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ON VOLTERRA COMPOSITION OPERATORS FROM BERGMAN-TYPE SPACE TO BLOCH-TYPE SPACE

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Abstract. Let φ be an analytic self-mapping of $\mathbb{D}$ and g an analytic function on $\mathbb{D}$. In this paper we characterize the bounded and compact Volterra composition operators from the Bergman-type space to the Bloch-type space. We also obtain an asymptotical expression of the essential norm of these operators in terms of the symbols g and φ.

Keywords: Bergman-type space, Volterra composition operator, Bloch-type space, little Bloch-type space, essential norm

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. If u is a positive continuous function on $[0,1)$ and there exist positive numbers $\delta \in [0,1)$, s and t, $0 < s < t$, such that $u(r)/(1-r)^s$ is decreasing on $[\delta,1)$ and $\lim\limits_{r\to 1^-} u(r)/(1-r)^s = 0$, $u(r)/(1-r)^t$ is increasing on $[\delta,1)$ and $\lim\limits_{r\to 1^-} u(r)/(1-r)^t = \infty$, then u is called a normal weight function (see [5]). For such normal weights, one can consider the following examples:

$$u(r) = (1-r^2)^\alpha, \quad \alpha \in (0,\infty);$$
$$u(r) = (1-r^2)^\alpha\{\log 2(1-r^2)^{-1}\}^\beta, \quad \alpha \in (0,1),\ \beta \in \Big[\frac{\alpha-1}{2}\log 2, 0\Big];$$

and

$$u(r) = (1-r^2)^\alpha\{\log\log \mathrm{e}^2(1-r^2)^{-1}\}^\gamma, \quad \alpha \in (0,1),\ \gamma \in \Big[\frac{\alpha-1}{2}\log 2, 0\Big].$$

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For $0 < p < \infty$ and a normal weight function u, the Bergman-type space A^p_u is defined by

$$A^p_u = \left\{ f \in H(\mathbb{D}) \colon \ \|f\|^p = \int_{\mathbb{D}} |f(z)|^p \frac{u(|z|)^p}{1-|z|} \, \mathrm{d}A(z) < \infty \right\}.$$

When $1 \leqslant p < \infty$, A^p_u is a Banach space with the norm $\|\cdot\|$. If $0 < p < 1$, it is a Fréchet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p.$$

Let ν be a radial bounded continuous positive function on the open unit disk $\mathbb{D}$. Recall that the Bloch-type space is defined by

$$\beta_\nu = \left\{ f \in H(\mathbb{D}) \colon \ \|f\|_\nu = \sup_{z \in \mathbb{D}} \nu(z)|f'(z)| < \infty \right\}$$

and the little Bloch-type space by

$$\beta_{\nu,0} = \left\{ f \in \beta_\nu \colon \ \lim_{|z| \to 1^-} \nu(z)|f'(z)| = 0 \right\}.$$

The Bloch-type space and the little Bloch-type space play an important role in the theory of Bergman-type spaces as the BMOA does in Hardy spaces. When normed by $\|f\| = |f(0)| + \|f\|_\nu$, the Bloch-type space β_ν is a Banach space and the little Bloch-type space $\beta_{\nu,0}$ is a closed subspace of β_ν.

Let φ be an analytic self-mapping of $\mathbb{D}$. Then the composition operator on $H(\mathbb{D})$ is given by

$$C_\varphi f = f \circ \varphi.$$

Composition operators acting on various spaces of analytic functions have been the object of study for recent several years, especially the problems of relating the operator-theoretic properties of C_φ to function theoretic properties of φ. See the books written by Cowen and MacCluer [4] and by Shapiro [8] for discussions of composition operators on classical spaces of analytic functions.

