Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 3, 623–639

Persistent URL: `http://dml.cz/dmlcz/141626`

Czechoslovak Mathematical Journal, 61 (136) (2011), 623–639

A CLASS OF TIGHT FRAMELET PACKETS

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(Received March 5, 2010)

Abstract. This paper obtains a class of tight framelet packets on $L^2(\mathbb{R}^d)$ from the extension principles and constructs the relationships between the basic framelet packets and the associated filters.

Keywords: wavelet frames, framelet packets, framelets, extension principles

MSC 2010: 42C15, 42C40

1. Introduction

Recently there has been an interest in the applications of redundant dyadic wavelet systems. Although many applications of wavelets use wavelet bases, other types of applications work better with redundant wavelet families, of which tight wavelet frames are the easiest to use. Tight wavelet frames are different from orthonormal wavelet bases in one important respect; they are (in general) redundant systems but with the same fundamental structure as wavelet systems. To mention only a few references on tight wavelet frames, the reader is referred to [11]–[16], [18] and [19]. The most common method to construct tight wavelet frames relies on the so-called extension principles. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called mother framelets. The construction of multiresolution-based wavelet frames has been extensively studied and well developed, see [2], [6], [11], [12], [14], [16], [18] or [19] .

However, wavelet frames provide poor frequency localization in applications. Take signal processing as example. The pyramid-structured framelet transform decomposes the signal into a set of frequency channels that have narrower bandwidths in

Supported by the Chinese National Programs for High Technology Research and Development (No. 2009AA12Z203) and Key Program of National Natural Science Foundation of China (No. 40930532).

the lower frequency region. The transform is suitable for a signal whose main information is concentrated in the low frequency regions. But it may not be suitable for information whose domain frequency channels are focused on the middle frequency region. To overcome this disadvantage, the concept of wavelet frames must be generalized to include a library of wavelet frames, called framelet packets or wavelet frame packets.

The original idea of framelet packets was introduced by Coifman, Meyer and Wickerhauser in [9] and [10], where orthonormal wavelet packets were considered; and then lots of results on wavelet packets emerged, see [1], [4], [5], [7], [8], [17] or [21]. As for the redundant wavelet packets, we refer the reader to [1] and [17] for the detailed discussion. In this paper, we first construct a class of tight framelet packets on $L^2(\mathbb{R}^d)$ from the unitary principle of Ron and Shen in [18], and then extend the results to the case of the oblique extension principle of Daubechies, Han, Ron and Shen in [12]. The results of [2, Section 7.2] show that, at a fixed dilation level, the spaces spanned by the basic framelet packets overlap and are not independent or even orthogonal anymore, which is different from the traditional wavelet frame packets given in [1] and [17].

2. Preliminaries

We begin by introducing some notation and a few results that we shall use. $\mathbb{Z}$ denotes the collection of all integers, $\mathbb{R}$ refers to the real line, and C represents the set of all complex numbers. Thoughout this paper, d and L are two positive integers. Translation by $h \in \mathbb{R}^d$, T_h is defined by $(T_h f)(x) = f(x-h)$ and dilation by $j \in \mathbb{Z}$, D^j is defined by $(D^j f)(x) = 2^{jd/2} f(2^j x)$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\mathbb{R}^d)$, i.e.,

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, \mathrm{d}x, \tag{2.1}$$

which can be extended to other f and g, e.g., when $fg \in L^1(\mathbb{R}^d)$. We normalize the Fourier transform as follows: $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-\mathrm{i}\xi\cdot x} \, \mathrm{d}x$. Given a function $\psi \in L^2(\mathbb{R}^d)$, we set $\psi_{j,k}\colon x \mapsto 2^{jd/2}\psi(2^j x - k)$. If the function ψ_i already carries an enumerative index, we write $\psi_{i,j,k}$ instead.

Let Ψ be a finite subset of $L^2(\mathbb{R}^d)$. The dyadic wavelet system generated by the mother wavelets Ψ is the family

$$X(\Psi) := \{\psi_{j,k} = D^j T_k \psi\colon\ \psi \in \Psi,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}. \tag{2.2}$$

The wavelet system $X(\Psi)$ is called a dyadic wavelet frame, or simply a wavelet frame, if there exist positive numbers $0 < A \leqslant B < \infty$ such that for all $f \in L^2(\mathbb{R}^d)$

$$A\|f\|^2 \leqslant \sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 \leqslant B\|f\|^2. \tag{2.3}$$

The largest A and smallest B satisfying (2.3) are the optimal wavelet frame bounds. We call $X(\Psi)$ a tight wavelet frame if $A = B$ and a Parseval wavelet frame if $A = B = 1$.

Next we shall introduce some results corresponding to FMRA. FMRA is just one way to construct wavelet frames via multiscale techniques. In this article, we shall follow a fundamental idea of Ron and Shen, which (in its first version) appeared in [18]. The idea is to modify the definition of the classical multiresolution analysis by requiring φ to satisfy a refinement equation instead of $\{T_k\varphi\}_{k\in\mathbb{Z}^d}$ being an orthonormal sequence.

The other conditions will be stated in the general setup as follows.

General setup: Let $\varphi := \psi_0 \in L^2(\mathbb{R}^d)$.

(1) There exists a $2\pi\mathbb{Z}^d$-periodic measurable function $H_0 \in L^\infty([-\pi, \pi)^d)$ such that

$$\hat{\psi}_0(2\xi) = H_0(\xi)\hat{\psi}_0(\xi). \tag{2.4}$$

(2) $\lim\limits_{\xi\to 0} \hat{\psi}_0(\xi) = 1$.

