t^{-\alpha}, & \text{if } 0 < t < 1, \\ 1, & \text{if } t \geq 1, \end{cases}$$

where $0 \leq \alpha < 1$;

- (iii) w is a positive constant (in the case $p > 1$).

For the weight w in (i) and (ii), it is easy to see that the weighted Lorentz space $\Lambda^{p,q}(w)$ for $0 < p, q < \infty$ does not coincide with any Lorentz space $L^{r,s}$.

Theorem 3.3. Let $r_1, \dots, r_n \in \mathbb{N}$, $1 < p < q < \infty$, $1/p - 1/q < r/n$, $r = n \left(\sum_{i=1}^n r_i^{-1} \right)^{-1}$, and $\alpha_i = r_i [1 - nr^{-1}(1/p - 1/q)]$, $i = 1, \dots, n$. If w satisfies the conditions (i), (ii) in Theorem 3.1 and there exists a constant β with $\beta < 1$ such that

$$(3.4) \quad W\left(\frac{t}{\xi}\right)^{p/q-1} w\left(\frac{t}{\xi}\right) \leq C\xi^\beta W(t)^{p/q-1} w(t), \quad \forall t > 0, \quad \forall \xi > 1,$$

then for every $f \in W_{\Lambda^p(w)}^{r_1, \dots, r_n}$, the following inequality holds

$$(3.5) \quad \sum_{j=1}^n \left(\int_0^\infty [h^{-\alpha_j} \|\Delta_j^{r_j}(h)f\|_{\Lambda^{q,p}(w)}]^p \frac{dh}{h} \right)^{1/p} \leq C \sum_{k=1}^n \|D_k^{r_k} f\|_{\Lambda^p(w)},$$

where C is a constant that does not depend on f .

Proof. Due to Theorem 3.1, we know that $f \in \Lambda^{q,p}(w)$ and $\Delta_j^{r_j}(h)f \in \Lambda^{q,p}(w)$, $j = 1, \dots, n$. We estimate the first term of the left-hand side of (3.5). Suppose

$$\varphi_h(x) = \Delta_1^{r_1}(h)f(x).$$

By virtue of (2.1), for any $A > 1$ it follows that

$$(3.6) \quad \varphi_h^*(t) \leq B \left[\varphi_h^*(At) + A^r W(t)^{r/n} \sum_{i=1}^n g_i^{**}(t) \right],$$

where $g_i = D_i^{r_i} f$ and B is a constant only depending on r_i , $i = 1, \dots, n$. On the other hand, there holds [3, Vol. 1, pp. 101] that

$$(3.7) \quad \varphi_h(x) = \int_0^h \dots \int_0^h g_1(x + (u_1 + \dots + u_{r_1})e_1) du_1 \dots du_{r_1}$$

for almost all x , and hence by (3.7) (see the proof of Theorem 3.1 in [8])

$$(3.8) \quad \varphi_h^{**}(t) \leq h^{r_1} g_1^{**}(t).$$

Let $\beta(h)$ be an increasing function with respect to $h > 0$, which will be chosen later. Furthermore, it is obvious that the following inequality holds

$$(3.9) \quad \|\varphi_h\|_{\Lambda^{q,p}(w)} \leq \left(\int_0^{\beta(h)} W(t)^{p/q-1} w(t) \varphi_h^{*p}(t) dt \right)^{1/p} + \left(\int_{\beta(h)}^\infty W(t)^{p/q-1} w(t) \varphi_h^{*p}(t) dt \right)^{1/p}.$$