For an analytic function $g \colon \mathbb{D} \to \mathbb{C}$, the Volterra-type operator J_g on $H(\mathbb{D})$ is defined as

$$J_g f(z) = \int_0^z f(\xi) g'(\xi) \, \mathrm{d}\xi, \quad z \in \mathbb{D}.$$

The Volterra-type operators have been studied in [1]–[3], [9] and [11]. In this paper, we consider the Volterra composition operator $J_{g,\varphi}$ which is defined by

$$J_{g,\varphi} f(z) = \int_0^z (f \circ \varphi)(\xi)(g \circ \varphi)'(\xi) \, \mathrm{d}\xi, \quad z \in \mathbb{D}.$$

Li has characterized the bounded and compact Volterra composition operators between a weighted Bergman space and a Bloch space in [6]. Recently, Wolf has also characterized the boundedness and compactness of the Volterra composition operators from a Bergman-type space with an analytic weight to a Bloch-type space in [10].

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively, and let $T\colon X \to Y$ be a linear operator. Recall that T is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Tf, 0) \leqslant K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. If Y is a Banach space, then the quantity $\|T\|_{A^p_u \to Y}$ is given by

$$\|T\|_{A^p_u \to Y} := \sup_{\|f\| \leqslant 1} \|Tf\|_Y.$$

It is easy to see that this quantity is finite if and only if the operator $T\colon A^p_u \to Y$ is metrically bounded. For the case $p \geqslant 1$ this is the standard definition of the norm of the operator $T\colon A^p_u \to Y$ between two Banach spaces. If we say that an operator is bounded it means that it is metrically bounded.

Recall that $T\colon X \to Y$ is *metrically compact* if it maps bounded sets into relatively compact sets. If X and Y are Banach spaces then metrical compactness becomes usual compactness. If we say that an operator is compact it means that it is metrically compact.

In this paper, we characterize the boundedness and compactness of the Volterra composition operator from the Bergman-type space to the Bloch-type space and the little Bloch-type space. We also obtain an asymptotical expression of the essential norm for these operators. This paper can be considered a continuation of investigations of these operators.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to another. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \leqslant b \leqslant Ca$.

2. Boundedness and compactness of Volterra composition operators

To deal with the compactness of Volterra composition operators, we need the following lemma which characterizes the compactness of Volterra composition operators in terms of sequential convergence.

Lemma 2.1. *Let φ be an analytic self-mapping of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then the bounded operator $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact if and only if for every bounded sequence $\{f_n\}$ in A^p_u which converges to zero uniformly on compact subsets of $\mathbb{D}$, it follows that $\|J_{g,\varphi} f_n\| \to 0$ as $n \to \infty$.*

P r o o f. Suppose that for a bounded sequence $\{f_n\}$ in A^p_u which converges to zero uniformly on compact subset of $\mathbb{D}$, it follows that $\|J_{g,\varphi} f_n\| \to 0$ as $n \to \infty$. For a fixed point $z_0 \in \mathbb{D}$ and $r > 0$, taking $z \in \mathbb{D}$ with $|z - z_0| \leqslant r$, by [7] or Lemma 2.2 we can find a positive constant C such that $|f_n(z)| \leqslant C$ for each $n \in \mathbb{N}$. Since every compact subset K of $\mathbb{D}$ is contained in $\{z\colon |z - z_0| \leqslant r\}$ for some $r > 0$, it follows that $\{f_n\}$ is uniformly bounded on every compact subset of $\mathbb{D}$. Montel's theorem allows us to pick out a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges uniformly on every compact subset of $\mathbb{D}$ to an analytic function f. It is easy to see that f must be in A^p_u. Then $\{f_{n_k} - f\}$ is a bounded sequence in A^p_u and $f_{n_k} - f \to 0$ uniformly on every compact subset of $\mathbb{D}$. The hypothesis of the lemma guarantees that $\|J_{g,\varphi}(f_{n_k} - f)\| \to 0$. This shows that $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact.

