Czechoslovak Mathematical Journal

Zahra Kamali; Bahram Khani Robati; Karim Hedayatian

Cyclicity of the adjoint of weighted composition operators on the Hilbert space of analytic functions

*Czechoslovak Mathematical Journal*, Vol. 61 (2011), No. 2, 551–563

Persistent URL: http://dml.cz/dmlcz/141553

**Czechoslovak Mathematical Journal, 61 (136) (2011), 551–563**

# CYCLICITY OF THE ADJOINT OF WEIGHTED COMPOSITION OPERATORS ON THE HILBERT SPACE OF ANALYTIC FUNCTIONS

Zahra Kamali, Bahram Khani Robati, Karim Hedayatian, Shiraz

(Received April 15, 2010)

*Abstract.* In this paper, we discuss the hypercyclicity, supercyclicity and cyclicity of the adjoint of a weighted composition operator on a Hilbert space of analytic functions.

*Keywords*: hypercyclicity, supercyclicity, cyclicity, weighted composition operators

*MSC 2010*: 47A16, 47B33, 47B38

## 1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane. Let $\mathscr{H}$ be a Hilbert space of analytic functions defined on $\mathbb{D}$ such that for each $\lambda$ in $\mathbb{D}$ the linear functional of point evaluation at $\lambda$ given by $f \mapsto f(\lambda)$ is bounded. By a Hilbert space of analytic functions $\mathscr{H}$ we mean one satisfying the above conditions.

For any $\lambda \in \mathbb{D}$, let $e_\lambda$ denote the linear functional of point evaluation at $\lambda$ on $\mathscr{H}$, that is, $e_\lambda(f) = f(\lambda)$ for every $f$ in $\mathscr{H}$. Since $e_\lambda$ is a bounded linear functional, the Riesz representation theorem states that

$$e_\lambda(f) = \langle f, k_\lambda \rangle$$

for some $k_\lambda \in \mathscr{H}$.

A well-known example of a Hilbert space of analytic functions is the weighted Hardy space. Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is defined as the space of analytic functions $f =$

This research was in part supported by a grant from Shiraz University Research Council.

$\sum_{n=0}^{\infty} \hat{f}(n)z^n$ on $\mathbb{D}$ satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2[\beta(n)]^2 < \infty.$$

These spaces are Hilbert spaces with the inner product defined by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}[\beta(n)]^2$$

for every $f$ and $g$ in $H^2(\beta)$.

The classical Hardy space, the Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(j) = 1$, $\beta(j) = (j+1)^{-1/2}$ and $\beta(j) = (j+1)^{1/2}$, respectively. Weighted Bergman and Dirichlet spaces are also weighted Hardy spaces. For further information on these spaces see [4].

Let $\varphi$ be an automorphism of the disc. Recall that $\varphi$ is elliptic if it has one fixed point in the disc and the other in the complement of the closed disc, hyperbolic if both of its fixed points are on the unit circle, and parabolic if it has one fixed point on the unit circle (of multiplicity two).

Recall that a multiplier of $\mathscr{H}$ is a complex-valued function $w$ on $\mathbb{D}$ such that $w\mathscr{H} \subseteq \mathscr{H}$. The set of all multipliers of $\mathscr{H}$ is denoted by $M(\mathscr{H})$. If $w$ is a multiplier, then the multiplication operator $M_w$, defined by $M_w f = wf$, is bounded on $\mathscr{H}$. Also, note that for each $\lambda \in \mathbb{D}$, $M_w^* k_\lambda = \overline{w(\lambda)} k_\lambda$. It is known that $M(\mathscr{H}) \subseteq H^\infty$. In fact, suppose that $w \in M(\mathscr{H})$ and $f$ is a nonzero function in $\mathscr{H}$. If $f$ has a zero of order $n$ at $z \in \mathbb{D}$ and $wf$ has a zero of order $m$ at $z$, then $(j-1)n \leqslant jm$ for every $j \geqslant 1$ thanks to the fact that $f^{j-1}(fw^j) = (fw)^j$ and $fw^j$ is analytic on $\mathbb{D}$. Therefore,

$$\frac{j-1}{j} \leqslant \frac{m}{n}, \quad j = 1, 2, 3, \ldots$$

letting $j \to \infty$, we conclude that $n \leqslant m$, which implies that $w = wf/f$ is analytic on $\mathbb{D}$. On the other hand, if $\lambda \in \mathbb{D}$ then

$$|w(\lambda)k_\lambda(\lambda)| = |\langle M_w k_\lambda, k_\lambda\rangle| \leqslant \|M_w\|\|k_\lambda\|^2.$$

This implies that $|w(\lambda)| \leqslant \|M_w\|$ for every $\lambda \in \mathbb{D}$ and so $w \in H^\infty$.

If $w \in M(\mathscr{H})$ and $\varphi$ is an analytic self-map of $\mathbb{D}$ such that $(f \circ \varphi)(z) = f(\varphi(z))$ is in $\mathscr{H}$ for every $f \in \mathscr{H}$, then an application of the closed graph theorem shows that the weighted composition operator $C_{w,\varphi}$ defined by $C_{w,\varphi}(f)(z) = M_w C_\varphi(f)(z) = w(z)f(\varphi(z))$ is bounded. The mapping $\varphi$ is called the composition map and $w$ is called the weight. From now on, we assume that $w$ is a multiplier of $\mathscr{H}$ and $\varphi$ has the above property. For a positive integer $n$, $\varphi_n$ is the $n$th iterate of $\varphi$; also $\varphi_0$ is the identity function and when $\varphi$ is invertible $\varphi_{-n}$ is the $n$th iterate of $\varphi^{-1}$. The weighted composition operators come up naturally. In 1964, Forelli [6] showed that every isometry on $H^p$ for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator. Recently, there has been a great interest in studying composition and weighted composition operators on the unit disc, polydisc or the unit ball, see for example monographs [4], [13]. In this paper, we discuss the cyclic behavior of the adjoint of these operators.