Furthermore, let $H_1, \ldots, H_L \in L^\infty([-\pi, \pi)^d)$ be $2\pi\mathbb{Z}^d$-periodic measurable functions. Define $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R}^d)$ by

$$\hat{\psi}_l(2\xi) = H_l(\xi)\hat{\psi}_0(\xi), \quad l = 1, 2, \ldots, L. \tag{2.5}$$

Then we call $\{\psi_l, H_l\}_{l=0}^L$ a general setup.

The spectrum $\sigma(\psi_0)$ associated to ψ_0 is defined by

$$\sigma(\psi_0) = \{\xi \in [-\pi, \pi]^d\colon\ \hat{\psi}_0(\xi + 2k\pi) \neq 0 \text{ for some } k \in \mathbb{Z}^d\}. \tag{2.6}$$

Daubechies *et al.* in [12] gave a complete characterization of the tight frames which can be obtained via the general setup. The following is the fundamental tool they gave to construct Parseval wavelet frames.

Theorem 2.1 The Oblique Extension Principle (OEP). *Let $\{\psi_l, H_l\}_{l=0}^L$ be as in the general setup. Assume that there exists a measurable and $2\pi\mathbb{Z}^d$-periodic function $\theta(\omega)$ which is strictly positive, essentially bounded, continuous at the origin, and $\theta(0) = 1$. If for almost all $\xi \in \sigma(\psi_0)$ and $\nu \in \{0, \pi\}^d$ satisfying $\xi + \nu \in \sigma(\psi_0)$, we have*

$$H_0(\xi)\overline{H_0(\xi+\nu)}\theta(2\xi) + \sum_{l=1}^{L} H_l(\xi)\overline{H_l(\xi+\nu)} = \begin{cases} \theta(\xi), & \text{if } \nu = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.7}$$

then the resulting wavelet system $X(\Psi)$ is a Parseval wavelet frame for $L^2(\mathbb{R}^d)$, where $\Psi = \{\psi_1, \ldots, \psi_L\}$.

Remark 2.2. If $\theta \equiv 1$, Theorem 2.1 reduces to the Unitary Extension Principle (UEP) of Ron and Shen in [18].

Remark 2.3. In many (most) interesting cases the spectrum $\sigma(\psi_0)$ is equal to $[-\pi, \pi]^d$. For example, if the integer translates of the scaling function ψ_0 are Riesz sequences, this is the case. So we suppose that $\sigma(\psi_0) = [-\pi, \pi]^d$ in this paper.

A wavelet system $X(\Psi)$ is said to be MRA-based if it is generated by the OEP or the UEP. The elements in $X(\Psi)$ are called framelets, and the elements in Ψ are called mother framelets. We call H_0 the refinement mask and the functions H_l, $l = 1, 2, \ldots, L$, wavelet masks.

3. Basic framelet packets

In this section, we shall show the construction of the basic framelet packets for $L^2(\mathbb{R}^d)$ via a frame multiresolution analysis generated by the UEP.

Let $\{\psi_l, H_l\}_{l=0}^L$ satisfy the conditions of the UEP and $w_0 := \varphi = \psi_0$. Define the functions $w_n(x)$, $n = 1, 2, \ldots$, associated with the refinable function φ recursively by

$$\hat{w}_{n(L+1)+l}(2\xi) = H_l(\xi)\hat{w}_n(\xi), \quad l = 0, 1, \ldots, L,\ n = 0, 1, 2, \ldots. \tag{3.1}$$

When $n = 0$ and $l = 1, 2, \ldots, L$, we obtain

$$\hat{w}_l(2\xi) = H_l(\xi)\hat{w}_0(\xi) = H_l(\xi)\hat{\psi}_0(\xi), \tag{3.2}$$

which shows that $w_l = \psi_l$, $l = 1, 2, \ldots, L$.

Theorem 3.1. *Let w_n, $n=0,1,\ldots$, be as in equation* (3.1). *Then, for all $n\geqslant 0$ and $j\in\mathbb{Z}$, we have*

$$(3.3)\qquad \sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_kw_n\rangle|^2=\sum_{l=0}^{L}\sum_{k\in\mathbb{Z}^d}|\langle f,D^{j-1}T_kw_{n(L+1)+l}\rangle|^2\quad\forall f\in L^2(\mathbb{R}^d).$$

Proof. Let

$$C_c(\mathbb{R}^d)=\{f\in L^2(\mathbb{R}^d)\colon\ \hat f \text{ is continuous and has compact support}\}.$$

Then $\forall f\in C_c(\mathbb{R}^d)$,

$$\begin{aligned}\langle f,D^{j-1}T_kw_{n(L+1)+l}\rangle&=\frac{1}{(2\pi)^d}2^{-(j-1)d/2}\int_{\mathbb{R}^d}\hat f(\xi)\overline{\hat w_{n(L+1)+l}(2^{-(j-1)}\xi)\mathrm{e}^{2^{-(j-1)}\mathrm{i}k\cdot\xi}}\,\mathrm{d}\xi\\&=\frac{1}{(2\pi)^d}2^{-(j-1)d/2}\int_{[-2^{j-1}\pi,2^{j-1}\pi]^d}\sum_{\alpha\in\mathbb{Z}^d}\hat f(\xi+2^j\pi\alpha)\\&\qquad\times\overline{\hat w_{n(L+1)+l}(2^{-(j-1)}\xi+2\pi\alpha)\mathrm{e}^{2^{-(j-1)}\mathrm{i}k\cdot\xi}}\,\mathrm{d}\xi.\end{aligned}$$