Now choosing a suitable number A , we get by (3.9), (3.6), (3.4), and (3.8)

$$\begin{aligned} \|\varphi_h\|_{\Lambda^{q,p}(w)} &\leq \frac{1}{2} \|\varphi_h\|_{\Lambda^{q,p}(w)} \\ &\quad + C \left[\sum_{i=1}^n \left(\int_0^{\beta(h)} W(t)^{p/q+rp/n-1} w(t) g_i^{**p}(t) dt \right)^{1/p} \right. \\ &\quad \left. + h^{r_1} \left(\int_{\beta(h)}^\infty W(t)^{p/q-1} w(t) g_1^{**p}(t) dt \right)^{1/p} \right], \end{aligned}$$

which gives

$$\begin{aligned} \|\varphi_h\|_{\Lambda^{q,p}(w)} &\leq C \left[\sum_{i=1}^n \left(\int_0^{\beta(h)} W(t)^{p/q+rp/n-1} w(t) g_i^{**p}(t) dt \right)^{1/p} \right. \\ &\quad \left. + h^{r_1} \left(\int_{\beta(h)}^\infty W(t)^{p/q-1} w(t) g_1^{**p}(t) dt \right)^{1/p} \right]. \end{aligned}$$

Thus

$$\begin{aligned} (3.10) \quad \int_0^\infty h^{-\alpha_1 p-1} \|\varphi_h\|_{\Lambda^{q,p}(w)}^p dh &\leq C \left[\sum_{i=1}^n \int_0^\infty h^{-\alpha_1 p-1} \int_0^{\beta(h)} W(t)^{p/q+rp/n-1} w(t) g_i^{**p}(t) dt \right. \\ &\quad \left. + \int_0^\infty h^{-\alpha_1 p+r_1 p-1} \int_{\beta(h)}^\infty W(t)^{p/q-1} w(t) g_1^{**p}(t) dt \right]. \end{aligned}$$

With the help of Fubini's Theorem, (3.10) establishes

$$\begin{aligned} &\left(\int_0^\infty h^{-\alpha_1 p-1} \|\varphi_h\|_{\Lambda^{q,p}(w)}^p dh \right)^{1/p} \\ &\leq C \left[\sum_{i=1}^n \left(\int_0^\infty [\beta^{-1}(t)]^{-\alpha_1 p} W(t)^{p/q+rp/n-1} w(t) g_i^{**p}(t) dt \right)^{1/p} \right. \\ &\quad \left. + \left(\int_0^\infty [\beta^{-1}(t)]^{-\alpha_1 p+r_1 p} W(t)^{p/q-1} w(t) g_1^{**p}(t) dt \right)^{1/p} \right]. \end{aligned}$$

Now setting $\beta^{-1}(t) = W(t)^{r/(nr_1)}$, we have

$$\left(\int_0^\infty h^{-\alpha_1 p-1} \|\varphi_h\|_{\Lambda^{q,p}(w)}^p dh \right)^{1/p} \leq C \sum_{k=1}^n \|g_k^{**}\|_{\Lambda^p(w)}.$$

For $w \in B_p$, the inequality (3.5) is proved. \square

Before stating the following theorem, we recall some facts related to the rearrangement invariant spaces.

A rearrangement invariant space (r.i. space), $X = X(\mathbb{R}^n)$, is a Banach space of Lebesgue measurable functions on \mathbb{R}^n with a norm $\|\cdot\|_X$ that satisfies the Fatou property and is such that if $g^* = f^*$, then $\|f\|_X = \|g\|_X$. Every r.i. space X has a representation (see [2]) as a function space on $\hat{X}(0, \infty)$ such that

$$\|f\|_{X(\mathbb{R}^n)} = \|f^*\|_{\hat{X}(0, \infty)}.$$

Since the measure space will be always clear from the context it is convenient to drop the hat and use the same letter X to indicate the different versions of the space that we use.

The upper and lower Boyd indices associated with a r.i. space X are defined by

$$(3.11) \quad \bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s}, \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm on $X(0, \infty)$ of the dilation operator E_s ($s > 0$), defined by

$$E_s f(t) = f(t/s).$$

By [2, Ch. 3], we know

$$(3.12) \quad \bar{\alpha}_X = \lim_{s \rightarrow \infty} \frac{\ln h_X(s)}{\ln s}, \quad \underline{\alpha}_X = \lim_{s \rightarrow 0} \frac{\ln h_X(s)}{\ln s}.$$