If $x$ is a vector in $\mathscr{H}$ and $T$ is an operator on $\mathscr{H}$, the set $\{x, Tx, T^2x, \ldots\}$, denoted by $\operatorname{Orb}\{T, x\}$, will be called the orbit of $x$ under $T$. If some orbit is dense in $\mathscr{H}$, then $T$ is called a hypercyclic operator and the vector $x$ is called a hypercyclic vector for $T$. The operator $T$ is called supercyclic, if the set of scalar multiples of the elements of $\operatorname{Orb}\{T, x\}$ is dense, and cyclic if the linear span of $\operatorname{Orb}\{T, x\}$ is dense; the vector $x$ is called, respectively, a supercyclic vector or a cyclic vector for $T$.

Hypercyclicity of operators has been studied a lot in literature. The classical hypercyclic operator is $2B$ on the space $\ell^2(\mathbb{N})$ where $B$ is the backward shift [10]. It is proved that many famous operators are hypercyclic. For instance, certain operators in the classes of composition operators [2], the adjoints of subnormal, hyponormal and multiplication operators ([5], [3]) and weighted shift operators ([11]) are hypercyclic. As a good source on hypercyclicity and supercyclicity of operators one can see [1].

The hypercyclicity of the adjoint of a weighted composition operator on a Hilbert space of analytic functions is investigated in [14] and [15]. In this paper, we first give counterexamples to the main result of [14] and then establish sufficient conditions for hypercyclicity, supercyclicity and cyclicity of the adjoint of a weighted composition operator.

## 2. Hypercyclicity of the adjoint of weighted composition operators

In [14] the authors claimed "as a theorem" that the adjoint of a weighted composition operator $M_w C_\varphi$ is hypercyclic on $\mathscr{H}$ if $\varphi$ is an automorphism, the composition operator $C_\varphi$ is bounded and $w$ is a non-constant multiplier such that the sets $\{\lambda \in \mathbb{D}\colon \sup_n |w \circ \varphi_n(\lambda)| < 1\}$ and $\{\lambda \in \mathbb{D}\colon \inf_n |w \circ \varphi_n^{-1}(\lambda)| > 1\}$ have limit points in $\mathbb{D}$. But here we present many examples to show that the above result is not correct on the Hardy space $H^2$ and the Bergman space $L^2_a(\mathbb{D})$. For $a \geqslant 1$, put $\alpha = a/(a+1)$ and let

$$\varphi(z) = \frac{z + \alpha}{1 + \alpha z}$$

and

$$w(z) = \frac{\sqrt{2a+1}}{az+a+1}$$

for all $z \in \mathbb{D}$.

If $a_1 = a+1$, $a_{n+1} = (2a+1)a_n - a$ for $n \geqslant 1$, then using induction we see that $a+1 \leqslant a_n$; moreover,

$$\varphi_n(z) = \frac{z+\alpha_n}{1+\alpha_n z}, \quad \text{where } \alpha_n = \frac{a_n - 1}{a_n}$$

and

$$w \circ \varphi_n(z) = \frac{((a_n-1)z + a_n)\sqrt{2a+1}}{(2aa_n + a_n - a - 1)z + 2aa_n + a_n - a}.$$

Suppose that $0 < x < 1$. So $x \leqslant xa_n$ and $a \leqslant a_n + ax$, which implies that

$$|w \circ \varphi_n(x)| = w \circ \varphi_n(x) \leqslant \frac{\sqrt{2a+1}}{2a} < 1.$$

Therefore, the open interval $(0,1)$ is a subset of the set $\bigl\{\lambda \in \mathbb{D}\colon \sup\limits_{n\geqslant 0} |w\circ\varphi_n(\lambda)| < 1\bigr\}$. On the other hand, it is obvious that

$$\varphi_{-n}(z) = \frac{z-\alpha_n}{1-\alpha_n z}$$

and

$$w \circ \varphi_{-n}(z) = \frac{((a_n-1)z - a_n)\sqrt{2a+1}}{(a_n - a - 1)z - a_n - a}$$

for all $n \geqslant 1$.

Suppose that $-1 < x \leqslant -\frac{1}{2}$. Then $2a(1+x) \leqslant a \leqslant a_n$ and $xa_n \leqslant x$. It follows that

$$|w \circ \varphi_{-n}(x)| = w \circ \varphi_{-n}(x) = \frac{((1-a_n)x + a_n)\sqrt{2a+1}}{(a - a_n + 1)x + a_n + a} \geqslant \frac{2\sqrt{2a+1}}{3} > 1;$$

therefore,

$$\Bigl(-1, -\frac{1}{2}\Bigr] \subseteq \bigl\{\lambda \in \mathbb{D}\colon \inf_{n\geqslant 1} |w \circ \varphi_{-n}(\lambda)| > 1\bigr\}.$$