The exchange of the integral and the summation is legitimate in the above formula. Since $\{(2^j\pi)^{-d/2}\mathrm{e}^{2^{-(j-1)}\mathrm{i}k\cdot\xi}\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2([-2^{j-1}\pi,2^{j-1}\pi]^d)$, we have

$$\begin{aligned}I&:=\sum_{l=0}^{L}\sum_{k\in\mathbb{Z}^d}|\langle f,D^{j-1}T_kw_{n(L+1)+l}\rangle|^2\\&=\frac{1}{(2\pi)^d}\sum_{l=0}^{L}\int_{[-2^{j-1}\pi,2^{j-1}\pi]^d}\Bigl|\sum_{\alpha\in\mathbb{Z}^d}\hat f(\xi+2^j\pi\alpha)\overline{\hat w_{n(L+1)+l}(2^{-(j-1)}\xi+2\pi\alpha)}\Bigr|^2\,\mathrm{d}\xi.\end{aligned}$$

By equation (3.1) we obtain

$$\begin{aligned}I&=\frac{1}{(2\pi)^d}\sum_{l=0}^{L}\int_{[-2^{j-1}\pi,2^{j-1}\pi]^d}\Bigl|\sum_{\alpha\in\mathbb{Z}^d}\hat f(\xi+2^j\pi\alpha)\overline{\hat w_n(2^{-j}\xi+\pi\alpha)H_l(2^{-j}\xi+\pi\alpha)}\Bigr|^2\,\mathrm{d}\xi\\&=\frac{1}{(2\pi)^d}\sum_{l=0}^{L}\int_{[-2^{j-1}\pi,2^{j-1}\pi]^d}\Bigl|\sum_{\alpha'\in\mathbb{Z}^d}\sum_{\nu\in\{0,1\}^d}\hat f(\xi+2^j\pi(2\alpha'+\nu))\\&\qquad\qquad\times\overline{\hat w_n(2^{-j}\xi+\pi(2\alpha'+\nu))H_l(2^{-j}\xi+\pi(2\alpha'+\nu))}\Bigr|^2\,\mathrm{d}\xi\end{aligned}$$

$$= \frac{1}{(2\pi)^d} \sum_{l=0}^{L} \int_{[-2^{j-1}\pi, 2^{j-1}\pi]^d} \Bigg| \sum_{\alpha' \in \mathbb{Z}^d} \sum_{\nu \in \{0,1\}^d} \hat{f}(\xi + 2^j\pi(2\alpha' + \nu)) \times \overline{\hat{w}_n(2^{-j}\xi + \pi(2\alpha' + \nu)) H_l(2^{-j}\xi + \pi\nu)} \Bigg|^2 \,\mathrm{d}\xi$$

$$:= \frac{1}{(2\pi)^d} \sum_{l=0}^{L} \int_{[-2^{j-1}\pi, 2^{j-1}\pi]^d} \Bigg| \sum_{\nu \in \{0,1\}^d} P^j_{f,w_n}(\xi, \nu) \overline{H_l(2^{-j}\xi + \pi\nu)} \Bigg|^2 \,\mathrm{d}\xi,$$

where

$$P^j_{f,w_n}(\xi, \nu) = \sum_{\alpha' \in \mathbb{Z}^d} \hat{f}(\xi + 2^j\pi(2\alpha' + \nu)) \overline{\hat{w}_n(2^{-j}\xi + \pi(2\alpha' + \nu))}.$$

Recall that equation (2.7), when $\theta \equiv 1$, can be written as

$$\sum_{l=0}^{L} H_l(\xi)\overline{H_l(\xi + \nu\pi)} = \begin{cases} 1, & \text{if } \ \nu = 0, \\ 0, & \text{otherwise}, \end{cases}$$

for $\xi \in [-\pi, \pi]^d$ and $\nu \in \{0,1\}^d$. Hence

$$\begin{aligned} I &= \frac{1}{(2\pi)^d} \sum_{l=0}^{L} \int_{[-2^{j-1}\pi, 2^{j-1}\pi]^d} \sum_{\nu \in \{0,1\}^d} P^j_{f,w_n}(\xi, \nu) \overline{H_l(2^{-j}\xi + \pi\nu)} \\ &\quad \times \sum_{\nu' \in \{0,1\}^d} \overline{P^j_{f,w_n}(\xi, \nu')} H_l(2^{-j}\xi + \pi\nu') \,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d} \int_{[-2^{j-1}\pi, 2^{j-1}\pi]^d} \sum_{\nu, \nu' \in \{0,1\}^d} P^j_{f,w_n}(\xi, \nu) \overline{P^j_{f,w_n}(\xi, \nu')} \\ &\quad \times \sum_{l=0}^{L} H_l(2^{-j}\xi + \pi\nu') \overline{H_l(2^{-j}\xi + \pi\nu)} \,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d} \int_{[-2^{j-1}\pi, 2^{j-1}\pi]^d} \sum_{\nu \in \{0,1\}^d} |P^j_{f,w_n}(\xi, \nu)|^2 \,\mathrm{d}\xi. \end{aligned}$$

