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# THE FOURIER INTEGRAL OPERATORS ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

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*Abstract.* In this paper, it is proved that the Fourier integral operators of order $m$, with $-n < m \leqslant -(n-1)/2$, are bounded from three kinds of Hardy spaces associated with Herz spaces to their corresponding Herz spaces.

*Keywords*: Fourier integral operator, Hardy spaces, Herz spaces

*MSC 2010*: 47G10, 35S30, 42B30

## 1. Introduction

Fourier integral operators have in the last 50 years become an important tool in certain areas of analysis, and in particular in a variety of problems arising in partial differential equations. In its basic form, the Fourier integral operator of order $m$ is given by

$$Tf(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\Phi(x,\xi)} a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi. \tag{1.1}$$

Here $\hat{f}$ denotes the Fourier transform of $f$; the functions $a$ and $\Phi$ are defined as in [7]. That is, $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}_n \setminus \{0\})$ and it has compact and connected support in $x$. The phase $\Phi$ is real-valued, homogeneous of degree 1 in $\xi$, and smooth in $(x,\xi)$ for $\xi \neq 0$, on the support of $a$. We also assume that $0 \in \operatorname{supp}_x a$ and $\Phi$ satisfies the

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crucial nondegeneracy condition, that is, for $\xi \neq 0$,

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j}\Big) \neq 0 \tag{1.2}$$

on the support of $a$.

A well known result of Hörmander in [1] state that the Fourier integral operator $T$ of order 0 is a bounded operator from $L^2$ to $L^2$. In [6], A. Seeger, C. D. Sogge and E. M. Stein showed that $T$ of order $-(n-1)/2$ is bounded from $H^1$ to $L^1$, and then by complex interpolation they obtained the boundedness from $L^p$ to $L^p$ of $T$ when $-(n-1)/2 < m \leqslant 0$ and $|1/2 - 1/p| \leqslant -m/(n-1)$. Recently, Marco M. P. and Silvia Secco proved the boundedness of $T$ from $h^p(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$ in [5].

On the other aspect, some new Hardy spaces $HK_q$ associated with Herz spaces $K_q$ are introduced by Shanzhen Lu and Dachun Yang [2], [8]. An interesting fact shown in [8] is that $HK_q$ is the localization of $H^1$ at the origin. It is easy to see that the relation between $HK_q$ and $K_q$ is similar to the one between $H^1$ and $L^1$, and the relation between $K_q^{\alpha,p}$ and $K_q^{\alpha,p}$ is similar to the one between $L^q$ and $L^q$.

In this paper we investigate the properties of the operator $T$ defined by (1.1) on Herz spaces and Hardy spaces associated to Herz spaces. To state our results, let us introduce some definitions and facts.

**Definition 1.1.** Let $1 < q < \infty$ and $1/q + 1/q' = 1$. The Herz space $K_q(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_q} := \sum_{k \in \mathbb{Z}} 2^{kn/q'} \cdot \|f\chi_k\|_{L^q} < \infty,$$

where $\chi_k$ denotes the characteristic function of $C_k$ and $C_k = B_k \setminus B_{k-1}$, $B_k = \{x\colon\ |x| \leqslant 2^k\}$.

The Hardy space $HK_q$ associated with Herz space $K_q$ is defined by

$$HK_q = \{f \in L^1\colon\ Gf \in K_q\},$$

where $Gf$ is the Grand maximal function of $f$ and $q > 1$.

**Definition 1.2.** Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$.

(i) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \|f\chi_k\|^p_{L^q}\Big\}^{1/p} < \infty.$$

(ii) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_q^{\alpha,p}} := \left\{ \|f\chi_{B_0}\|^p_{L^q} + \sum_{k=1}^{\infty} 2^{k\alpha p} \cdot \|f\chi_k\|^p_{L^q} \right\}^{1/p} < \infty,$$

where $\chi_k$ is as above.

**Definition 1.3.** Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$.

(i) The homogeneous Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ associated with $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n)\colon\ Gf \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ associated with $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n)\colon\ Gf \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$, where $Gf$ is the Grand maximal function of $f$. Now, we can state our results as follows.

**Theorem 1.1.** *Let $1 < q < \infty$ and let $T$ be the Fourier integral operator, as in* (1.1), *whose symbol $a$ is of order $m$, with $-n < m \leqslant -(n-1)/2$. If $q$ satisfies*

$$\frac{1}{q} \geqslant 2 + \frac{2(m-1)}{(n+1)}, \tag{1.3}$$

*then $T$, originally defined on $S$, extends to a bounded operator from $HK_q$ to $K_q$, that is*

$$\|Tf\|_{K_q} \leqslant C \cdot \|f\|_{HK_q}.$$

**Theorem 1.2.** *Let $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < n(1 - 1/q)$. Let $T$ be the Fourier integral operator as in* (1.1), *whose symbol $a$ is of order $m$, with $-n < m \leqslant -(n-1)/2$. If $q$ satisfies* (1.3) *in Theorem* 1.1, *then $T$, originally defined on $S$, extends to a bounded operator on $\dot{K}_q^{\alpha,p}$* and *$K_q^{\alpha,p}$, respectively.*