Sometimes one considers a slightly different set of indices by means of replacing $h_X(s)$ in (3.11) by

$$M_X(s) = \sup_{t>0} \frac{\Phi_X(ts)}{\Phi_X(t)}, \quad s > 0,$$

where $\Phi_X(s)$ is the fundamental function of X which is defined by

$$\Phi_X(s) = \|\chi_E\|_X, \quad \text{with } |E| = s.$$

The corresponding indices are denoted by $\bar{\beta}_X$, $\underline{\beta}_X$ and will be referred to as the upper and the lower fundamental indices of X . According to [2, Ch. 3, pp. 178], we obtain that

$$(3.13) \quad \bar{\beta}_X = \lim_{s \rightarrow \infty} \frac{\ln M_X(s)}{\ln s}, \quad \underline{\beta}_X = \lim_{s \rightarrow 0} \frac{\ln M_X(s)}{\ln s},$$

$$0 \leq \underline{\alpha}_X \leq \underline{\beta}_X \leq \bar{\beta}_X \leq \bar{\alpha}_X \leq 1.$$

Let $r_i \in \mathbb{N}$, $i = 1, \dots, n$ and X be a r.i. space in \mathbb{R}^n . We denote $W_X^{r_1, \dots, r_n}$ the class of all functions $f \in X$ for which there exist generalized partial derivatives $D_i^{r_i} f$ that belong to X ($i = 1, \dots, n$).

Using a similar method as in the proof of (3.2) and considering (3.12), we can get

Theorem 3.4. Let X be a r.i. space, $\underline{\alpha}_X > r/n$, and $\bar{\alpha}_X < 1$. Then for any $f \in W_X^{r_1, \dots, r_n}$,

$$\|t^{-r/n} f^*(t)\|_X \leq C \sum_{i=1}^n \|D_i^{r_i} f\|_X.$$

Remark 3.5. If $X = L^{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$, then

$$\underline{\alpha}_X = \bar{\alpha}_X = 1/p.$$

Applying Theorem 3.4, we get that if $0 < p < n/r$, $1 \leq q < \infty$, and f belongs to $W_{L^{p,q}}^{r_1, \dots, r_n}$, then

$$\|f\|_{L^{q^*,q}} \leq C \sum_{i=1}^n \|D_i^{r_i} f\|_{L^{p,q}}, \quad \text{where } q^* = np/(n - rp),$$

which is the result of Theorem 3 in [6] in the context of r.i. space. Let us define (see [11])

$$X_n(\infty, k) \equiv \{f: t^{-k/n}(f^{**}(t) - f^*(t)) \in X(\mathbb{R}^n)\},$$

and

$$\|f\|_{X_n(\infty, k)} = \|t^{-k/n}(f^{**}(t) - f^*(t))\|_X.$$

On the other hand, we define the space $W_{0,X}^{r_1, \dots, r_n}$ as follows:

$$W_{0,X}^{r_1, \dots, r_n} = \{f \in \mathcal{M}(\mathbb{R}^n): f \in W_X^{r_1, \dots, r_n}, \text{ and vanishes at infinity}\}.$$

By Lemma 2.6 and Lemma 2.7 in [11], we know that if $\underline{\alpha}_X > r/n$ (equivalent to the $Q(r)$ -condition there), then for all measurable functions f with $f^{**}(\infty) = 0$,

$$\|t^{-r/n} f^*(t)\|_X \approx \|t^{-r/n} f^{**}(t)\|_X \approx \|t^{-r/n}(f^{**}(t) - f^*(t))\|_X = \|f\|_{X_n(\infty, r)}.$$

Hence, by Theorem 3.4, we get that if $\underline{\alpha}_X > r/n$, $\bar{\alpha}_X < 1$ (equivalent to the P -condition in [11]), then

$$\|f\|_{X_n(\infty, r)} \leq C \|f\|_{W_X^{r_1, \dots, r_n}}, \quad \forall f \in W_{0,X}^{r_1, \dots, r_n},$$

that is to say,

$$W_{0,X}^{r_1, \dots, r_n} \subset X_n(\infty, r),$$

which is the generalization, to some extent, of Theorem 1.2 and Corollary 1.3 in [11].