Noticing that $P^j_{f,w_n}(\xi, \nu)$ are $2^{j+1}\pi\mathbb{Z}^d$-periodic functions and

$$\bigcup_{\nu \in \{0,1\}^d} ([-2^{j-1}\pi, 2^{j-1}\pi]^d + 2^j\pi\nu) = [-2^{j-1}\pi, 3 \times 2^{j-1}\pi]^d,$$

we have

$$
\begin{aligned}
I &= \frac{1}{(2\pi)^d} \sum_{\nu\in\{0,1\}^d} \int_{[-2^{j-1}\pi,2^{j-1}\pi]^d} \Bigg| \sum_{\alpha'\in\mathbb{Z}^d} \hat{f}(\xi+2^j\pi(2\alpha'+\nu)) \\
&\qquad\qquad \times \overline{\hat{w}_n(2^{-j}\xi+\pi(2\alpha'+\nu))}\Bigg|^2 \,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \sum_{\nu\in\{0,1\}^d} \int_{[-2^{j-1}\pi,2^{j-1}\pi]^d+2^j\pi\nu} \Bigg| \sum_{\alpha'\in\mathbb{Z}^d} \hat{f}(\xi+2^{j+1}\pi\alpha')\overline{\hat{w}_n(2^{-j}\xi+2\pi\alpha')}\Bigg|^2 \,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \int_{[-2^{j-1}\pi,3\times 2^{j-1}\pi]^d} \Bigg| \sum_{\alpha'\in\mathbb{Z}^d} \hat{f}(\xi+2^{j+1}\pi\alpha')\overline{\hat{w}_n(2^{-j}\xi+2\pi\alpha')}\Bigg|^2 \,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d} \int_{[-2^{j}\pi,2^{j}\pi]^d} \Bigg| \sum_{\alpha'\in\mathbb{Z}^d} \hat{f}(\xi+2^{j+1}\pi\alpha')\overline{\hat{w}_n(2^{-j}\xi+2\pi\alpha')}\Bigg|^2 \,\mathrm{d}\xi.
\end{aligned}
$$

With the same method we can obtain

$$
\sum_{k\in\mathbb{Z}^d} |\langle f, D^jT_kw_n\rangle|^2 = \frac{1}{(2\pi)^d} \int_{[-2^{j}\pi,2^{j}\pi]^d} \Bigg| \sum_{\alpha'\in\mathbb{Z}^d} \hat{f}(\xi+2^{j+1}\pi\alpha')\overline{\hat{w}_n(2^{-j}\xi+2\pi\alpha')}\Bigg|^2 \,\mathrm{d}\xi.
$$

So equation (3.3) holds for all $f\in C_c(\mathbb{R}^d)$. The proof is complete since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. $\square$

Define a family of subspaces of $L^2(\mathbb{R}^d)$ by

$$
\mathbf{U}_j^n := \overline{\operatorname{span}}\left\{D^jT_kw_n\colon\ k\in\mathbb{Z}^d\right\} \tag{3.4}
$$

for $j\in\mathbb{Z}$ and $n=0,1,2,\ldots$. We have the following results on the subspaces $\mathbf{U}_j^n$.

Theorem 3.2. *For $n=0,1,2,\ldots$, we have*

$$
\mathbf{U}_{j+1}^n = \mathbf{U}_j^{n(L+1)} + \ldots + \mathbf{U}_j^{n(L+1)+L}, \quad j\in\mathbb{Z}. \tag{3.5}
$$

Proof. Recall that

$$
\hat{w}_{n(L+1)+l}(2\xi) = H_l(\xi)\hat{w}_n(\xi), \quad l=0,1,\ldots,L,\ n=0,1,\ldots.
$$

So we conclude that, for all $n=0,1,2,\ldots$ and $j\in\mathbb{Z}$,

$$
\mathbf{U}_j^{n(L+1)+l} \subseteq \mathbf{U}_{j+1}^n, \quad l=0,1,\ldots,L,
$$

which means that

$$\Delta_j := \mathbf{U}_j^{n(L+1)} + \ldots + \mathbf{U}_j^{n(L+1)+L} \subseteq \mathbf{U}_{j+1}^n.$$

To show equation (3.5), we argue by contradiction. If there exists $0 \neq f \in \mathbf{U}_{j+1}^n$ such that $f \in \Delta_j^{\perp}$, where $\Delta_j^{\perp}$ denotes the orthogonal complement of Δ_j in $\mathbf{U}_{j+1}^n$, then

$$\sum_{l=0}^{L} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k w_{n(L+1)+l} \rangle|^2 = 0.$$

However, $\sum\limits_{k \in \mathbb{Z}^d} |\langle f, D^{j+1} T_k w_n \rangle|^2 \neq 0$, which contradicts equation (3.3). So we complete the proof. $\square$

Using equation (3.5) repeatedly, we have the following results.

Theorem 3.3. *For each $j = 1, 2, \ldots$, we have*

$$V_j = \overline{\operatorname{span}}\{D^j T_k \varphi \colon k \in \mathbb{Z}^d\} = \overline{\operatorname{span}}\{D^j T_k w_0 \colon k \in \mathbb{Z}^d\} = \mathbf{U}_j^0 \tag{3.6}$$

and

$$\begin{cases} \mathbf{U}_j^0 &= \mathbf{U}_{j-1}^0 + \mathbf{U}_{j-1}^1 + \ldots + \mathbf{U}_{j-1}^L, \\ \mathbf{U}_j^0 &= \mathbf{U}_{j-2}^0 + \mathbf{U}_{j-2}^1 + \ldots + \mathbf{U}_{j-2}^{(L+1)^2-1}, \\ &\vdots \\ \mathbf{U}_j^0 &= \sum\limits_{l=0}^{(L+1)^k-1} \mathbf{U}_{j-k}^l, \\ &\vdots \\ \mathbf{U}_j^0 &= \sum\limits_{l=0}^{(L+1)^j-1} \mathbf{U}_0^l, \end{cases} \tag{3.7}$$

where $\mathbf{U}_j^n$ are defined as in equation (3.4).