**Theorem 1.3.** *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geqslant n(1-1/q)$. Let $T$ be the Fourier integral operator as in* (1.1), *whose symbol $a$ is of order $m$, with $-n-s-1 < m \leqslant -(n-1)/2 - s - 1$. If $q$ satisfies*

$$\frac{1}{q} \geqslant 2 + \frac{2(m+s)}{(n+1)}, \tag{1.4}$$

*then $T$, originally defined on $S$, extends to a bounded operator from $H\dot{K}_q^{\alpha,p}$ to $\dot{K}_q^{\alpha,p}$ and $HK_q^{\alpha,p}$ to $K_q^{\alpha,p}$, respectively.*

Obviously, if $\alpha = n(1-1/q)$ and $p = 1$, then we denote $K_q^{\alpha,p}$ by $K_q$, so Theorem 1.1 is a special case of Theorem 1.3. As concerns the proofs of Theorems 1.2 and 1.3, we only prove the homogeneous case.

The paper is organized as follows. In the next section, we list some definitions and lemmas which will be used throughout the paper. In Section 3 we give the estimate of kernel. In Section 4, we present the proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Section 5 and we conclude the paper in Section 6.

## 2. Definitions and lemmas

Let us first introduce some lemmas on the characterization of $HK_q$.

**Definition 2.1.** Let $1 < q < \infty$. A function $a$ on $\mathbb{R}^n$ is called a central $(1,q)$-atom if

(1) $\operatorname{supp} a \subset B$, where $B$ is a ball centered at the origin;
(2) $\|a\|_q \leqslant |B|^{1/q-1}$;
(3) $\int a(x)\,\mathrm{d}x = 0$.

In [2], [8], S. Z. Lu and D. C. Yang have proved the following result.

**Lemma 2.1.** *Let $f \in L^1(\mathbb{R}^n)$ and $1 < q < \infty$. Then $f \in HK_q(\mathbb{R}^n)$ if and only if $f$ can be represented as*

$$f(x) = \sum_{l=-\infty}^{\infty} \lambda_l a_l(x),$$

*where each $a_l$ is a central $(1,q)$-atom which satisfies $\operatorname{supp} a_l \subset B(0,2^l)$, and $\sum\limits_{l=-\infty}^{+\infty} |\lambda_l| < \infty$. Moreover,*

$$\|f\|_{HK_q} \sim \inf\Big\{\sum_{l=-\infty}^{\infty} |\lambda_l|\Big\},$$

*where the infimum is taken over all decompositions of $f$ as above.*

Now let us introduce some lemmas on the characterization of $HK_q^{\alpha,p}$ and $K_q^{\alpha,p}$.

**Definition 2.2.** Let $0 < \alpha < \infty$ and $1 < q < \infty$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q)$-atom, if

(1) $\operatorname{supp} a \subset B(0, r)$;

(2) $\|a\|_q \leqslant |B|^{-\alpha/n}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q)$-atom of restrictive type, if

(1) $\operatorname{supp} a \subset B(0, r)$, $r \geqslant 1$;

(2) $\|a\|_q \leqslant |B|^{-\alpha/n}$, where $B(0, r) = \{x \in \mathbb{R}^n \colon |x| < r\}$.

**Lemma 2.2.** *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$. Then $f \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if $f$ can be represented as*

$$f(x) = \sum_{l=-\infty}^{+\infty} \lambda_l a_l(x),$$

*where each $a_l$ is a central $(\alpha, q)$-atom with support $B_l$, and $\sum\limits_{l=-\infty}^{+\infty} |\lambda_l|^p < \infty$. Moreover,*

$$\|f\|_{\dot{K}_q^{\alpha,p}} \sim \inf \left( \sum_{l=-\infty}^{+\infty} |\lambda_l|^p \right)^{1/p},$$

*where the infimum is taken over all decompositions of $f$ as above.*

We have found the proof of Lemma 2.2 in [3].

**Definition 2.3.** Let $n(1-1/q) \leqslant \alpha < \infty$, $1 < q < \infty$ and let $s$ be a non-negative integer, $s \geqslant \lfloor \alpha + n(1/q - 1) \rfloor$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s)$-atom, if it satisfies

(1) $\operatorname{supp} a \subset B(0, r)$;

(2) $\|a\|_q \leqslant |B|^{-\alpha/n}$ where $B(0, r) = \{x \in \mathbb{R}^n \colon |x| < r\}$;

(3) $\int a(x) x^\beta \, dx = 0$, $|\beta| \leqslant s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s)_0$-atom, if it satisfies (1) through (3), and $a(x) = 0$ on some neighborhood of 0;

(iii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s)$-atom of restrictive type, if it satisfies (1) with $r \geqslant 1$, (2) and (3).

**Lemma 2.3.** *Let $n(1 - 1/q) \leqslant \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$. Then $f \in H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ if and only if $f$ can be represented as*

$$f(x) = \sum_{l=-\infty}^{+\infty} \lambda_l a_l(x),$$

*where each $a_l$ is a central $(\alpha, q)$-atom with support $B_l$, and $\sum\limits_{l=-\infty}^{+\infty} |\lambda_l|^p < \infty$. Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} \sim \inf \left( \sum_{l=-\infty}^{+\infty} |\lambda_l|^p \right)^{1/p},$$

*where the infimum is taken over all decompositions of $f$ as above.*

We have found the proof of Lemma 2.3 in [4].