Now, all conditions are satisfied for $H^2$. Moreover, if $\varphi(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}t}$ then $|\varphi'(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta = \mathrm{d}t$. Also, an easy calculation shows that $|\varphi'(\mathrm{e}^{\mathrm{i}\theta})| = |w(\mathrm{e}^{\mathrm{i}\theta})|^2$ for all $\theta$ in $[0, 2\pi]$. Therefore,

$$\|M_w C_\varphi f\|_{H^2}^2 = \int_0^{2\pi} |w(\mathrm{e}^{\mathrm{i}\theta}) f \circ \varphi(\mathrm{e}^{\mathrm{i}\theta})|^2 \frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}t})|^2 \frac{\mathrm{d}\theta}{2\pi} = \|f\|_{H^2}^2$$

for all $f \in H^2$. So $\|(M_wC_\varphi)^*\|_{H^2} = \|(M_wC_\varphi)\|_{H^2} = 1$ and consequently, $(M_wC_\varphi)^*$ cannot be hypercyclic on the Hardy space $H^2$. On the other hand, if $v(z) = w(z)^2$ then it follows that the open interval $(0,1)$ is a subset of $\bigl\{\lambda \in \mathbb{D}\colon \sup\limits_{n\geqslant 0} |v\circ\varphi_n(\lambda)| < 1\bigr\}$; moreover,

$$\Bigl(-1, -\frac{1}{2}\Bigr] \subseteq \bigl\{\lambda \in \mathbb{D}\colon \inf_{n\geqslant 1} |v \circ \varphi_{-n}(\lambda)| > 1\bigr\}.$$

Also, an easy calculation shows that $\varphi'(z) = v(z)$ for all $z$ in $\mathbb{D}$. So if $\eta = \varphi(z)$, the usual change of variable formula implies that

$$\begin{aligned}\|M_vC_\varphi f\|^2_{L^2_a(\mathbb{D})} &= \int_{\mathbb{D}} |v(z) f \circ \varphi(z)|^2 \,\mathrm{d}A(z)\\ &= \int_{\varphi(\mathbb{D})} |f(\eta)|^2 \,\mathrm{d}A(\eta)\\ &= \int_{\mathbb{D}} |f(\eta)|^2 \,\mathrm{d}A(\eta) = \|f\|^2_{L^2_a(\mathbb{D})}\end{aligned}$$

for all $f \in L^2_a(\mathbb{D})$ where $\mathrm{d}A$ denotes the Lebesgue area measure on the unit disc. Thus,

$$\|(M_vC_\varphi)^*\|_{L^2_a(\mathbb{D})} = \|(M_vC_\varphi)\|_{L^2_a(\mathbb{D})} = 1;$$

consequently, $(M_vC_\varphi)^*$ cannot be hypercyclic on the Bergman space $L^2_a(\mathbb{D})$.

The first example of a hypercyclic operator on a Banach space was given by Rolewicz [10] in 1969. The study of hypercyclicity was really begun with Kitai's work [9] in 1982. She gave a hypercyclicity criterion. In [7], Gethner and Shapiro rediscovered this criterion independently and generalized it. We state this criterion and use it for indicating the hypercyclicity of the adjoint of $C_{w,\varphi}$.

**Hypercyclicity criterion**

Suppose that $X$ is a separable Banach space and $T$ is a bounded operator on $X$. If there exist an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and two dense sets $Y$ and $Z$ such that

1. $T^{n_k}x \to 0$ for every $x \in Y$, and
2. there exists a function $S\colon Z \to Z$ such that $TSx = x$ for all $x \in Z$, and $S^{n_k}x \to 0$ for every $x \in Z$

then $T$ is hypercyclic.

In the next theorem we give sufficient conditions for hypercyclicity of the adjoint of a weighted composition operator. We say that a sequence $\{a_j\}_j$ of complex numbers is not a Blaschke sequence, if there exists $j_0$ such that $a_j \in \mathbb{D}$ for all $j \geqslant j_0$ and $\sum\limits_{j=1}^{\infty}(1 - |a_j|)$ diverges.

**Theorem 1.** *Let $\varphi$ be a disc automorphism such that the sets*

$$E = \{\lambda \in \mathbb{D}\colon \ \{w \circ \varphi_n(\lambda)\}_{n=0}^{\infty} \text{ is not a Blaschke sequence}\}$$

*and*

$$F = \{\lambda \in \mathbb{D}\colon \ \{(w \circ \varphi_{-n}(\lambda))^{-1}\}_{n=1}^{\infty} \text{ is not a Blaschke sequence}\}$$

*have limit points in $\mathbb{D}$. If for every $\lambda_1 \in E$ and every $\lambda_2 \in F$ the sequences $\{k_{\varphi_n(\lambda_1)}\}_{n\geqslant 1}$ and $\{k_{\varphi_{-n}(\lambda_2)}\}_{n\geqslant 1}$ are bounded, then $C^*_{w,\varphi}$ is hypercyclic on $\mathscr{H}$.*

P r o o f. First we note that the sets $H_E = \operatorname{span}\{k_\lambda\colon \ \lambda \in E\}$ and $H_F = \operatorname{span}\{k_\lambda\colon \ \lambda \in F\}$ are dense in $\mathscr{H}$. To see this, suppose that $f \in \mathscr{H}$ and $\langle f, k_\lambda\rangle = 0$ for all $\lambda \in E$. Then the zero set of $f$ has a limit point in $\mathbb{D}$ and so $f \equiv 0$; i.e., $\operatorname{cl} H_E = \mathscr{H}$. Similarly $\operatorname{cl} H_F = \mathscr{H}$. Since $C^*_\varphi k_\lambda = k_{\varphi(\lambda)}$ and $M^*_w k_\lambda = \overline{w(\lambda)} k_\lambda$, we have $C^*_{w,\varphi} k_\lambda = \overline{w(\lambda)} k_{\varphi(\lambda)}$. By using the mathematical induction we get,

$$C^{*n}_{w,\varphi} k_\lambda = \prod_{j=0}^{n-1} \overline{w(\varphi_j(\lambda))} k_{\varphi_n(\lambda)}.$$