Theorem 3.4. *Let $\mathbf{U}_j^n$ be as in equation* (3.4). *Then*

$$L^2(\mathbb{R}^d) = \sum_{l=0}^{\infty} \mathbf{U}_0^l, \tag{3.8}$$

and the collection $\{T_k w_n \colon k \in \mathbb{Z}^d;\ n = 0, 1, \ldots\}$ generates a Parseval frame for $L^2(\mathbb{R}^d)$.

Definition 3.5. The functions w_n, $n = 0, 1, 2, \ldots$, are called the basic framelet packets associated with the refinable function φ.

In order to prove Theorem 3.4 we need some lemmas as follows.

Lemma 3.6 (see [3]). *Let $\varphi \in L^2(\mathbb{R}^d)$. For $j \in \mathbb{Z}$, define*

$$V_j = \overline{\operatorname{span}}\{D^j T_k \varphi \colon\ k \in \mathbb{Z}^d\}.$$

If $V_j \subseteq V_{j+1}$, $j \in \mathbb{Z}$, and $|\hat{\varphi}| > 0$ on a neighborhood of 0, then

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d) \quad \text{and} \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

Lemma 3.7. *Assume that $\varphi \in L^2(\mathbb{R}^d)$ satisfies $\lim\limits_{\xi\to 0} \hat{\varphi}(\xi) = 1$. If $f \in C_c(\mathbb{R}^d)$, then for any $\varepsilon > 0$ there exists $J \in \mathbb{Z}$ such that*

$$(1-\varepsilon)\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k \varphi\rangle|^2 \leqslant (1+\varepsilon)\|f\|^2 \quad \text{for all} \ \ j \geqslant J. \tag{3.9}$$

Indeed, the proof of Lemma 3.7 is similar to that of its analogue in the case of $d = 1$ (see [6, Lemma 14.2.2]). However, for the readers' convenience, we give the argument as follows:

P r o o f. Let $j \in \mathbb{Z}$ and $f \in C_c(\mathbb{R}^d)$. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k \varphi\rangle|^2 &= \frac{1}{(2\pi)^{2d}} \sum_{k\in\mathbb{Z}^d} \left| 2^{jd/2} \int_{\mathbb{R}^d} \hat{f}(2^j\xi)\overline{\hat{\varphi}(\xi)} \mathrm{e}^{\mathrm{i}k\cdot\xi}\,\mathrm{d}\xi \right|^2 \\ &= \frac{1}{(2\pi)^{2d}} \sum_{k\in\mathbb{Z}^d} \left| 2^{jd/2} \int_{[-\pi,\pi]^d} \sum_{\alpha\in\mathbb{Z}^d} \hat{f}(2^j(\xi+2\pi\alpha))\overline{\hat{\varphi}(\xi+2\pi\alpha)} \mathrm{e}^{\mathrm{i}k\cdot\xi}\,\mathrm{d}\xi \right|^2.\end{aligned}$$

It is easy to know that $[f,\varphi](\xi) := \sum\limits_{\alpha\in\mathbb{Z}^d} \hat{f}(2^j(\xi+2\pi\alpha))\overline{\hat{\varphi}(\xi+2\pi\alpha)}$ is well defined. When we only consider $\xi \in [-\pi,\pi]^d$, $[f,\varphi]$ can be bounded by a finite linear combination of translates of $\overline{\hat{\varphi}}$, so $[f,\varphi] \in L^2([-\pi,\pi]^d)$. Recall that $\{(2\pi)^{-d/2}\mathrm{e}^{\mathrm{i}k\cdot\omega}\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2([-\pi,\pi]^d)$. So

$$\sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k \varphi\rangle|^2 = 2^{jd} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left| \sum_{\alpha\in\mathbb{Z}^d} \hat{f}(2^j(\xi+2\pi\alpha))\overline{\hat{\varphi}(\xi+2\pi\alpha)} \right|^2 \mathrm{d}\xi.$$

Now let $\varepsilon > 0$ be given. By assumption, we can choose $b \in (0,\pi)$ such that $1-\varepsilon \leqslant |\hat{\varphi}(\omega)|^2 \leqslant 1+\varepsilon$ whenever ξ with $\|\xi\|_{\mathbb{R}^d} \leqslant b$. By taking $J \in \mathbb{Z}$ such that $D^j\hat{f}$ has support in $B_b := \{\xi\colon \|\xi\|_{\mathbb{R}^d} \leqslant b\}$ for $j \geqslant J$, we obtain that for all $j \geqslant J$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k\varphi\rangle|^2 &= 2^{jd}\frac{1}{(2\pi)^d}\int_{B_b} |\hat{f}(2^j\xi)\overline{\hat{\varphi}(\xi)}|^2 \,\mathrm{d}\xi \\ &\leqslant (1+\varepsilon)\frac{1}{(2\pi)^d}\int_{B_b} |D^j\hat{f}(\xi)|^2\,\mathrm{d}\xi \\ &= (1+\varepsilon)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |D^j\hat{f}(\xi)|^2\,\mathrm{d}\xi \\ &= (1+\varepsilon)\|f\|^2.\end{aligned}$$

On the other hand, we have

$$(1-\varepsilon)\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k\varphi\rangle|^2.$$

So equation (3.9) holds for all $f \in C_c(\mathbb{R}^d)$. □

Proof of Theorem 3.4. We first consider equation (3.8). By Lemma 3.6 and Theorem 3.3 we get

$$L^2(\mathbb{R}^d) = \overline{\bigcup_{j\in\mathbb{Z}} V_j} = \lim_{j\to\infty} V_j = \lim_{j\to\infty}\sum_{l=0}^{(L+1)^j-1} \mathbf{U}_0^l = \sum_{l=0}^{\infty}\mathbf{U}_0^l.$$