For the nonhomogeneous spaces $K_q^{\alpha,p}$ and $HK_q^{\alpha,p}$, there are results similar to Lemma 2.2 and Lemma 2.3. However, the central atom must be replaced by the central atom of restrictive type.

**Lemma 2.4.** *Let $p \in (0,\infty)$, $q \in (1,\infty)$, $\alpha \in [n(1-1/q), \infty)$ and let $s$ be a nonnegative integer, $s \geqslant \lfloor \alpha - n(1-1/q) \rfloor$. Then $\dot{F}_p^{\alpha,q,s}$ and $F_p^{\alpha,q,s}$ are dense in $H\dot{K}_p^{\alpha,q,s}$ and $HK_p^{\alpha,q,s}$ respectively, where*

$$\dot{F}_p^{\alpha,q,s} \equiv \left\{ f = \sum_{j=1}^{m} \lambda_j a_j \colon \ m \in \mathrm{N}, \{a_j\}_{j=1}^m \text{ are central } (\alpha, q, s)_0\text{-atoms} \right\},$$

*and*

$$\|f\|_{\dot{F}_p^{\alpha,q,s}} \equiv \inf \left\{ \left( \sum_{j=1}^{m} |\lambda_j|^p \right)^{1/p} \colon \ f \in \dot{F}_p^{\alpha,q,s} \right\}.$$

*We also introduce $F_p^{\alpha,q,s}$ and $\|f\|_{F_p^{\alpha,q,s}}$ as in the above two formulas just replacing central $(\alpha, q, s)$-atoms by central $(\alpha, q, s)$-atoms of restrictive type.*

**Lemma 2.5.** *Let $p \in (0,1]$, $r \in [p,1]$, $q \in (1,\infty)$, $\alpha \in [n(1-1/q), \infty)$ and let $s$ be a nonnegative integer, $s \geqslant \lfloor \alpha - n(1-1/q) \rfloor$.*

(i) *If $T$ is a linear operator defined on $\dot{F}_p^{\alpha,q,s}$ such that*

$$S \equiv \sup\{\|Ta\|_{B_r} \colon \ a \text{ is any central } (\alpha, q, s)_0 \text{ atom } \} < \infty$$

*then $T$ uniquely extends to be a bounded $B_r$-sublinear operator from $H\dot{K}_p^{\alpha,q}$ to $B_r$.*

(ii) *If $T$ is a linear operator defined on $F_p^{\alpha,q,s}$ such that*

$$S \equiv \sup\{\|Ta\|_{B_r} \colon \ a \text{ is any central } (\alpha, q, s)_0 \text{ atom of restrictive type}\} < \infty$$

*then $T$ uniquely extends to a bounded $B_r$-sublinear operator from $HK_p^{\alpha,q}$ to $B_r$.*

*Here $B_r$ is an $r$-quasi-Banach space and quasi-Banach spaces $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$, $K_p^{\alpha,q}$, $\dot{K}_p^{\alpha,q}$, $HK_p^{\alpha,q}$ and $H\dot{K}_p^{\alpha,q}$ with $p \in (0,1)$ are typically $p$-quasi-Banach spaces.*

The proofs of the above two lemmas are in [9].

## 3. The estimation of Kernel

By the definition of $T$, we see that its kernel is given by

$$K(x,y) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}(\Phi(x,\xi)-y\cdot\xi)} a(x,\xi)\,\mathrm{d}\xi. \tag{3.1}$$

As usual, we first construct an exceptional set $B_l^*$ for every ball $B_l (l \leqslant 0)$, which is something like

$$B_l^* = \{x\colon\ \mathrm{dist}(\Sigma_x, B_l) \leqslant c2^l\},$$

where $\Sigma_x = \{y\colon\ y = \Phi_\xi(x,\xi) \text{ for some } \xi\}$, are the singular sets of $T$. For every positive integer $j$ we choose atom vectors $\{\xi_j^\nu\}$, $\nu = 1,\ldots,N(j)$, such that $|\xi_j^\nu - \xi_j^{\nu'}| \geqslant 2^{-j/2}$ for different $\nu$, $\nu'$. Then for every $\xi \in S^{n-1}$ there exists a $\xi_j^\nu$ such that $|\xi - \xi_j^\nu| < 2^{-j/2}$. Observe that

$$N(j) \sim 2^{j(n-1)/2}.$$

For $B(0,2^l)$ with radius $2^l \leqslant 1$ we define the "rectangles"

$$\widetilde{\mathbb{R}}_j^\nu = \{y; |y-0| \leqslant \bar{c}\cdot 2^{-j/2}, |\pi_j^\nu(y-0)| \leqslant \bar{c}\cdot 2^{-j}\},$$

where $\pi_j^\nu$ is the orthogonal projection in the direction $\xi_j^\nu$ and $\bar{c}$ is a large constant (independent of $j$). Obviously, $|\widetilde{\mathbb{R}}_j^\nu| \sim 2^{j(n+1)/2}$. Next, the mapping

$$x \mapsto y = \Phi_\xi(x,\xi)$$

has for each $\xi$ a nonvanishing Jacobian, by virtue of (1.2). So we take $\mathbb{R}_j^\nu$ to be the inverse under $\Phi_\xi$, with $\xi = \xi_j^\nu$ from the rectangle $\widetilde{\mathbb{R}}_j^\nu$:

$$\mathbb{R}_j^\nu = \{x; |\Phi_\xi(x,\xi_j^\nu) - 0| \leqslant \bar{c}\cdot 2^{-j/2}, |\pi_j^\nu(\Phi_\xi(x,\xi_j^\nu)-0)| \leqslant \bar{c}\cdot 2^{-j}\}. \tag{3.2}$$