Conversely, suppose $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact. Let B be the closed unit ball in A^p_u. The compactness of $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ means that $\{J_{g,\varphi}(B)\}$ is a relatively compact subset of β_ν. Hence we are given a sequence $\{f_n\}$ that lies in rB and converges to zero uniformly on every compact subset of $\mathbb{D}$. We wish to show that $\|J_{g,\varphi} f_n\| \to 0$. For this goal it is enough to prove that the zero function is the unique limit point of the sequence $\{J_{g,\varphi} f_n\}$. Since $\{J_{g,\varphi} f_n\}$ is relatively compact, there must be a function $f \in \beta_\nu$ such that $\|J_{g,\varphi} f_n - f\| \to 0$ as $n \to \infty$. This implies that for each $z \in \mathbb{D}$,

$$|f(0)| + \nu(z)|f_n(\varphi(z))g'(\varphi(z))\varphi'(z) - f'(z)| \to 0 \tag{2.1}$$

as $n \to \infty$. By (2.1) and since $\{f_n\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$, we have $|f(0)| + \nu(z)|f'(z)| = 0$ for each $z \in \mathbb{D}$, which means $f \equiv 0$. This completes the proof. $\square$

Similarly to the proof of Lemma 2.1, we can show that Lemma 2.1 is true for $J_{g,\varphi}\colon A^p_u \to \beta_{\nu,0}$, and so we omit its proof here. The following lemma was obtained in [7].

Lemma 2.2. *There is a positive constant C independent of $f \in A^p_u$ such that for every $z \in \mathbb{D}$, the following inequality holds:*

$$|f(z)| \leqslant \frac{C\|f\|}{u(|z|)(1 - |z|^2)^{1/p}}.$$

Now we formulate and prove one of the main results of this section.

Theorem 2.3. *Let φ be an analytic self-mapping of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then we have the following statements:*

(i) *The operator $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \infty. \tag{2.2}$$

(ii) *The operator $J_{g,\varphi}\colon A^p_u \to \beta_{\nu,0}$ is bounded if and only if $g\circ\varphi \in \beta_{\nu,0}$ and*

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \infty. \tag{2.3}$$

Proof. Suppose that $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is bounded. For fixed $w\in\mathbb{D}$, taking the function

$$f_w(z) = \frac{(1-|w|^2)^{t+1}}{u(|\varphi(w)|)(1-\overline{w}z)^{1/p+t+1}}, \quad z\in\mathbb{D},$$

by Theorem 3.1 in [7] we have $f_w \in A^p_u$ and $\|f_w\| \leqslant C$. By the boundedness of $J_{g,\varphi}\colon A^p_u \to \beta_\nu$, we obtain

$$C\|J_{g,\varphi}\| \geqslant \|J_{g,\varphi}f_w\|_\nu \geqslant \frac{\nu(w)|g'(\varphi(w))||\varphi'(w)|}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/p}}.$$

This implies that

$$\sup_{z\in\mathbb{D}} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \infty,$$

from which the desired condition (2.2) follows.

Conversely, suppose that condition (2.2) holds. For $f\in A^p_u$, by Lemma 2.2 we have

$$\begin{aligned}\|J_{g,\varphi}f\| &= \sup_{z\in\mathbb{D}} \nu(z)|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)|\\ &\leqslant C\sup_{z\in\mathbb{D}} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}}.\end{aligned}$$

From this inequality and condition (2.2) we deduce that $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is bounded. The result (ii) is proved similarly and we omit its proof. □

Theorem 2.4. *Let φ be an analytic self-mapping of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then we have the following statements:*

(i) *The bounded operator $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} = 0. \tag{2.4}$$

(ii) *The bounded operator* $J_{g,\varphi}\colon A^p_u \to \beta_{\nu,0}$ *is compact if and only if* $g\circ\varphi \in \beta_{\nu,0}$ *and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} = 0. \tag{2.5}$$

P r o o f. First we prove that the function 1 belongs to A^p_u. Since

$$\begin{aligned}\|1\|^p &= \int_{\mathbb{D}} \frac{u(|z|)^p}{1-|z|}\,\mathrm{d}A(z) = \int_0^{2\pi}\int_0^1 \frac{u(r)^p}{1-r}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi\int_0^1 \frac{u(r)^p}{1-r}\,r\,\mathrm{d}r = 2\pi\int_0^1 \frac{u(r)^p}{1-r}\,\mathrm{d}r - 2\pi\int_0^1 u(r)^p\,\mathrm{d}r\end{aligned} \tag{2.6}$$

and $\int_0^1 u(r)^p\,\mathrm{d}r < \infty$, to prove $1 \in A^p_u$ it is enough to prove that the first integral on the right of (2.6) is finite. Since $u(r)/(1-r)^s$ is decreasing on $[\delta,1)$,