Fix $\lambda \in E$. Since $\{w(\varphi_j(\lambda))\}_j$ is not a Blaschke sequence, $\sum\limits_{j=0}^{\infty}(1 - |w(\varphi_j(\lambda))|) = \infty$; or equivalently, $\lim\limits_{n\to\infty} \prod\limits_{j=0}^{n-1} \overline{w(\varphi_j(\lambda))} = 0$. This together with the fact that $\{k_{\varphi_n(\lambda)}\}_n$ is a bounded sequence implies that $\lim\limits_{n\to\infty} C^{*n}_{w,\varphi} k_\lambda = 0$. Set $G_F = \{k_\lambda\colon \ \lambda \in F\}$ and first suppose that $G_F$ is a linearly independent set. Also define $B\colon G_F \to \mathscr{H}$ by $Bk_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} k_{\varphi^{-1}(\lambda)}$. Clearly, $\varphi^{-1}(\lambda) \in F$ whenever $\lambda \in F$, and so we can define $B^n$ for all $n \geqslant 1$. Since $G_F$ is linearly independent, we can extend $B$ by linearity on $H_F$. An easy computation shows that

$$B^n k_\lambda = \prod_{j=1}^{n} (\overline{w(\varphi_{-j}(\lambda))})^{-1} k_{\varphi_{-n}(\lambda)} \quad \text{for all } \lambda \in F.$$

If $\lambda \in F$ then we have $C^*_{w,\varphi} B k_\lambda = k_\lambda$, that is, $C^*_{w,\varphi} B = I$ on the dense set $H_F$ of $\mathscr{H}$. A similar argument shows that $\lim\limits_{n\to\infty} B^n k_\lambda = 0$ for all $\lambda \in F$. Now assume that $G_F$ is not necessarily linearly independent. In this case, we use the same method as the one used by Godefroy and Shapiro in Theorem 4.5 of [8] or [14]. Consider a countable dense subset $F_1 = \{\lambda_n\colon \ n > 0\}$ of $F$, and using induction choose a sequence $\{z_n\}_n$ as follows. Take $z_1 = \lambda_1$, $F_2 = F_1 - \{\lambda \in F_1\colon \ k_\lambda \in \operatorname{span}\{k_{z_1}\}\}$. Denote the first element of $F_2$ by $z_2$ and let $F_3 = F_2 - \{\lambda \in F_2\colon \ k_\lambda \in \operatorname{span}\{k_{z_1}, k_{z_2}\}\}$. Continuing this process we obtain a subset $L = \{z_n\colon \ n > 0\}$ of $F$ for which the set $H_L = \{k_\lambda\colon \ \lambda \in L\}$ is linearly independent and dense in $\mathscr{H}$. Define $S_n\colon H_L \mapsto \mathscr{H}$

by $S_n k_\lambda = \Big(\overline{\prod_{j=1}^{n} w \circ \varphi_{-j}(\lambda)}\Big)^{-1} k_{\varphi_{-n}(\lambda)}$. Clearly $C^{*n}_{w,\varphi} S_n k_\lambda = k_\lambda$ for all $k_\lambda \in H_L$. Furthermore, to prove the fact that $S_n \to 0$ pointwise on $H_L$ is similar to the proof of the fact that $B^n \to 0$ pointwise on $H_G$. So in both cases the conditions of hypercyclicity criterion are satisfied for $C^*_{w,\varphi}$ and the proof is completed. □

Recall that if $\varphi$ is an analytic self-map of $\mathbb{D}$ that is neither the identity nor an elliptic automorphism, then there is a point $a$ in $\overline{\mathbb{D}}$, called the Denjoy-Wolff point of $\varphi$, such that $\{\varphi_n\}_n$ converges to $a$ uniformly on compact subsets of $\mathbb{D}$. If $\varphi$ is a hyperbolic automorphism one of its fixed point is the Denjoy-Wolff point of $\varphi$ and the other is repulsive, i.e., it is the Denjoy-Wolff point of $\varphi^{-1}$. Furthermore, the angular derivative of $\varphi$ at the Denjoy-Wolff point $a$, $\varphi'(a)$ is less than 1.

**Corollary 1.** *Suppose that $\mathscr{H}$ is a Hilbert space of analytic functions such that the set $\{k_\lambda\colon \lambda \in \mathbb{D}\}$ is bounded and $\varphi$ is a hyperbolic automorphism with the Denjoy-Wolff point $a$ and the repulsive fixed point $b$. Moreover, suppose that $w$ has nontangential limits $w(a)$ at $a$ and $w(b)$ at $b$. If $|w(a)| < 1 < |w(b)|$, then $C^*_{w,\varphi}$ is hypercyclic.*

Proof. Since $\varphi'(a) < 1$, for every $z \in \mathbb{D}$ there is a nontangential approach region containing all the iterates $\varphi_n(z)$; so $\lim\limits_{n\to\infty} w(\varphi_n(z)) = w(a)$ which implies that $\sum\limits_{j=0}^{\infty}(1 - |w(\varphi_j(z))|) = \infty$. Therefore, the set $E$, in the preceding theorem, has a limit point in $\mathbb{D}$. Similarly, since $\varphi'_{-1}(b) < 1$, we have $\sum\limits_{j=1}^{\infty}(1 - |w(\varphi_{-j}(z))|^{-1}) = \infty$ and so the set $F$ in the preceding theorem has a limit point in $\mathbb{D}$. Hence, the proof is completed by applying Theorem 1. □

For $a \in \mathbb{D}$, consider an automorphism of $\mathbb{D}$ defined by $\psi_a(z) = (a - z)/(1 - \bar{a}z)$ ($z \in \mathbb{D}$). A Hilbert space $\mathscr{H}$ of analytic functions is called *automorphism invariant*, if for every $a \in \mathbb{D}$, $f \circ \psi_a \in \mathscr{H}$ whenever $f \in \mathscr{H}$, and also, $w \circ \psi_a \in M(\mathscr{H})$ whenever $w \in M(\mathscr{H})$. Many spaces such as the Hardy and Bergman spaces are automorphism invariant.