Now let $B_l^* = \bigcup\limits_{2^{-j}\leqslant 2^l} \bigcup\limits_{\nu} \mathbb{R}_j^\nu$. Since $\widetilde{\mathbb{R}}_j^\nu$ is compact and $\Phi(x,\xi)$ satisfies the crucial nondegeneracy condition, $\mathbb{R}_j^\nu$ is also compact. So there exists $\{x_i\}_{i=1}^n \subset \mathbb{R}_j^\nu$ such that $\mathbb{R}_j^\nu \subset \bigcup\limits_{i=1}^n U(x_i)$, where $U(x_i)$ is the neighborhood of $x_i$. For every $x \in \mathbb{R}_j^\nu$ there exists $x_{i_0}$ such that $x \in U(x_{i_0})$, while for all $x \in U(x_{i_0})$ we have

$$\Phi_\xi(x,\xi_j^\nu) = \Phi_\xi(x_{i_0},\xi_j^\nu) + \Phi_{\xi x}(x',\xi_j^\nu)(x - x_{i_0}),$$

where $x'$ is a point on the segment between $x$ and $x_{i_0}$. Because $\Phi(x,\xi)$ satisfies the crucial nondegeneracy condition, there exist $C_1, C_2$ such that $C_1 \leqslant \|\Phi_{\xi x}(x,\xi_j^\nu)\| \leqslant$

$C_2$, and because of the relation between $\widetilde{\mathbb{R}}_j^\nu$ and $\mathbb{R}_j^\nu$, there exist $y = \Phi_\xi(x, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$ and $y_{i_0} = \Phi_\xi(x_{i_0}, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$. So we have

$$|x| \leqslant |x_{i_0}| + c \cdot |y - y_{i_0}|,$$

and because of the proposition on finite covering, for all $x \in \mathbb{R}_j^\nu$, we have

$$|x| \leqslant \max_{1 \leqslant i \leqslant n} \{|x_i| + c \cdot |y - y_i|\},$$

where $y_i = \Phi_\xi(x_i, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$. Since $0 \in \operatorname{supp}_x a$, we have

$$\Phi_\xi(x_i, \xi_j^\nu) = \Phi_\xi(0, \xi_j^\nu) + \Phi_{\xi x}(x'', \xi_j^\nu)(x_i - 0),$$

where $x''$ is a point on the segment between $x_i$ and 0. If $\operatorname{supp}_x a$ is a multiply connected domain, then use finite Talor expansion. Since the mapping

$$\Phi_\xi(0, \cdot)\colon\ |\xi| = 1 \to \Sigma_0$$

is smooth and the atom sphere is compact, we know $\Sigma_0$ is compact. So there exists a const $r$, which is independent of $l, j$, such that $\Sigma_0 \subset B_r$. That is, for any $j, \nu$, we have

$$|\Phi_\xi(0, \xi_j^\nu)| \leqslant 2^r. \tag{3.3}$$

So for all $x \in \mathbb{R}_j^\nu$ there exists a constant $k_0$ such that

$$\begin{aligned} |x| &\leqslant \max_{1 \leqslant i \leqslant n} \{c|y_i| + 2^r + |y - y_i|\} \\ &\leqslant c \cdot |y| + 2^r \leqslant \bar{c} 2^{-j/2} + 2^r \\ &\leqslant 2^{-j/2 + k_0} + 2^r \\ &\leqslant 2^{k_1}(2^{-j/2} + 1), \end{aligned}$$

where $k_1 = \max\{k_0, r\}$. Then we have

$$B_l^* = \bigcup_{2^{-j} \leqslant 2^l} \bigcup_\nu \mathbb{R}_j^\nu \subset \bigcup_{2^{-j} \leqslant 2^l} B(0, 2^{k_1}(2^{-j/2} + 1)) \subset B(0, 2^{k_1}(2^{l/2} + 1));$$

since $l \leqslant 0$, we obtain

$$B_l^* \subset B_{k_1 + 1}, \tag{3.4}$$

where $k_1$ is a constant and is independent of $l$.