Before outlining the next theorem, we give an important property of the r.i. spaces. Suppose X is a r.i. space with the fundamental function Φ , then we obtain

$$E_1 = \{\alpha: \|\Phi(t)^\alpha \chi_{(0,1)}(t)\|_X < \infty\} \neq \emptyset$$

and if $\underline{\beta}_X > 0$, then we also get

$$E_2 = \{\alpha: \|\Phi(t)^\alpha \chi_{(1,\infty)}(t)\|_X < \infty\} \neq \emptyset.$$

First, we note that [2, Th. II.6.6]

$$(3.14) \quad L^1 \cap L^\infty \subset X.$$

Now it is easy to get $E_1 \neq \emptyset$ by (3.14) and the property that $\Phi(t)$ is increasing on \mathbb{R}_+ . On the other hand, if $\underline{\beta}_X > 0$, it follows (see [10]) that for every $0 < \gamma < \underline{\beta}_X$, there exists positive constant Q such that

$$\Phi(t) \geq Qt^\gamma, \quad \forall 1 < t < \infty,$$

which in combination with (3.14) implies $E_2 \neq \emptyset$.

Let

$$L_1 = \sup\{\xi: \|\Phi(t)^{-\xi-1+r/(n\bar{\beta}_X)} \chi_{(0,1)}(t)\|_X < \infty\}$$

and

$$L_2 = \inf\{\xi: \|\Phi(t)^{-\xi-1} \chi_{(1,\infty)}(t)\|_X < \infty\}.$$

Note that if $\bar{\beta}_X < 1$, then for every $\bar{\beta}_X < \eta < 1$, there exists a constant \tilde{C} (see Lemma 1 in [10]) such that

$$\Phi(t) \geq \tilde{C}t^\eta, \quad \forall 0 < t < 1.$$

Applying a method similar to the proof of (3.3), while considering (3.12) and (3.13), we have the following theorem. The details are omitted.

Theorem 3.6. *Let $\underline{\beta}_X > 0$, $\bar{\beta}_X < 1$. Suppose $f \in W_X^{r_1, \dots, r_n}$. If $E \neq \emptyset$ where*

$$E = \{\alpha: L_2 < \alpha < \min(L_1, \underline{\alpha}_X / \bar{\beta}_X)\},$$

then for every $\alpha \in E$ there holds

$$\|\Phi(t)^{-\alpha} f^*(t)\|_X \leq C \left(\|f\|_X + \sum_{i=1}^n \|D_i^{r_i} f\|_X \right),$$

where C is independent of f .

Remark 3.7. (1) If $X = L^{p,q}$ ($1 < p < \infty$, $1 \leq q \leq p < \infty$), then $\Phi(t) = ((p/q)t)^{1/p}$ and $L_1 = rp/n$, $L_2 = 0$, $\underline{\alpha}_X = \bar{\beta}_X = 1/p$. In the case of $1/p > r/n$, the above theorem implies that for $0 < \alpha < rp/n$,

$$\|t^{-\alpha} f^*(t)\|_{L^{p,q}} \leq C \left(\|f\|_{L^{p,q}} + \sum_{i=1}^n \|D_i^{r_i} f\|_{L^{p,q}} \right),$$

which coincides with Theorem 3 in [6].

(2) If $X = \Lambda^p(w)$ ($p \geq 1$) where

$$w = \begin{cases} t^{-\alpha}, & \text{if } 0 < t \leq 1, \\ t^{-\beta}, & \text{if } t > 1, \end{cases}$$

and $0 < \alpha < \beta < 1$, then $L_1 = rp/n$, $L_2 = 0$, $\underline{\alpha}_X = (1 - \beta)/p$, $\bar{\beta}_X = (1 - \alpha)/p$ (use Theorem 4.1 in [4] to calculate the indices), which implies

$$E = \left(0, \min\left(\frac{rp}{n}, \frac{1 - \beta}{1 - \alpha}\right) \right).$$

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