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ON LEHMER'S PROBLEM AND DEDEKIND SUMS

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Abstract. Let p be an odd prime and c a fixed integer with $(c,p)=1$. For each integer a with $1\leqslant a\leqslant p-1$, it is clear that there exists one and only one b with $0\leqslant b\leqslant p-1$ such that $ab\equiv c \pmod p$. Let $N(c,p)$ denote the number of all solutions of the congruence equation $ab\equiv c \pmod p$ for $1\leqslant a, b\leqslant p-1$ in which a and $\overline{b}$ are of opposite parity, where $\overline{b}$ is defined by the congruence equation $b\overline{b}\equiv 1 \pmod p$. The main purpose of this paper is to use the properties of Dedekind sums and the mean value theorem for Dirichlet L-functions to study the hybrid mean value problem involving $N(c,p)-\frac12\varphi(p)$ and the Dedekind sums $S(c,p)$, and to establish a sharp asymptotic formula for it.

Keywords: Lehmer's problem, error term, Dedekind sums, hybrid mean value, asymptotic formula

MSC 2010: 11L40, 11F20

1. Introduction

Let p be an odd prime and c be a fixed integer with $(c,p)=1$. For each integer a with $1\leqslant a\leqslant p-1$, it is clear that there exists one and only one b with $0\leqslant b\leqslant p-1$ such that $ab\equiv c \pmod p$. Let $M(c,p)$ denote the number of cases in which a and b are of opposite parity. In [4], Professor D. H. Lehmer encouraged the authors to study $M(1,p)$ or at least to say something nontrivial about it. It is known that $M(1,p)\equiv 2$ or $0 \pmod 4$ when $p\equiv \pm1 \pmod 4$. For general odd number $q\geqslant 3$, Zhang Wenpeng [10] studied the asymptotic properties of $M(1,q)$, and obtained a sharp asymptotic formula

$$M(1,q)=\tfrac12\varphi(q)+O(q^{\frac12}d^2(q)\ln^2 q),$$

where $\varphi(q)$ denotes the Euler function, and $d(q)$ is the number of divisors of q.

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Zhang Wenpeng [9] also studied the asymptotic properties of the mean square value of the error term $M(a,p)-\frac{1}{2}(p-1)$, and gave the asymptotic formula

$$\sum_{a=1}^{p-1}\Bigl(M(a,p)-\frac{p-1}{2}\Bigr)^2=\frac{3}{4}p^2+O\Bigl(p\exp\Bigl(\frac{3\ln p}{\ln\ln p}\Bigr)\Bigr).$$

Now, let p be an odd prime, c any integer with $(c,p)=1$, $N(c,p)$ the number of pairs of integers a, b with $ab\equiv c \pmod p$ for $1\leqslant a,b\leqslant p-1$ in which a and $\overline{b}$ are of opposite parity, and

$$E(c,p)=N(c,p)-\frac{p-1}{2}.$$

In this paper, we find that there are some close relations between $E(c,p)$ and the classical Dedekind sums, which are defined by

$$S(h,q)=\sum_{a=1}^{q}\Bigl(\Bigl(\frac{a}{q}\Bigr)\Bigr)\Bigl(\Bigl(\frac{ah}{q}\Bigr)\Bigr),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

Several authors have studied various properties of $S(h,k)$, namely, L. Carlitz [2], J. B. Conrey *et* al. [3], Jia Chaohua [5], and Wenpeng Zhang [8]. For example, Wenpeng Zhang [8] proved the asymptotic formula

$$\sum_{h=1}^{k}{}'|S(h,k)|^2=\frac{5}{144}\,k\varphi(k)\cdot\prod_{p^\alpha\|k}\Bigl(\Bigl(1+\frac{1}{p}\Bigr)^2-\frac{1}{p^{3\alpha+1}}\Bigr)\Bigm/\prod_{p|k}\Bigl(1+\frac{1}{p}+\frac{1}{p^2}\Bigr)$$
$$+O\Bigl(k\exp\Bigl(\frac{4\ln k}{\ln\ln k}\Bigr)\Bigr),$$

where $p^\alpha\parallel k$ denotes that p^α divides k, and $p^{\alpha+1}$ does not divide k.

In this paper, we consider the following hybrid mean value problem related to the error term $N(c,p)$ and the Dedekind sums:

$$\sum_{c=1}^{p-1}E(c,p)S(c,p). \tag{1}$$

Concerning this mean value, it seems that no one had studied it yet, at least we have not seen any related result before. In this paper, we use the properties of character sums and the analytic method to study the asymptotic properties of (1), and obtain the following two conclusions:

Theorem 1. *For any prime $p \geqslant 3$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1} E(c,p)S(c,p) = -\frac{1}{8}p^2 + O\Big(p \cdot \exp\Big(\frac{4\ln p}{\ln\ln p}\Big)\Big),$$

where $\exp(y) = \mathrm{e}^y$.