We recall the atom vectors $\{\xi_j^\nu\}$ used above; they give an essentially uniform grid on the atom sphere, with separation $2^{-j/2}$. Let

$$\Gamma_j^\nu = \{\xi\colon\ |\xi/|\xi| - \xi_j^\nu| \leqslant 2\cdot 2^{-j/2}\}.$$

For every $j$ we shall choose $C^\infty$ functions $\chi_j^\nu$, $\nu = 1, \dots, N(j)$, each homogeneous of degree $0$ in $\xi$ and supported in $\Gamma_j^\nu$, with

$$\sum_\nu \chi_j^\nu = 1 \qquad \text{for all } \xi \neq 0 \text{ and all } j. \tag{3.5}$$

So it is easy to obtain the refined Littlewood-Paley decomposition

$$1 = \widehat{\Psi}_0(\xi) + \sum_{j=1}^\infty \sum_\nu \chi_j^\nu(\xi)\cdot \widehat{\Psi}_j(\xi)$$

where $\widehat{\Psi}_j(\xi) = \eta(2^{-j}\xi) - \eta(2^{-j+1}\xi)$, and $\eta \in C^\infty$,

$$\eta(\xi) = \begin{cases} 1, & |\xi| \leqslant 1,\\ 0, & |\xi| > 2.\end{cases}$$

With this decomposition, we define operators $T_j^\nu$ by

$$T_j^\nu f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\Phi(x,\xi)} a_j^\nu(x,\xi)\hat{f}(x)\,\mathrm{d}\xi, \tag{3.6}$$

where $a_j^\nu(x,\xi) = \chi_j^\nu(\xi)\cdot\widehat{\Psi}_j(\xi)\cdot a(x,\xi)$. We also define the corresponding operators $T_j$, using symbols $a_j(x) = \widehat{\Psi}_j(\xi)\cdot a(x,\xi)$. Clearly

$$T_j = \sum_\nu T_j^\nu. \tag{3.7}$$

We let $K_j$ denote the kernel of the operator $T_j$. For $y \in B(0,2^l)$, the key estimates we shall derive are

$$\Big(\int_{C_k} |K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q} \leqslant C2^{j(n+m-(n+1)/2q)} \quad \text{for all } k \in \mathbb{Z}, \tag{3.8}$$

$$\Big(\int_{C_k} |K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q} \leqslant C2^{j(n+m-1/2q-N)}, \quad k > k_1+1, \tag{3.9}$$

$$\Big(\int_{C_k} |D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q} \leqslant C2^{j(n+m+s+1-(n+1)/2q)} \quad \text{for all } k \in \mathbb{Z}, \tag{3.10}$$

$$\Big(\int_{C_k} |D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q} \leqslant C2^{j(n+m+s+1-1/2q-N)}, \quad k > k_1+1, \tag{3.11}$$

where the bound C is independent of $j,k$, $\gamma=(\gamma_1,\dots,\gamma_n)$ is a multi-index with $|\gamma|=\sum\limits_{k=1}^{n}\gamma_k=s+1$ and $D_y^{\gamma}=\sum\limits_{|\gamma|=s+1}D_{y_1}^{\gamma_1}\dots D_{y_n}^{\gamma_n}$, $y=(y_1,\dots,y_n)$.

Because of (3.7), it suffices to derive similar estimates for the kernels of the $T_j^{\nu}$'s, which we denote by $K_j^{\nu}$. By the definition of $T_j^{\nu}$, we see that its kernel is given by

$$K_j^{\nu}(x,y)=\int_{\mathbb{R}^n}\mathrm{e}^{2\pi\mathrm{i}(\Phi(x,\xi)-y\cdot\xi)}a_j^{\nu}(x,\xi)\,\mathrm{d}\xi.$$

From the discussion by E. M. Stein in [7], we know that

$$|K_j^{\nu}(x,y)|\leqslant C2^j\cdot 2^{j(n-1)/2+mj} \times\{1+2^j\cdot|(\Phi_\xi(x,\xi_j^{\nu})-y)_1|+2^{j/2}\cdot|(\Phi_\xi(x,\xi_j^{\nu})-y)'|\}^{-2N}, \tag{3.12}$$

where $(\cdot)_1$ indicates the component in the direction $\xi_j^{\nu}$, and $(\cdot)'$ denotes the orthogonal component.

So for $q>1$,

$$\begin{aligned}&\Bigl(\int_{C_k}|K_j^{\nu}(x,y)|^q\,\mathrm{d}x\Bigr)^{1/q}\\&\leqslant C2^{j((n+1)/2+m)}\cdot\Bigl(\int_{C_k}(1+2^j|(\Phi_\xi(x,\xi_j^{\nu})-y)_1|+2^{j/2}|(\Phi_\xi(x,\xi_j^{\nu})-y)'|)^{-2Nq}\,\mathrm{d}x\Bigr)^{1/q}\\&\leqslant C\cdot 2^{j((n+1)/2+m-(n+1)/2q)}\Bigl(\int_{\mathbb{R}^n}(1+|z|)^{-2Nq}\,\mathrm{d}z\Bigr)^{1/q}\\&\leqslant C\cdot 2^{j((n+1)/2+m-(n+1)/2q)},\end{aligned}$$

if we choose $-2Nq<-n$.