$$\begin{aligned}\int_0^1 \frac{u(r)^p}{1-r}\,\mathrm{d}r &= \int_0^\delta \frac{u(r)^p}{1-r}\,\mathrm{d}r + \int_\delta^1 \frac{u(r)^p}{(1-r)^{sp}}\frac{(1-r)^{sp}}{1-r}\,\mathrm{d}r \\ &\leqslant \int_0^\delta \frac{u(r)^p}{1-r}\,\mathrm{d}r + \frac{u(\delta)^p}{(1-\delta)^{sp}}\int_\delta^1 \frac{(1-r)^{sp}}{1-r}\,\mathrm{d}r \\ &= \int_0^\delta \frac{u(r)^p}{1-r}\,\mathrm{d}r + \frac{u(\delta)^p}{sp} < \infty,\end{aligned}$$

which shows that the first integral on the right of (2.6) is finite and thus $1 \in A^p_u$.

Suppose that the bounded operator $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact. Let $\{z_n\}$ be a sequence with $|\varphi(z_n)| \to 1$ as $n\to\infty$. For every $z\in\mathbb{D}$, set

$$f_n(z) = \frac{(1-|\varphi(z_n)|^2)^{t+1}}{u(|\varphi(z_n)|)(1-\overline{\varphi(z_n)}z)^{1/p+t+1}}.$$

We know that $f_n \in A^p_u$, $\|f_n\| \leqslant C$ for each $n\in\mathbb{N}$, and $\{f_n\}$ converges uniformly to zero on compact subset of $\mathbb{D}$. Since $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is compact, by Lemma 2.1 we get

$$\lim_{n\to\infty} \frac{\nu(z_n)|g'(\varphi(z_n))||\varphi'(z_n)|}{u(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/p}} = 0.$$

From this and the arbitrariness of the sequence $\{z_n\}$ it follows that condition (2.4) holds.

Now suppose that condition (2.4) is satisfied. Let $\{f_n\}$ be a bounded sequence in A^p_u such that $\|f_n\| \leqslant C$ and $\{f_n\}$ converges uniformly to zero on the compact subset of $\mathbb{D}$. In view of Lemma 2.1, it is enough to show $\|J_{g,\varphi}f_n\| \to 0$ as $n\to\infty$. By condition (2.4), for every $\varepsilon > 0$ we can choose r_0, $0 < r_0 < 1$, such that $|\varphi(z)| > r_0$, and it follows that

$$\frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \varepsilon. \tag{2.7}$$

Since $J_{g,\varphi}\colon A_u^p \to \beta_\nu$ is bounded and $1 \in A_u^p$, we get $g \circ \varphi \in \beta_\nu$. Then by the definition of β_ν we can choose $M > 0$ such that

$$\sup_{|\varphi(z)| \leqslant r_0} \nu(z)|g'(\varphi(z))||\varphi'(z)| \leqslant M.$$

Since $\{f_n\}$ converges uniformly to zero on the compact subset of $\mathbb{D}$, there is some $N_0 > 0$ such that $\sup\limits_{|\varphi(z)| \leqslant r_0} |f_n(\varphi(z))| < \varepsilon$ for $n \geqslant N_0$. By combining these facts and Lemma 2.2, it follows that for all $n \geqslant N_0$,

$$\begin{aligned}
\|J_{g,\varphi} f_n\| &= \|J_{g,\varphi} f_n\|_\nu \\
&= \sup_{z \in \mathbb{D}} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\leqslant \sup_{|\varphi(z)| \leqslant r_0} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\quad + \sup_{|\varphi(z)| > r_0} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\leqslant \sup_{|\varphi(z)| \leqslant r_0} \nu(z)|g'(\varphi(z))||\varphi'(z)| \sup_{|\varphi(z)| \leqslant r_0} |f_n(\varphi(z))| \\
&\quad + \varepsilon \sup_{|\varphi(z)| > r_0} |f_n(\varphi(z))| u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p} \\
&\leqslant (M + C)\varepsilon.
\end{aligned}$$

This shows that $\|J_{g,\varphi} f_n\| \to 0$ as $n \to \infty$, and the proof of (i) is complete.