**Theorem 2.** *Suppose that $\mathscr{H}$ is automorphism invariant and $\varphi$ is an elliptic disc automorphism. If the sets $E$ and $F$ in the preceding theorem have limit points in $\mathbb{D}$, then $C^*_{w,\varphi}$ is hypercyclic.*

Proof. First suppose that $\varphi(0) = 0$. Then $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Thus, for every $\lambda \in E$, $\{\varphi_n(\lambda)\colon n \in \mathbb{N}\} \subseteq \lambda\partial\mathbb{D}$. But $\lambda\partial\mathbb{D}$ is a compact subset of $\mathbb{D}$ and so for $f \in \mathscr{H}$ the continuity of $f$ implies that $\{f(\varphi_n(\lambda))\}_n$ is a bounded sequence. This, by virtue of the uniform boundedness principle, implies that $\{k_{\varphi_n(\lambda)}\}_n$ is also bounded. Similarly, the sequence $\{k_{\varphi_{-n}(\lambda)}\}_n$ is bounded for each $\lambda \in F$. Hence, by Theorem 1, $C^*_{w,\varphi}$ is hypercyclic.

Now, suppose that $0 \neq a \in \mathbb{D}$ is a fixed point of $\varphi$. Set

$$\psi_a(z) = \frac{a-z}{1-\overline{a}z}, \quad \phi = \psi_a \circ \varphi \circ \psi_a, \quad W = w \circ \psi_a,$$
$$E_a = \{\lambda \in \mathbb{D}\colon\ \{W \circ \phi_j(\lambda)\}_j \text{ is not a Blaschke sequence}\},$$

and

$$F_a = \{\lambda \in \mathbb{D}\colon\ \{(W \circ \phi_{-j}(\lambda))^{-1}\}_j \text{ is not a Blaschke sequence}\}.$$

Now, an application of the closed graph theorem shows that $C_{W,\phi}$ is bounded. Moreover, $W \circ \phi_j(\lambda) = w \circ \varphi_j(\psi_a(\lambda))$ for all $j \geqslant 1$ and all $\lambda \in \mathbb{D}$; thus, $\psi_a^{-1}(E) = E_a$ and $\psi_a^{-1}(F) = F_a$. Hence, the sets $E_a$ and $F_a$ have limit points in $\mathbb{D}$. But since $\phi(0) = 0$, according to the first part of the proof, $C^*_{W,\phi}$ is hypercyclic. Thus, taking into account that $C^*_{w,\varphi}$ is similar to $C^*_{W,\phi}$, the result follows. □

**Corollary 2** ([15], Theorem 3.3). *Let $\mathscr{H}$ be automorphism invariant, let $\varphi$ be an elliptic automorphism with an interior fixed point $a$ and suppose that $w$ satisfies the inequality $|w(a)| < 1 < \liminf\limits_{|z|\to 1^-} |w(z)|$. Then $C^*_{w,\varphi}$ is hypercyclic.*

Proof. As is seen in the preceding theorem, we can assume that $a = 0$ and $\varphi(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ satisfying $|\alpha| = 1$. From the inequality $|w(0)| < 1 < \liminf\limits_{|z|\to 1^-} |w(z)|$, we conclude that there exist $\delta_1$, $\delta_2$, $\lambda_1$, $\lambda_2$ in $\mathbb{C}$ such that if $|z| < \delta_1$ then $|w(z)| < \lambda_1 < 1$, and if $|z| > 1 - \delta_2$ then $|w(z)| > \lambda_2 > 1$. But since $|\alpha| = 1$, we have $|\varphi_j(z)| = |z|$ for all $j \in \mathbb{N}$. Thus, whenever $|z| < \delta_1$, we observe that $|w \circ \varphi_j(z)| < \lambda_1 < 1$ and so $\left|\prod\limits_{j=0}^{n-1} w \circ \varphi_j(z)\right| < \lambda_1^n < 1$. Furthermore, if $|z| > 1 - \delta_2$, we have $\left|\prod\limits_{j=1}^{n} w \circ \varphi_{-j}(z)\right|^{-1} < \lambda_2^n < 1$. These two inequalities imply that

$$\{z \in \mathbb{D}\colon\ |z| < \delta_1\} \subseteq \{z \in \mathbb{D}\colon\ \{w \circ \varphi_n(z)\}_n \text{ is not a Blaschke sequence}\},$$

and

$$\{z \in \mathbb{D}\colon\ |z| > 1 - \delta_2\} \subseteq \{z \in \mathbb{D}\colon\ \{(w \circ \varphi_{-n}(z))^{-1}\}_n \text{ is not a Blaschke sequence}\}.$$

Now, the corollary follows from the preceding theorem. □

Recall that if $\varphi$ is an elliptic automorphism, a rotation through a rational multiple of $\pi$, then there is $m > 0$ such that $\varphi_m(z) = z$ for all $z \in \mathbb{D}$. Also, an automorphism $\varphi$ of $\mathbb{D}$ is an involution if $(\varphi\circ\varphi)(z)=z$ for all $z\in\mathbb{D}$. For example $\psi_a(z)=(a-z)/(1-\overline{a}z)$ where $|a|<1$ and $\varphi(z)=z$ are involutions. In the next proposition for a subset $E$ of the complex plane, by $\overline{E}$ we mean the set $\{\overline{z}\colon z\in E\}$.