Let $\varepsilon > 0$ and $f \in C_c(\mathbb{R}^d)$. By Lemma 3.7, we can choose $J > 0$ such that for all $j > J$,

$$(1-\varepsilon)\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k\varphi\rangle|^2 \leqslant (1+\varepsilon)\|f\|^2.$$

For any $j \geqslant 0$, Theorem 3.1 shows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k\varphi\rangle|^2 = \sum_{k\in\mathbb{Z}^d} |\langle f, D^j T_k w_0\rangle|^2 &= \sum_{l=0}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, D^{j-1} T_k w_l\rangle|^2 \\ &= \sum_{l=0}^{L}\sum_{\mu=0}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, D^{j-2} T_k w_{l(L+1)+\mu}\rangle|^2 \\ &= \sum_{n=0}^{(L+1)^2-1}\sum_{k\in\mathbb{Z}^d} |\langle f, D^{j-2} T_k w_n\rangle|^2 = \ldots \\ &= \sum_{n=0}^{(L+1)^j-1}\sum_{k\in\mathbb{Z}^d} |\langle f, T_k w_n\rangle|^2.\end{aligned}$$

It follows that for all $j > J$

$$(1-\varepsilon)\|f\|^2 \leqslant \sum_{n=0}^{(L+1)^j-1} \sum_{k\in\mathbb{Z}^d} |\langle f, T_k w_n\rangle|^2 \leqslant (1+\varepsilon)\|f\|^2.$$

Letting $j \to \infty$,

$$(1-\varepsilon)\|f\|^2 \leqslant \sum_{n=0}^{\infty} \sum_{k\in\mathbb{Z}^d} |\langle f, T_k w_n\rangle|^2 \leqslant (1+\varepsilon)\|f\|^2.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\sum_{n=0}^{\infty} \sum_{k\in\mathbb{Z}^d} |\langle f, T_k w_n\rangle|^2 = \|f\|^2.$$

The proof is complete since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. □

For $w_0 = \varphi$ with $\lim\limits_{\xi\to 0} \hat{w}_0(\xi) = 1$, we have

$$\begin{aligned}
\hat{w}_0(\xi) &= H_0\Big(\frac{\xi}{2}\Big)\hat{w}_0\Big(\frac{\xi}{2}\Big) \\
&= H_0\Big(\frac{\xi}{2}\Big)H_0\Big(\frac{\xi}{2^2}\Big)\hat{w}_0\Big(\frac{\xi}{2^2}\Big) = \dots \\
&= \hat{w}_0\Big(\frac{\xi}{2^n}\Big)\prod_{j=1}^{n} H_0\Big(\frac{\xi}{2^j}\Big).
\end{aligned}$$

If the finite product $\prod\limits_{j=1}^{n} H_0(\xi/2^j)$ is convergent as $n \to \infty$ for each $\xi \in \mathbb{R}^d$, then

$$\hat{w}_0(\xi) = \prod_{j=1}^{\infty} H_0\Big(\frac{\xi}{2^j}\Big). \tag{3.10}$$

To generalize the result to the basic framelet packets we need to consider the unique "a-adic expansion" (i.e., expansion in the base a) for an integer $n \geqslant 1$:

$$n = \sum_{j=1}^{k} \varepsilon_j a^{j-1}, \tag{3.11}$$

where $\varepsilon_j \in \{0, 1, 2, \dots, a-1\}$ for all $j = 1, 2, \dots, k$ and $\varepsilon_k \neq 0$. Let $a = L+1$. Suppose that $\varepsilon_j = 0$ if $j \geqslant k+1$. Then we have

$$n = \sum_{j=1}^{\infty} \varepsilon_j (L+1)^{j-1} \tag{3.12}$$

for all $n \geqslant 0$, where $\varepsilon_j \in \{0, 1, 2, \dots, L\}$.

Theorem 3.8. *Let n be a non-negative integer with "$(L+1)$-adic expansion" given by* (3.12). *Then the Fourier transform of the basic framelet packets given by* (3.1) *satisfies*

$$\hat{w}_n(\xi) = \prod_{j=1}^{\infty} H_{\varepsilon_j}(2^{-j}\xi) = \Big\{ \prod_{j=1}^{k} H_{\varepsilon_j}(2^{-j}\xi) \Big\} \hat{\varphi}(2^{-k}\xi) \tag{3.13}$$

if $H_0(\xi)$ is a continuously differentiable function.

P r o o f. The infinite product

$$\prod_{j=1}^{\infty} H_{\varepsilon_j}(2^{-j}\xi)$$

clearly converges for each $\xi \in \mathbb{R}^d$. In fact, from the definition of the basic framelet packets we know that when k is sufficiently large, $H_{\varepsilon_k} = H_0$. We also have, from the general setup, $H_0(0) = 1$. Let $\Pi_k(\xi) = \prod\limits_{j=1}^{k} H_{\varepsilon_j}(2^{-j}\xi)$. Note that equation (2.7) implies that $|H_{\varepsilon_j}(\xi)| \leqslant 1$ for all ξ, which shows that $|\Pi_k(\xi)| \leqslant 1$ for all $k \geqslant 1$. Consequently,

$$\begin{aligned} |\Pi_{k+1}(\xi) - \Pi_k(\xi)| &= |\Pi_k(\xi)(H_0(2^{-k-1}\xi) - 1)| \\ &\leqslant |H_0(2^{-k-1}\xi) - H_0(0)| \leqslant \|H_0'\|_{L^\infty(\mathbb{R}^d)} 2^{-(k+1)} |\xi|. \end{aligned}$$

Hence,

$$\begin{aligned} |\Pi_{k+m}(\xi) - \Pi_k(\xi)| &\leqslant \|H_0'\|_{L^\infty(\mathbb{R}^d)} |\xi| (2^{-(k+1)} + \ldots + 2^{-(k+m)}) \\ &\leqslant \|H_0'\|_{L^\infty(\mathbb{R}^d)} |\xi| 2^{-k} \end{aligned}$$

for all $m \in \mathbb{N}$ and all sufficiently large $k \in \mathbb{N}$. This shows that the sequence $\{\Pi_k(\xi) \colon k \in \mathbb{N}\}$ not only converges but it does so uniformly on bounded sets.