Because $B_k$ is compact, similarly to the consideration concerning $B_l^*\subset B_{k_1+1}$, we have that the mapping $x\mapsto z=\Phi_\xi(x,\xi_j^{\nu})$ projects $C_k$ into $\{x\colon\ 2^{k_1}+2^{k-1}<|x|\leqslant 2^{k_1}+2^k\}$, so the mapping

$$x\mapsto z=(2^j(\Phi_\xi(x,\xi_j^{\nu}))_1,2^{j/2}(\Phi_\xi(x,\xi_j^{\nu}))')$$

projects $C_k$ into $D_{k,j}=\{x\colon\ (2^{k_1}+2^{k-1})2^{j/2}<|x|\leqslant(2^{k_1}+2^k)2^j\}$. When $k>k_1+1$, we know $C_k\subset^c B_l^*$, thus $|\Phi_\xi(x,\xi_j^{\nu})-y|\sim|\Phi_\xi(x,\xi_j^{\nu})|$, so we have

$$\begin{aligned}&\Bigl(\int_{C_k}|K_j^{\nu}(x,y)|^q\,\mathrm{d}x\Bigr)^{1/q}\\&\leqslant C2^{j((n+1)/2+m)}\cdot\Bigl(\int_{C_k}(1+2^j|(\Phi_\xi(x,\xi_j^{\nu})-y)_1|+2^{j/2}|(\Phi_\xi(x,\xi_j^{\nu})-y)'|)^{-2Nq}\,\mathrm{d}x\Bigr)^{1/q}\\&\leqslant C2^{j((n+1)/2+m-(n+1)/2q)}\Bigl(\int_{D_{k,j}}(1+|z|)^{-2Nq}\,\mathrm{d}z\Bigr)^{1/q},\end{aligned}$$

and taking $N$ so large that $-2Nq < -n$, we have

$$\begin{aligned}\left(\int_{C_k} |K_j^\nu(x,y)|^q \,\mathrm{d}x\right)^{1/q} & \\ &\leqslant C\cdot 2^{j((n+1)/2+m-(n+1)/2q)}\cdot\left((2^{k_1}+2^{k-1})\cdot 2^{j/2}\right)^{(-2N+n/q)} \\ &\leqslant C\cdot 2^{j((n+1)/2+m-1/2q-N)}\cdot 2^{(k_1+1)(-2N+n/q)} \\ &\leqslant C\cdot 2^{j((n+1)/2+m-1/2q-N)};\end{aligned}$$

the second inequality follows by virtue of $2^{k_1+1} < 2^{k_1}+2^{k-1} < 2^k$ when $k > k_1+1$.

Since $N(j)\sim 2^{j(n-1)/2}$, (3.8) and (3.9) are obtained.

A similar estimate holds for $D_y^\gamma K_j^\nu(x,y)$, once we observe that the differentiation in $y$ introduces factors bounded by $2^j$. As a result, (3.10) and (3.11) are obtained.

## 4. Boundedness on $HK_q$

Now let us present the proof of Theorem 1.1.

By Lemma 2.4 and lemma 2.5, it suffices to show that

$$\|Ta_l\|_{K_q}\leqslant C \tag{4.1}$$

for all $(1,q)$-atom $a_l$ associated with the ball $B(0,2^l)$ and the $C$ is independent of $l$.

To begin with, if the radius of $B_l$ exceeds 1, the estimate of (4.1) is easy, because of our assumption that the symbol of $T$ has compact support in the variable $x$: hence there exists a $K_0$ such that $k > K_0$ implies $Ta_l = 0$. So

$$\begin{aligned}\|Ta_l\|_{K_q} &= \sum_{k=-\infty}^{K_0} 2^{kn/q'}\|Ta_l\chi_k\|_{L^q} \\ &\leqslant C\cdot\sum_{k=-\infty}^{K_0} 2^{kn/q'}\|a_l\|_{L^q} \\ &\leqslant C\cdot 2^{ln(1/q-1)}\sum_{k=-\infty}^{K_0} 2^{kn/q'} \\ &\leqslant C;\end{aligned}$$

the first inequality follows from the $L^q$ boundedness property already obtained by A. Seeger in [6].

Next, assuming that $l < 0$, we have

$$\begin{aligned}
\|Ta_l\|_{K_q} &= \sum_{k=-\infty}^{k_1+1} 2^{kn/q'} \|Ta_l\chi_k\|_{L^q} + \sum_{k=k_1+2}^{K_0} 2^{kn/q'} \|Ta_l\chi_k\|_{L^q} \\
&\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn/q'} \sum_{j=1}^{\infty} \|T_j a_l\chi_k\|_{L^q} + \sum_{k=k_1+2}^{K_0} 2^{kn/q'} \sum_{j=1}^{\infty} \|T_j a_l\chi_k\|_{L^q} \\
&= I + II.
\end{aligned}$$

By using the Minkowski inequality, we arrive at

$$\begin{aligned}
\|T_j a_l \chi_k\|_{L_q} &= \left( \int_{C_k} \left| \int_{B_l} K_j(x,y) a_l(y)\, \mathrm{d}y \right|^q \mathrm{d}x \right)^{1/q} \\
&\leqslant \int_{B_l} \left( \int_{C_k} |K_j(x,y)|^q \, \mathrm{d}x \right)^{1/q} |a_l(y)| \, \mathrm{d}y \\
&\leqslant \sup_{y \in B_l} \left( \int_{C_k} |K_j(x,y)|^q \, \mathrm{d}x \right)^{1/q} \cdot \|a_l\|_1.
\end{aligned} \tag{4.2}$$

By (3.8), (4.2) and $\|a_l\|_1 \leqslant C$ we get

$$\begin{aligned}
II &\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} \|T_j a_l \chi_k\|_{L^q} \\
&\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} 2^{j(m+n-(n+1)/2q)}, \\
&\leqslant C,
\end{aligned}$$

when $m$ satisfies $m \leqslant (n+1)/2q - n$.