**Proposition 1.** *Suppose that there is $m>0$ such that $\varphi_m(z)=z$ for all $z\in\mathbb{D}$.*

(a) *If $C^*_{w,\varphi}$ is hypercyclic then* $\mathrm{cl}\Big(\overline{\mathrm{ran}\Big(\prod_{j=0}^{m-1} w\circ\varphi_j\Big)}\Big)\cap\partial\mathbb{D}\neq\emptyset$.

(b) *If $w$ is non-constant and* $\mathrm{ran}\Big(\prod_{j=0}^{m-1} w\circ\varphi_j\Big)\cap\partial\mathbb{D}\neq\emptyset$ *then $C^*_{w,\varphi}$ is hypercyclic.*

Proof. (a) If $C^*_{w,\varphi}$ is hypercyclic then so is $(C^*_{w,\varphi})^m$. Consequently, $\sigma((C^*_{w,\varphi})^m)\cap\partial\mathbb{D}\neq\emptyset$ (see [1], page 11). But $(C^*_{w,\varphi})^m=M^*_\psi$, where $\psi=\prod_{j=0}^{m-1} w\circ\varphi_j$ and $\sigma(M^*_\psi)=\mathrm{cl}(\overline{\mathrm{ran}\,\psi})$; so the result follows.

(b) We observe that the sequences $\{\varphi_n(\lambda)\}_{n=0}^{\infty}$ and $\{\varphi_{-n}(\lambda)\}_{n=0}^{\infty}$ are, indeed, finite sets consisting of $\{\varphi_j(\lambda)\}_{j=0}^{m-1}$. Thus, the sequences $\{k_{\varphi_n(\lambda)}\}_n$ and $\{k_{\varphi_{-n}(\lambda)}\}_n$ are bounded for every $\lambda$ in $\mathbb{D}$. Furthermore, since $\mathrm{ran}\Big(\prod_{j=0}^{m-1} w\circ\varphi_j\Big)\cap\partial\mathbb{D}\neq\emptyset$ and $\prod_{j=0}^{m-1} w\circ\varphi_j$ is analytic on $\mathbb{D}$, the open mapping theorem implies that $U=\Big\{\lambda\in\mathbb{D}\colon \Big|\prod_{j=0}^{m-1} w\circ\varphi_j(\lambda)\Big|<1\Big\}$ and $V=\Big\{\lambda\in\mathbb{D}\colon \Big|\prod_{j=0}^{m-1} w\circ\varphi_j(\lambda)\Big|>1\Big\}$ are non-empty open sets. Fix $\lambda\in U$, and let $P_n=\prod_{j=0}^{n-1} w\circ\varphi_j(\lambda)$ for $n\in\mathbb{N}$. We have $P_{km}=\Big(\prod_{j=0}^{m-1} w\circ\varphi_j(\lambda)\Big)^k$ and so $P_{km}\to 0$ as $k\to\infty$. Also, if $M=\max\Big\{\Big|\prod_{j=0}^{i} w\circ\varphi_j(\lambda)\Big|\colon i=0,1,\ldots,m-1\Big\}$ and $n\geqslant m$ then $|P_n|\leqslant|P_{km}|M$ for some $k\in\mathbb{N}$, which implies that $P_n\to 0$ as $n\to\infty$. Similarly, if $\lambda\in V$ and $Q_n=\prod_{j=1}^{n}(w\circ\varphi_{-j}(\lambda))^{-1}$ for $n\in\mathbb{N}$ then $Q_n\to 0$ and the result follows by using Theorem 1. □

If we let $\varphi(z)=z$ then in virtue of part (b) of the preceding proposition we obtain the following result due to Godefroy and Shapiro [8].

**Corollary 3.** *If $w$ is a non-constant multiplier of $\mathscr{H}$ such that* $\mathrm{ran}\,w$ *intersects the unit circle then $M^*_w$ is hypercyclic.*

## 3. Supercyclicity of the adjoint of weighted composition operators

To discuss the supercyclicity of the adjoint of a weighted composition operator we need the supercyclicity criterion. See [12], or more generally [5].

**Supercyclicity criterion**

Suppose that $X$ is a separable Banach space and $T$ is a bounded operator on $X$. If there is an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$, and two dense sets $Y$ and $Z$ of $X$ such that

1. there exists a function $S\colon Z \to Z$ satisfying $TSx = x$ for all $x \in Z$, and
2. $\|T^{n_k}x\| \cdot \|S^{n_k}y\| \to 0$ for every $x \in Y$ and $y \in Z$,

then $T$ is supercyclic.

**Theorem 3.** *Let $\varphi$ be a disc automorphism. Set*

$$E = \left\{\lambda \in \mathbb{D}\colon \left\{\prod_{j=0}^{n-1} w \circ \varphi_j(\lambda)\right\}_n \text{ is a bounded sequence}\right\},$$

$$F = \{\lambda \in \mathbb{D}\colon \{(w \circ \varphi_{-n}(\lambda))^{-1}\}_n \quad \text{is not a Blaschke sequence}\},$$

$$G = \{\lambda \in \mathbb{D}\colon \{w \circ \varphi_n(\lambda)\}_n \quad \text{is not a Blaschke sequence}\}$$

*and*

$$H = \left\{\lambda \in \mathbb{D}\colon \left\{\left(\prod_{j=1}^{n} w \circ \varphi_{-j}(\lambda)\right)^{-1}\right\}_n \text{ is a bounded sequence}\right\}.$$