We now turn to the equality (3.13) and proceed by induction on n.

Let $n \in \{0,\ 1, \ldots, L\}$. The "$(L+1)$-adic expansion" of n is $\varepsilon_1 = n$, $\varepsilon_j = 0$, $j \geqslant 2$. By equation (3.1) we have

$$\begin{aligned} \hat{w}_n(\xi) &= H_n\Big(\frac{\xi}{2}\Big) \hat{w}_0\Big(\frac{\xi}{2}\Big) = H_n\Big(\frac{\xi}{2}\Big) \prod_{j=1}^{\infty} H_0\Big(\frac{\xi}{2^{j+1}}\Big) \\ &= H_n\Big(\frac{\xi}{2}\Big) \prod_{j=2}^{\infty} H_0\Big(\frac{\xi}{2^j}\Big) = \prod_{j=1}^{\infty} H_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big). \end{aligned}$$

Hence equation (3.13) holds for $n \in \{0, 1, \ldots, L\}$.

Suppose equation (3.13) is true for every non-negative integer $m < n$, where $n \geqslant L+1$.

For some $\mu \in \{0,1,\ldots,L\}$, n can be written as $n = t(L+1)+\mu$. It is easy to know that $t < n$. Suppose the "$(L+1)$-adic expansion" of t is

$$t = \sum_{j=1}^{\infty} \varepsilon_j (L+1)^{j-1}.$$

By assumption we have

$$\hat{w}_t(\xi) = \prod_{j=1}^{\infty} H_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big).$$

On the one hand,

$$\begin{aligned} n = t(L+1)+\mu &= \sum_{j=1}^{\infty} \varepsilon_j (L+1)^j + \mu \\ &= \sum_{j=2}^{\infty} \varepsilon_{j-1}(L+1)^{j-1} + \mu := \sum_{j=1}^{\infty} \varepsilon'_j (L+1)^{j-1}, \end{aligned}$$

where $\varepsilon'_1 = \mu$, $\varepsilon'_j = \varepsilon_{j-1}$, $j \geqslant 2$. On the other hand,

$$\begin{aligned} \hat{w}_n(\xi) = \hat{w}_{t(L+1)+\mu}(\xi) &= H_\mu\Big(\frac{\xi}{2}\Big)\hat{w}_t\Big(\frac{\xi}{2}\Big) \\ &= H_\mu\Big(\frac{\xi}{2}\Big)\prod_{j=1}^{\infty} H_{\varepsilon_j}\Big(\frac{\xi}{2^{j+1}}\Big) = H_\mu\Big(\frac{\xi}{2}\Big)\prod_{j=2}^{\infty} H_{\varepsilon_{j-1}}\Big(\frac{\xi}{2^j}\Big) \\ &= \prod_{j=1}^{\infty} H_{\varepsilon'_j}\Big(\frac{\xi}{2^j}\Big). \end{aligned}$$

This completes the proof. □

The idea of the basic framelet packets enables us to construct lots of tight frames for $L^2(\mathbb{R}^d)$ by replacing some mother framelets.

Theorem 3.9. *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$. Suppose $X(\Psi)$ is a Parseval wavelet frame generated by the UEP, and $H_0, H_1, \ldots, H_L$ are the refinement mask and wavelet masks, respectively. Given $m_0 \in \{1,2,\ldots,L\}$, define functions $\psi^\mu_{m_0}$, $\mu = 0,1,\ldots,L$, by*

$$\hat{\psi}^\mu_{m_0}(2\xi) = H_\mu(\xi)\hat{\psi}_{m_0}(\xi). \tag{3.14}$$

Then the collection $\{D^jT_k\psi_m, D^{j-1}T_k\psi^{\mu}_{m_0}\colon m=1,2,\ldots,L$ *and* $m\neq m_0,\ \mu=0,1,2,\ldots,L;\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$ *generates a new Parseval frame for* $L^2(\mathbb{R}^d)$.

Proof. Let $w_n=\psi_{m_0}$. It is obvious that $w_{n(L+1)+\mu}=\psi^{\mu}_{m_0}$. For $j\in\mathbb{Z}$, by Theorem 3.1 we know that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_k\psi_{m_0}\rangle|^2&=\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_kw_n\rangle|^2\\&=\sum_{\mu=0}^{L}\sum_{k\in\mathbb{Z}^d}|\langle f,D^{j-1}T_kw_{n(L+1)+\mu}\rangle|^2\\&=\sum_{\mu=0}^{L}\sum_{k\in\mathbb{Z}^d}|\langle f,D^{j-1}T_k\psi^{\mu}_{m_0}\rangle|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{\substack{m=1\\m\neq m_0}}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_k\psi_m\rangle|^2+\sum_{\mu=0}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,D^{j-1}T_k\psi^{\mu}_{m_0}\rangle|^2\\&\quad=\sum_{\substack{m=1\\m\neq m_0}}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_k\psi_m\rangle|^2+\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_k\psi_{m_0}\rangle|^2\\&\quad=\sum_{m=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,D^jT_k\psi_m\rangle|^2=\|f\|^2.\end{aligned}$$

This completes the proof. □

For illustration and completeness, we briefly and almost verbatim recall a simple example of an application of the UEP, see [18]. For a higher dimensional construction, see [12].