By (3.9), (4.2) and $\|a_l\|_1 \leqslant C$, we get

$$\begin{aligned}
II &\leqslant \sum_{k=k_1+2}^{K_0} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} \|T_j a_l \chi_k\|_{L^q} \\
&\leqslant \sum_{k=k_1+2}^{K_0} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} 2^{j(m+n-1/2q-N)}, \\
&\leqslant C,
\end{aligned}$$

when we choose $N$ such that $N > \max\{m+n-1/2q, n/2q\}$.

Because $q > 1$, we have $(n+1)/2q - n < -(n+1)/2$. So by the properties of symbols that if $\alpha < \beta$, $a \in S^\alpha$ implies $a \in S^\beta$, we know that when $m \leqslant (n+1)/2q - n$, the $(q,q)$ boundedness of the *Fourier integral operators* is certain. So when $-n < m \leqslant -(n-1)/2$ and $q$ satisfies

$$\frac{1}{q} \geqslant 2 + \frac{2(m-1)}{(n+1)},$$

we get (4.1). So Theorem 1.1 is proved.

## 5. Boundedness on $K_q^{\alpha,p}$ and $HK_q^{\alpha,p}$

Now, let us deal with the boundedness on $K_q^{\alpha,p}$.

Suppose $f \in \dot{K}_q^{\alpha,p}$. By Lemma 2.3, $f(x) = \sum_{l\in\mathbb{Z}} \lambda_l a_l(x)$, where $a_l(x)$ is a central $(\alpha,q)$-atom with the support $B_l$, and

$$\sum_{l\in\mathbb{Z}} |\lambda_l|^p \leqslant C\|f\|^p_{\dot{K}_q^{\alpha,p}}.$$

By Definition 1.2 we have

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_q^{\alpha,p}} &= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|Tf\chi_k\|^p_{L^q} \\
&= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0} + \sum_{l\leqslant 0} |\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p \\
&\leqslant C\cdot\left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\sum_{l>0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\right)^p + \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\right)^p\right\} \\
&= C(I_1 + I_2).
\end{aligned}$$

Let us first estimate $I_1$. Using the fact that $T$ maps $L^q$ into $L^q$ and the assumption that the symbol of $T$ has compact support in the variable $x$, we have

$$\begin{aligned}
I_1 &\leqslant C \sum_{k=-\infty}^{K_0} 2^{k\alpha p} \sum_{l>0} |\lambda_l|^p \|a_l\|^p_{L^q} \\
&\leqslant \begin{cases} C \sum\limits_{k=-\infty}^{K_0} 2^{k\alpha p} \sum\limits_{l>0} |\lambda_l|^p 2^{-l\alpha p}, & 0 < p \leqslant 1 \\[2ex] C \sum\limits_{k=-\infty}^{K_0} 2^{k\alpha p}\left(\sum\limits_{l>0}|\lambda_l|^p 2^{-l\alpha p/2}\right)\left(\sum\limits_{l>0} 2^{-l\alpha p'/2}\right)^{p/p'}, & p > 1 \end{cases} \\
&\leqslant C\sum_{l\in\mathbb{Z}} |\lambda_l|^p.
\end{aligned}$$

To estimate $I_2$, by Hölder's inequality and the proposition of $(\alpha,q)$-atom we get

$$\int |a_l(y)|\,\mathrm{d}y \leqslant \Big(\int |a_l(y)|^q\,\mathrm{d}y\Big)^{1/q} \cdot |B_l|^{1-1/q} \leqslant C2^{l(n-n/q-\alpha)}. \tag{5.1}$$

By the Minkowski inequality and the above we have

$$\|Ta_l\chi_k\|_{L^q} \leqslant C2^{l(n-n/q-\alpha)} \sup_{y\in B_l}\Big(\int_{C_k} |K(x,y)|^q\,\mathrm{d}x\Big)^{1/q}. \tag{5.2}$$

$I_2$ is dominated by

$$\begin{aligned} I_2 &= \sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p + \sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p \\ &= J_1 + J_2. \end{aligned}$$

For $J_1$, using (5.2) and (3.8) we have

$$J_1 \leqslant C\sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j(m+n-(n+1)/2q)}\Big)^p\Big(\sum_{l\leqslant 0}|\lambda_l|2^{l(n-n/q-\alpha)}\Big)^p.$$

By condition (1.3), $J_1$ can be controlled by

$$J_1 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot\Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0}2^{lp'(n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

As $\alpha < n(1-1/q)$, we obtain

$$J_1 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

For $J_2$, using (5.2) and (3.9) we have

$$J_2 \leqslant C\sum_{k=k_1+2}^{k_0+1} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j[m+n-1/2q-N]}\Big)^p\Big(\sum_{l\leqslant 0}|\lambda_l|2^{l(n-n/q-\alpha)}\Big)^p.$$

Choosing $N$ so that $N > \max\{n/2q, m+n-1/2q\}$, we have

$$J_2 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot\Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0}2^{lp'(n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

Since $\alpha < n(1-1/q)$, it is easy to see that

$$J_2 \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

Therefore

$$\|Tf\|^p_{\dot{K}_q^{\alpha,p}} \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

So the boundedness on $K_q^{\alpha,p}$ is obtained.