Theorem 2. *For any prime $p \geqslant 3$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1} E^2(c,p) = \frac{3}{4}p^2 + O\Big(p \cdot \exp\Big(\frac{5\ln p}{\ln\ln p}\Big)\Big).$$

As an application of our method, we can also give a series identity for the mean square value of the Dirichlet L-function. For example, we have

Corollary 1. *For any odd prime p, we have the identity*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24} \cdot \frac{(p-1)^2(p-5)}{p^2}.$$

Corollary 2. *Let $p \geqslant 3$ be an odd prime, then we have the identities*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 = \frac{\pi^2}{48} \cdot \frac{(p-1)^2(p-17)}{p^2},$$

if $p \equiv 1 \pmod 4$;

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 = \frac{\pi^2}{48} \cdot \frac{(p-1)(p^2-6p+17)}{p^2},$$

if $p \equiv 3 \pmod 4$.

It is very interesting that the value of the mean value $\sum\limits_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2$ depends on the congruence $p \equiv 1$ or $-1 \pmod 4$.

2. Several lemmas

To prove our theorems, we will make use of the following lemmas.

Lemma 1. *Let χ be a primitive character modulo m with $\chi(-1)=-1$. Then we have*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b) = \frac{\mathrm{i}}{\pi}\tau(\chi)L(1,\overline{\chi}),$$

where $\tau(\chi)$ is the Gaussian sum associated with χ, $e(y)=\mathrm{e}^{2\pi\mathrm{i}y}$, and $L(1,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. This can be easily deduce from Theorems 12.11 and 12.20 in [1]. □

Lemma 2. *Let $q \geqslant 3$ be an odd number. For any nonprincipal character $\chi \bmod q$, we have*

$$\frac{1}{q}\sum_{a=1}^{q} a\chi(a) = \frac{\chi(2)}{1-2\chi(2)}\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

Proof. See Lemma 1.5 in [6]. □

Lemma 3. *Let p be an odd prime. For any integer c with $(c,p)=1$, we have the identity*

$$E(c,p) = -\frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\overline{\chi}(c)|1-2\chi(2)|^2\cdot|L(1,\chi)|^2.$$

Proof. From the orthogonality relation for character sums modulo p and the definition of $N(c,p)$ we have

$$\begin{aligned}
(2)\qquad N(c,p) &= \frac{1}{2}\sum_{\substack{a=1\\ ab\equiv c(p)}}^{p-1}\sum_{b=1}^{p-1}(1-(-1)^{a+\overline{b}}) = \frac{1}{2}\varphi(p) - \frac{1}{2}\sum_{\substack{a=1\\ ab\equiv c(p)}}^{p-1}\sum_{b=1}^{p-1}(-1)^{a+\overline{b}}\\
&= \frac{p-1}{2} - \frac{1}{2(p-1)}\sum_{\chi \bmod p}\overline{\chi}(c)\sum_{a=1}^{p-1}(-1)^a\chi(a)\sum_{b=1}^{p-1}(-1)^{\overline{b}}\chi(b)\\
&= \frac{p-1}{2} - \frac{1}{2(p-1)}\sum_{\chi \bmod p}\overline{\chi}(c)\sum_{a=1}^{p-1}(-1)^a\chi(a)\overline{\sum_{b=1}^{p-1}(-1)^b\chi(b)}\\
&= \frac{p-1}{2} - \frac{1}{2(p-1)}\sum_{\chi \bmod p}\overline{\chi}(c)\left|\sum_{a=1}^{p-1}(-1)^a\chi(a)\right|^2.
\end{aligned}$$

If $\chi(-1)=1$, then

$$\sum_{a=1}^{p-1}(-1)^a\chi(a)=0. \tag{3}$$

If $\chi(-1)=-1$, then

$$\sum_{a=1}^{p-1}(-1)^a\chi(a)=\sum_{a=1}^{\frac{p-1}{2}}\chi(2a)-\sum_{a=1}^{\frac{p-1}{2}}\chi(p-2a)=2\chi(2)\sum_{a=1}^{\frac{p-1}{2}}\chi(a). \tag{4}$$

Combining (2), (3), (4), and applying Lemmas 1 and 2, we obtain

$$N(c,p)=\frac{p-1}{2}-\frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\overline{\chi}(c)|1-2\chi(2)|^2\cdot|L(1,\chi)|^2.$$

Hence

$$E(c,p)=-\frac{2p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\overline{\chi}(c)|1-2\chi(2)|^2\cdot|L(1,\chi)|^2.$$

This completes the proof of Lemma 3. □

Lemma 4. *Let $q\geqslant 3$ be a positive integer. Then for any integer c with $(c,q)=1$, we have the identity*

$$S(c,q)=\frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(c)|L(1,\chi)|^2.$$

Proof. See Lemma 2 in [11]. □

Lemma 5. *Let p be a prime, χ a Dirichlet character modulo p, and $m\geqslant 0$ a fixed integer. Then we have*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)|L(1,\chi)|^4=\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}p+O\Big(\exp\Big(\frac{(3+m)\ln p}{\ln\ln p}\Big)\Big),$$

where $\exp(y)=e^y$.