We are ready to prove (ii). Suppose the bounded operator $J_{g,\varphi}\colon A_u^p \to \beta_{\nu,0}$ is compact. Since $1 \in A_u^p$ and a constant function is in $\beta_{\nu,0}$, we have $g \circ \varphi = J_{\varphi,g} 1 + g \circ \varphi(0) \in \beta_{\nu,0}$. By the compactness of $J_{g,\varphi}\colon A_u^p \to \beta_{\nu,0}$ and since $\beta_{\nu,0}$ is a closed subspace of β_ν, this implies that $J_{g,\varphi}\colon A_u^p \to \beta_\nu$ is compact. Hence by (i) of this theorem, the desired condition (2.5) follows.

Conversely, suppose that $g \circ \varphi \in \beta_{\nu,0}$ and condition (2.5) holds. Let $\{f_n\}$ be a bounded sequence in A_u^p such that $\|f_n\| \leqslant C$ and $\{f_n\}$ converges uniformly to zero on the compact subset of $\mathbb{D}$. By Lemma 2.1, we only need to show that $\|J_{g,\varphi} f_n\| \to 0$ as $n \to \infty$. By condition (2.5), for every $\varepsilon > 0$ there exists r_0, $0 < r_0 < 1$, such that $|\varphi(z)| > r_0$, and it follows that

$$\frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \varepsilon.$$

Since $\{f_n\}$ converges uniformly to zero on the compact subset of $\mathbb{D}$, there is $N > 0$ such that

$$\sup_{|\varphi(z)|\leqslant r_0} |f_n(\varphi(z))| < \varepsilon$$

for all $n \geqslant N$. Since $g \circ \varphi \in \beta_{\nu,0}$ implies that $g \circ \varphi \in \beta_\nu$, we have

$$M := \sup_{|\varphi(z)|\leqslant r_0} \nu(z)|g'(\varphi(z))||\varphi'(z)| = \sup_{|\varphi(z)|\leqslant r_0} \nu(z)|(g\circ\varphi)'(z)| < \infty.$$

Once again using Lemma 2.2, we see that for every $n \geqslant N$,

$$\begin{aligned}
\|J_{g,\varphi} f_n\| = \|J_{g,\varphi} f_n\|_\nu &= \sup_{z\in\mathbb{D}} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\leqslant \sup_{|\varphi(z)|\leqslant r_0} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\quad + \sup_{|\varphi(z)|> r_0} \nu(z)|f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\leqslant \sup_{|\varphi(z)|\leqslant r_0} \nu(z)|g'(\varphi(z))||\varphi'(z)| \sup_{|\varphi(z)|\leqslant r_0} |f_n(\varphi(z))| \\
&\quad + \varepsilon \sup_{|\varphi(z)|> r_0} |f_n(\varphi(z))|u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p} \\
&\leqslant (M + C)\varepsilon,
\end{aligned}$$

which shows that $\|J_{g,\varphi} f_n\|_\nu \to 0$ as $n \to \infty$, and the proof of (ii) is complete. $\square$

3. The essential norm of Volterra composition operators

In this section we are dealign with calculating the essential norms of Volterra composition operators. First we recall the definition of the essential norm of the bounded linear operators.

Let X and Y be Banach spaces and $T\colon X \to Y$ a bounded linear operator. The essential norm of the operator $T\colon X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|\colon K \in \mathcal{K}\},$$

where $\mathcal{K}$ denotes the set of all compact linear operators from X to Y. From this definition it is easy to check that a bounded linear operator $T\colon X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

Theorem 3.1.