Now let us consider the boundedness on $HK_q^{\alpha,p}$.

Suppose $f \in H\dot{K}_q^{\alpha,p}$. By Lemma 2.3, $f(x) = \sum\limits_{l\in\mathbb{Z}} \lambda_l a_l(x)$, where $a_l(x)$ is a central $(\alpha,q)$-atom with the support $B_l$, and

$$\sum_{l\in\mathbb{Z}} |\lambda_l|^p \leqslant c\|f\|^p_{H\dot{K}_q^{\alpha,p}}.$$

By Definition 1.2, we have

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_q^{\alpha,p}} &:= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|Tf\chi_k\|^p_{L^q} \\
&= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0} + \sum_{l\leqslant 0} |\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p \\
&\leqslant C\cdot\Big\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p + \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p\Big\} \\
&= C(I_1 + I_2).
\end{aligned}$$

Similarly to the estimate of $I_1$ in Theorem 1.2 we have $I_1 \leqslant C\sum\limits_l |\lambda_l|^p$.

For the estimate of $I_2$, suppose $n+s \leqslant \alpha + n/q < n+s+1$, and let $\gamma$ be a multi-index with $|\gamma| = s+1$. we have

$$\begin{aligned}
\|T_j a_l\chi_k\|_{L^q} &= \Big(\int_{C_k}\Big|\int K_j(x,y)\cdot a_l(y)\,\mathrm{d}y\Big|^q\,\mathrm{d}x\Big)^{1/q} \\
&\leqslant C\Big(\int_{C_k}\Big|\int D_y^\gamma K_j(x,y)\cdot a_l(y)y^\gamma\,\mathrm{d}y\Big|^q\,\mathrm{d}x\Big)^{1/q} \\
&\leqslant C2^{l(s+1)}\cdot\sup\nolimits_{y\in B_l}\Big(\int_{C_k}|D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q}\cdot\int|a_l(y)|\,\mathrm{d}y,
\end{aligned}$$

by virtue of the vanishing moment condition of the atom and the Minkowski inequality. And then using (3.10), (3.11) and (5.1), we can dominate $\|T_j a_l\chi_k\|_{L^q}$ by

$$\|T_j a_l\chi_k\|_{L^q} \leqslant C2^{l(s+1+n-n/q-\alpha)}2^{j(s+1+m+n-(n+1)/2q)}, \quad k\in\mathbb{Z}, \tag{5.3}$$

and

$$\|T_j a_l \chi_k\|_{L^q} \leqslant C 2^{l(s+1+n-n/q-\alpha)} 2^{j(s+1+m+n-1/2q-N)}, \quad k > k_1+1. \tag{5.4}$$

Now let us estimate $I_2$. Write

$$\begin{aligned} I_2 &= \sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|T a_l\chi_k\|_{L^q}\Big)^p + \sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|T a_l\chi_k\|_{L^q}\Big)^p \\ &= J_1 + J_2. \end{aligned}$$

For $J_1$, using (5.3) we have

$$J_1 \leqslant C\sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j(s+1+m+n-(n+1)/2q)}\Big)^p\Big(\sum_{l\leqslant 0}|\lambda_l| 2^{l(s+1+n-n/q-\alpha)}\Big)^p.$$

By the fact that $m \leqslant (n+1)/2q - n - s - 1$, we have

$$J_1 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot\Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0}2^{lp'(s+1+n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

The assumption $\alpha + n/q < n+s+1$ implies

$$J_1 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

For $J_2$, using (5.4) we have

$$J_2 \leqslant C\sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j(s+1+m+n-1/2q-N)}\Big)^p\Big(\sum_{l\leqslant 0}|\lambda_l| 2^{l(s+1+n-n/q-\alpha)}\Big)^p.$$

Choosing $N$ such that $N > \max\{n/2q, s+1+m+n-1/2q\}$, we have

$$J_2 \leqslant \begin{cases} C\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0}2^{lp'(s+1+n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

By the assumption that $\alpha + n/q < n+s+1$, we get

$$J_2 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

Thus we get

$$\|Tf\|^p_{\dot{K}_q^{\alpha,p}} \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

so we obtain the boundedness on $HK_q^{\alpha,p}$,

## 6. Conclusions

In this paper, the main result shows that a classical Fourier $T$ is bounded from $HK_q$ to $K_q$ when the order $m$ of $T$ satisfies

$$-n < m \leqslant -\frac{n-1}{2}, \quad \text{and} \quad \frac{1}{q} \geqslant 2 + \frac{2(m-1)}{n+1}.$$

Also the boundedness from the other two kinds of Herz spaces $K_q^{\alpha,p}(\dot{K}_q^{\alpha,p})$ and $HK_q^{\alpha,p}(H\dot{K}_q^{\alpha,p})$ to Herz spaces $K_q^{\alpha,p}(\dot{K}_q^{\alpha,p})$ is obtained. A natural problem is that whether the boundedness of $T$ from Hardy spaces associated with Herz spaces into themselves is valid, on which we will keep an eye in the future.

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