Proof. See Lemma 6 in [7]. □

3. Proof of the theorems

In this section, we will use the lemmas from Section 2 to prove our theorems. First we prove Theorem 1. For any prime p, Lemmas 3 and 4 and the orthogonality relation for character sums modulo p, yield

$$\begin{aligned}
(5)\qquad & \sum_{c=1}^{p-1} E(c,p)S(c,p) \\
&= \frac{-2p^2}{\pi^4(p-1)^2} \sum_{c=1}^{p-1} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \overline{\chi_1}(c)\chi_2(c)|1-2\chi_1(2)|^2 \\
&\qquad\qquad \times |L(1,\chi_1)|^2 \cdot |L(1,\chi_2)|^2 \\
&= -\frac{2p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |1-2\chi(2)|^2 \cdot |L(1,\chi)|^4 \\
&= -\frac{2p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (1-2\chi(2)-2\overline{\chi}(2)+4) \cdot |L(1,\chi)|^4 \\
&= \frac{2p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (4\chi(2)-5) \cdot |L(1,\chi)|^4.
\end{aligned}$$

From Lemma 5 we know that

$$(6)\qquad \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{5\pi^4}{144} \cdot p + O\Big(\exp\Big(\frac{3\ln p}{\ln\ln p}\Big)\Big)$$

and

$$(7)\qquad \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^4 = \frac{\pi^4}{36} \cdot p + O\Big(\exp\Big(\frac{4\ln p}{\ln\ln p}\Big)\Big).$$

Then combining (5), (6) and (7), and taking into account the asymptotic formula

$$\frac{p^3}{8(1-p)} = -\frac{p^2}{8} + O(p),$$

we immediately deduce that

$$\begin{aligned}\sum_{c=1}^{p-1} E(c,p)S(c,p) &= \frac{2p^2}{\pi^4(p-1)}\Bigg(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (4\chi(2)-5)\,|L(1,\chi)|^4\Bigg)\\ &= -\frac{2p^2}{\pi^4(p-1)}\left[\frac{5^2\pi^4}{144}\cdot p - \frac{4\pi^4}{36}\cdot p\right] + O\Big(p\cdot\exp\Big(\frac{4\ln p}{\ln\ln p}\Big)\Big)\\ &= -\frac{1}{8}p^2 + O\Big(p\cdot\exp\Big(\frac{4\ln p}{\ln\ln p}\Big)\Big).\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From the orthogonality relation for character sums modulo p and Lemma 5 we have

$$\begin{aligned}\sum_{c=1}^{p-1} E^2(c,p) &= \frac{4p^2}{\pi^4(p-1)^2}\sum_{c=1}^{p-1}\Bigg(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(c)\,|1-2\chi(2)|^2\cdot|L(1,\chi)|^2\Bigg)^2\\ &= \frac{4p^2}{\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |1-2\chi(2)|^4\cdot|L(1,\chi)|^4\\ &= \frac{4p^2}{\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} (33-40\chi(2)+8\chi(4))\cdot|L(1,\chi)|^4\\ &= \frac{4p^2}{\pi^4}\Big(33\cdot\frac{5\pi^4}{144} - 40\cdot\frac{8\pi^4}{72\times 4} + 8\cdot\frac{11\pi^4}{72\times 8}\Big) + O\Big(p\cdot\exp\Big(\frac{5\ln p}{\ln\ln p}\Big)\Big)\\ &= \frac{3}{4}p^2 + O\Big(p\cdot\exp\Big(\frac{5\ln p}{\ln\ln p}\Big)\Big).\end{aligned}$$

This proves Theorem 2.

To prove Corollary 1, note that $E(1,p) = N(1,p) - \frac{1}{2}(p-1) = -\frac{1}{2}(p-1)$ and (see [11])

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}. \tag{8}$$

From Lemma 3 we immediately deduce the identity

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24}\cdot\frac{(p-1)^2(p-5)}{p^2}.$$

Now we prove Corollary 2. Note that $\frac{1}{2}(p+1)=\overline{2}$, $E(\frac{1}{2}(p+1),p)=N(\frac{1}{2}(p+1),p)-\frac{1}{2}(p-1)=\frac{1}{2}(p+1)-\frac{1}{2}(p-1)=1$ if $p\equiv 3 \pmod 4$; and $E(\frac{1}{2}(p+1),p)=N(\frac{1}{2}(p+1),p)-\frac{1}{2}(p-1)=\frac{1}{2}(p-1)-\frac{1}{2}(p-1)=0$ if $p\equiv 1 \pmod 4$. From (8), Lemma 3 and Corollary 1 we also have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 = \frac{\pi^2}{48}\cdot\frac{(p-1)^2(p-17)}{p^2}$$

if $p\equiv 1 \pmod 4$;

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(4)|L(1,\chi)|^2 = \frac{\pi^2}{48}\cdot\frac{(p-1)(p^2-6p+17)}{p^2}$$

if $p\equiv 3 \pmod 4$.

This completes the proof of our all conclusions. □

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