*If one of the following conditions holds then $C^*_{w,\varphi}$ is a supercyclic operator.*

(i) *The sets $E$ and $F$ have limit points in $\mathbb{D}$; moreover, $\{k_{\varphi_n(\lambda_1)}\}_n$ and $\{k_{\varphi_{-n}(\lambda_2)}\}_n$ are bounded sequences for all $\lambda_1 \in E$ and $\lambda_2 \in F$.*

(ii) *The sets $G$ and $H$ have limit points in $\mathbb{D}$; furthermore, $\{k_{\varphi_n(\lambda_1)}\}_n$ and $\{k_{\varphi_{-n}(\lambda_2)}\}_n$ are bounded sequences for all $\lambda_1 \in G$ and $\lambda_2 \in H$.*

Proof. The proof is similar to the proof of Theorem 1. □

**Corollary 4.** *Suppose that $\mathscr{H}$ is automorphism invariant and $\varphi$ is an elliptic disc automorphism with an interior fixed point $a$.*

(a) *If the sets $E$ and $F$ or the sets $G$ and $H$, defined in Theorem 3, have limit points in $\mathbb{D}$, then $C^*_{w,\varphi}$ is supercyclic.*

(b) *If $|w(a)| < 1$ and there exists $0 < \delta < 1$ satisfying $|w(z)| \geqslant 1$ for all $|z| > 1 - \delta$, then $C^*_{w,\varphi}$ is supercyclic.*

P r o o f. (a) By an argument similar to the proof of Theorem 2, one can show that $\{k_{\varphi_n(\lambda)}\}_n$ is bounded for every $\lambda$ in $E$, $F$, $G$ and $H$. So the result follows from the previous theorem.

(b) By an argument similar to the proof of Corollary 2, one can show that the sets $G$ and $H$ in Theorem 3 have limit points in $\mathbb{D}$. So the result follows. □

**Proposition 2.** *Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $\lambda \in \mathbb{D}$ and let $\{\lambda_m\}_m$ be a sequence in $\mathbb{D}$ satisfying*

$$\varphi_m(\lambda_m) = \lambda, \quad m = 1, 2, 3, \ldots.$$

*Suppose, further, that the set $\{\lambda_m \colon m \geqslant 1\}$ has a limit point in $\mathbb{D}$. If $w$ is not identically zero, but $w(\lambda) = 0$, then $C^*_{w,\varphi}$ is supercyclic.*

P r o o f. Taking $f \in \mathscr{H}$ such that

$$\langle f, k_{\lambda_m}\rangle = f(\lambda_m) = 0, \quad m = 1, 2, 3, \ldots,$$

we have $f \equiv 0$. Therefore,

$$\bigvee\{k_{\lambda_m} \colon m = 1, 2, 3, \ldots\} = \mathscr{H}.$$

On the other hand,

$$C^{*n}_{w,\varphi} k_z = \prod_{j=0}^{n-1} \overline{w(\varphi_j(z))} k_{\varphi_n(z)} \quad \forall\, z \in \mathbb{D}.$$

But then for every positive integer $n$

$$C^{*n}_{w,\varphi} k_{\lambda_m} = 0, \quad m = 0, 1, \ldots n-1$$

where $\lambda_0 = \lambda$, because of the fact that $w(\lambda) = 0$. Thus, $\operatorname{cl}\Big(\bigcup_{n=1}^{\infty} \ker C^{*n}_{w,\varphi}\Big) = \mathscr{H}$.

Let $U$ and $V$ be two nonempty open subsets of $\mathscr{H}$. To see that the operator $C^*_{w,\varphi}$ is supercyclic, it is sufficient to show that there exist a natural number $n$ and a scalar $\alpha$ such that $\alpha C^{*n}_{w,\varphi}(U) \cap V \neq \emptyset$ (see [1], page 9). Choose $f \in U$ such that $C^{*n}_{w,\varphi}(f) = 0$ for some $n$. Since for every $g \in \mathscr{H}$,

$$C^n_{w,\varphi}(g) = \left(\prod_{k=0}^{n-1} w \circ \varphi_k\right) g \circ \varphi_n,$$

it is easily seen that $\ker C^n_{w,\varphi} = 0$. Thus, $C^{*n}_{w,\varphi}$ has dense range. It follows that there is a function $h$ in $\mathscr{H}$ such that $C^{*n}_{w,\varphi}(h) \in V$. Now pick $\alpha$ such that $\alpha^{-1}h + f$ belongs to $U$. Then

$$\alpha C^{*n}_{w,\varphi}\Big(\frac{1}{\alpha}h + f\Big) = C^{*n}_{w,\varphi}(h) \in V$$

and the result follows. □

**Corollary 5.** *Let $\mathscr{H}$ be automorphism invariant and let $\varphi$ be an elliptic automorphism, conjugate to a rotation through an irrational multiple of $\pi$, with an interior fixed point $a$. If $w$ is not identically zero, but $w(\lambda)=0$ for some $\lambda \neq a$, then $C^*_{w,\varphi}$ is supercyclic.*

P r o o f. Since similarity preserves supercyclicity, we may assume that $a=0$. Then $\varphi(z)=\mathrm{e}^{\mathrm{i}\pi\theta}z$ where $\theta$ is an irrational number. Put

$$\lambda_m=\mathrm{e}^{\mathrm{i}(-m)\pi\theta}\lambda, \quad m=1,2,3,\ldots.$$

Therefore, $\varphi_m(\lambda_m)=\lambda$. Moreover, since $\mathrm{cl}\{\mathrm{e}^{\mathrm{i}(-m)\pi\theta}\colon m\geqslant 0\}=\partial\mathbb{D}$, $\{\lambda_m\}_m$ has a limit point in $\mathbb{D}$; hence the proof is completed by Proposition 2. □

## 4. Cyclicity of the adjoint of weighted composition operators

In this section, we are going to give sufficient conditions for the cyclicity of the adjoint of a weighted composition operator.