Let m be a positive integer, and define the 2π-periodic function $H_0(\xi)=\cos^{2m}(\frac12\xi)$. The polynomial H_0 is the refinement mask of the centered B-spline $\psi_0=\varphi:=B_{2m}$ of order $2m$ defined by its Fourier transform as follows:

$$\hat{\psi}_0(\xi)=\frac{\sin^{2m}(\xi/2)}{(\xi/2)^{2m}}. \tag{3.15}$$

Note that $\lim\limits_{\xi\to0}\hat{\psi}_0(\xi)=1$. Define $2m$ (2π-periodic) wavelet masks H_l, $l=1,2,\ldots,2m$, by

$$H_l(\xi)=-\mathrm{i}^l\sqrt{\begin{bmatrix}2m\\l\end{bmatrix}}\sin^l\Big(\frac{\xi}{2}\Big)\cos^{2m-l}\Big(\frac{\xi}{2}\Big). \tag{3.16}$$

Observe that firstly,

$$\sum_{l=0}^{2m} |H_l(\xi)|^2 = \Bigl(\cos^2\Bigl(\frac{\xi}{2}\Bigr) + \sin^2\Bigl(\frac{\xi}{2}\Bigr)\Bigr)^{2m} = 1,$$

and that secondly,

$$\sum_{l=0}^{2m} H_l(\xi)\overline{H_l(\xi+\pi)} = \Bigl(\sin\Bigl(\frac{\xi}{2}\Bigr)\cos\Bigl(\frac{\xi}{2}\Bigr)\Bigr)^{2m}(1-1)^{2m} = 0.$$

Therefore, the $2m$ wavelets defined by

$$\hat{\psi}_l(\xi) = -\mathrm{i}^l\sqrt{\begin{bmatrix}2m\\ l\end{bmatrix}}\frac{\cos^{2m-l}(\xi/4)\sin^{2m+l}(\xi/4)}{(\xi/4)^{2m}}, \quad l = 1,2,\ldots,2m, \tag{3.17}$$

generate a tight Parseval frame for $L^2(\mathbb{R})$.

Example 3.10. Let $m = 1$. Here, we obtain the refinable function $\varphi = \psi^0 = B_2$, and mother framelets ψ_1, ψ_2.

Let $w_0 = \varphi$. Define functions w_n, $n \geqslant 0$, as follows:

$$\hat{w}_{3n}(2\xi) = H_0(\xi)\hat{w}_n(\xi),$$
$$\hat{w}_{3n+1}(2\xi) = H_1(\xi)\hat{w}_n(\xi)$$

and

$$\hat{w}_{3n+2}(2\xi) = H_2(\xi)\hat{w}_n(\xi).$$

Then the collection $\{w_n\}_{n\geqslant 0}$ is a family of the basic framelet packets for $L^2(\mathbb{R})$.

By Theorem 3.9 we know that each of the collections

$$\{D^jT_k\psi_1, D^{j-1}T_kw_n\colon\ n = 6,7,8;\ j,k \in \mathbb{Z}\},$$
$$\{D^{j-1}T_kw_n, D^jT_k\psi_2\colon\ n = 3,4,5;\ j,k \in \mathbb{Z}\}$$

and

$$\{D^{j-1}T_kw_n\colon\ n = 3,4,\ldots,8;\ j,k \in \mathbb{Z}\},$$

generates a Parseval frame for $L^2(\mathbb{R})$.

4. Conclusion

Let $\Psi = \{\psi_l \colon l = 1, 2, \ldots, L\}$. Suppose the tight wavelet frame $X(\Psi)$ is generated by the OEP with the refinable function φ. Suppose H_0 and H_l, $l = 1, 2, \ldots, L$, are the refinable mask and wavelet masks, respectively. Then we can verify that

$$\widetilde{H}_0(\xi) = \sqrt{\frac{\theta(2\xi)}{\theta(\xi)}} H_0(\xi), \quad \widetilde{H}_l(\xi) = \sqrt{\frac{1}{\theta(\xi)}} H_l(\xi),\ l = 1, 2, \ldots, L,$$

satisfy the conditions of the UEP with the refinable function $\tilde{\varphi}$ defined by $\widehat{\tilde{\varphi}}(\xi) = \sqrt{\theta(\xi)}\hat{\varphi}(\xi)$. The resulting mother framelets $\tilde{\psi}_l$ via the UEP are equal to ψ_l, $l = 1, 2, \ldots, L$. So the results in Section 3 still hold for the refinable function $\tilde{\varphi}$, mother wavelets ψ^l, $l = 1, 2, \ldots, L$, the refinable mask $\widetilde{H}_0$ and wavelet masks $\widetilde{H}_l$, $l = 1, 2, \ldots, L$.

At the end of the paper, it is worthy to point out that as a generalization of tight wavelet frames, tight framelet packets generated in this paper preserve many nice constructive properties of tight wavelet frames. For example, when $d = 1$, the methods of construction for symmetric tight wavelet frames via the UEP given in [20] is still works for tight framelet packets with a slight modification by requiring the basic framelet packets instead of the refinable function.

Acknowledgement. The authors would like to express their gratitude to the referee for his (or her) valuable comments and suggestions that led to a significant improvement of the manuscript.

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