**Theorem 4.** *Suppose that $w$ is not identically zero on $\mathbb{D}$.*

(a) *If $\varphi$ is not a constant function and has a Denjoy-Wolff point $a\in\mathbb{D}$, or*
(b) *if $\mathscr{H}$ is automorphism invariant and $\varphi$ is an elliptic automorphism, conjugate to a rotation through an irrational multiple of $\pi$,*

*then $C^*_{w,\varphi}$ is cyclic.*

P r o o f. First, suppose that (a) holds. Since $w$ is not identically zero the set of all $\lambda\in\mathbb{D}$ such that $w\circ\varphi_n(\lambda)\neq 0$ for all $n\geqslant 0$ is uncountable. On the other hand, if for some $\lambda\in\mathbb{D}$ the set $\{\varphi_n(\lambda)\colon n\geqslant 1\}$ is finite then there is a positive integer $N$ such that for $n\geqslant N$, $\varphi_n(\lambda)=a$. This coupled with the assumption that $\varphi$ is not constant shows easily the existence of $\lambda\in\mathbb{D}$ satisfying $w\circ\varphi_n(\lambda)\neq 0$ for all $n\geqslant 0$ and so the set $\{\varphi_n(\lambda)\colon n\geqslant 1\}$ is infinite. Now, suppose that $f\in\mathscr{H}$ and

$$\langle f, C^{*n}_{w,\varphi}k_\lambda\rangle=\left\langle f, \prod_{j=0}^{n-1}\overline{w\circ\varphi_j(\lambda)}k_{\varphi_n(\lambda)}\right\rangle=0$$

for all $n\in\mathbb{N}$. Then $f(\varphi_n(\lambda))=0$ for every positive integer $n$. Since the Denjoy-Wolff point of $\varphi$ lies in $\mathbb{D}$, the zeros of $f$ have a limit point in $\mathbb{D}$, which implies that $f\equiv 0$; consequently,

$$\bigvee\{C^{*n}_{w,\varphi}k_\lambda\colon n\in\mathbb{N}\cup\{0\}\}=\mathscr{H}.$$

(b) Now, let (b) hold. Without loss of generality we can assume that $\varphi(z) = \mathrm{e}^{\mathrm{i}\pi\theta}z$ where $\theta$ is not a rational number. Applying an argument similar to the one used in (a) shows that if $\langle f, C_{w,\varphi}^{*n}k_\lambda\rangle = 0$, then $f(\mathrm{e}^{in\pi\theta}\lambda) = 0$ for all $n \geqslant 1$. But $\mathrm{cl}\{\mathrm{e}^{in\pi\theta}\colon n \geqslant 1\} = \partial\mathbb{D}$, and so the zero set of $f$ must have limit points in $\mathbb{D}$; this in turn implies that $f \equiv 0$. $\square$

*References*


[1] *F. Bayart, E. Matheron*: Dynamics of Linear Operators. Cambridge Tracts in Mathematics 179. Cambridge University Press, Cambridge, 2009.
[2] *P. S. Bourdon, J. H. Shapiro*: Cyclic Phenomena for Composition Operators. Mem. Am. Math. Soc. 596 (1997).
[3] *P. S. Bourdon, J. H. Shapiro*: Hypercyclic operators that commute with the Bergman backward shift. Trans. Am. Math. Soc. *352* (2000), 5293–5316.
[4] *C. C. Cowen, B. D. MacCluer*: Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995.
[5] *N. S. Feldman, V. G. Miller, T. L. Miller*: Hypercyclic and supercyclic cohyponormal operators. Acta Sci. Math. *68* (2002), 303–328.
[6] *F. Forelli*: The isometries of $H^p$. Can. J. Math. *16* (1964), 721–728.
[7] *R. M. Gethner, J. H. Shapiro*: Universal vectors for operators on spaces of holomorphic functions. Proc. Am. Math. Soc. *100* (1987), 281–288.
[8] *G. Godefroy, J. H. Shapiro*: Operators with dense, invariant, cyclic vector manifolds. J. Funct. Anal. *98* (1991), 229–269.
[9] *C. Kitai*: Invariant closed sets for linear operators. Thesis. Univ. of Toronto, Toronto, 1982.
[10] *S. Rolewicz*: On orbits of elements. Stud. Math. *32* (1969), 17–22.
[11] *H. N. Salas*: Hypercyclic weighted shifts. Trans. Am. Math. Soc. *347* (1995), 993–1004.
[12] *H. N. Salas*: Supercyclicity and weighted shifts. Stud. Math. *135* (1999), 55–74.
[13] *J. H. Shapiro*: Composition Operators and Classical Function Theory. Tracts in Mathematics. Springer, New York, 1993.
[14] *B. Yousefi, S. Haghkhah*: Hypercyclicity of special operators on Hilbert function spaces. Czech. Math. J. *57* (2007), 1035–1041.
[15] *B. Yousefi, H. Rezaei*: Hypercyclic property of weighted composition operators. Proc. Am. Math. Soc. *135* (2007), 3263–3271.



*Authors' address*: Z. Kamali, B. Khani Robati, K. Hedayatian (corresponding author), Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran, e-mail: `zkamali@shirazu.ac.ir`, `bkhani@shirazu.ac.ir`, `hedayati@shirazu.ac.ir`.