(i) *Let $1 \leqslant p < \infty$ and let $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ be bounded. Then*

$$\|J_{g,\varphi}\|_{e,A^p_u\to\beta_\nu} \asymp \limsup_{n\to\infty} \frac{\nu(z_n)|g'(\varphi(z_n))||\varphi'(z_n)|}{u(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/p}}. \tag{3.1}$$

(ii) *Let $1 \leqslant p < \infty$ and let $J_{g,\varphi}\colon A^p_u \to \beta_{\nu,0}$ be bounded and let $g\circ\varphi \in \beta_{\nu,0}$. Then*

$$\|J_{g,\varphi}\|_{e,A^p_u\to\beta_{\nu,0}} \asymp \limsup_{n\to\infty} \frac{\nu(z_n)|g'(\varphi(z_n))||\varphi'(z_n)|}{u(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/p}}. \tag{3.2}$$

Proof. Suppose that $\{\varphi(z_n)\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n\to\infty$. For this sequence $\{\varphi(z_n)\}$, we define

$$f_n(z) = \frac{(1-|\varphi(z_n)|^2)^{t+1}}{u(|\varphi(z_n)|)(1-\overline{\varphi(z_n)}z)^{1/p+t+1}}.$$

By the proof of Theorem 3.1 in [7] we know that $\|f_n\| \leqslant C$ and $\{f_n\}$ converges to zero uniformly on compact subset of $\mathbb{D}$. Hence for every compact operator $K\colon A^p_u \to \beta_\nu$, we have $\|Kf_n\| \to 0$ as $n\to\infty$. Thus it follows that

$$\begin{aligned} \|J_{g,\varphi}-K\| &= \sup_{\|f\|=1} \|(J_{g,\varphi}-K)f\| \geqslant \limsup_{n\to\infty} \frac{\|(J_{g,\varphi}-K)f_n\|}{\|f_n\|} \\ &\geqslant \limsup_{n\to\infty} \frac{\|J_{g,\varphi}f_n\| - \|Kf_n\|}{\|f_n\|} \\ &\geqslant C^{-1} \limsup_{n\to\infty} \nu(z_n)|f_n(\varphi(z_n))||g'(\varphi(z_n))||\varphi'(z_n)| \\ &= C^{-1} \limsup_{n\to\infty} \frac{\nu(z_n)|g'(\varphi(z_n))||\varphi'(z_n)|}{u(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/p}}. \end{aligned} \tag{3.3}$$

By taking the infimum in (3.3) over the set of all compact operators $K\colon A^p_u \to \beta_\nu$, it follows that

$$\|J_{g,\varphi}\|_{e,A^p_u\to\beta_\nu} \geqslant C^{-1} \limsup_{n\to\infty} \frac{\nu(z_n)|g'(\varphi(z_n))||\varphi'(z_n)|}{u(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/p}}. \tag{3.4}$$

Now suppose that $\{r_n\}$ is a positive sequence which increasingly converges to 1. For every r_n we define the operator

$$J_{g,r_n\varphi}f(z) = \int_0^z (f\circ r_n\varphi)(\xi)(g\circ\varphi)'(\xi)\,\mathrm{d}\xi, \quad z\in\mathbb{D}.$$

Since $J_{g,\varphi}\colon A^p_u \to \beta_\nu$ is bounded, by Theorem 2.3 (i) one can check that the operator $J_{g,r_n\varphi}\colon A^p_u \to \beta_\nu$ is also bounded. Since $|r_n\varphi(z)| \leqslant r_n < 1$, by Theorem 2.3 (i) and Lemma 2.1 we conclude that the operator $J_{g,r_n\varphi}\colon A^p_u \to \beta_\nu$ is bounded and compact. By using the definition of A^p_u, we get that $\|f - f_{r_n}\| \leqslant 2\|f\|$. Hence we have

$$
\begin{aligned}
(3.5)\quad \|J_{g,r_n\varphi} - J_{g,\varphi}\| &= \sup_{\|f\|=1}\sup_{z\in\mathbb{D}} \nu(z)|g'(\varphi(z))||\varphi'(z)||f(r_n\varphi(z)) - f(\varphi(z))| \\
&\leqslant \sup_{\|f\|=1}\sup_{|\varphi(z)|\leqslant r} \nu(z)|g'(\varphi(z))||\varphi'(z)||f(r_n\varphi(z)) - f(\varphi(z))| \\
&\quad + \sup_{\|f\|=1}\sup_{|\varphi(z)|> r} \nu(z)|g'(\varphi(z))||\varphi'(z)||f(r_n\varphi(z)) - f(\varphi(z))| \\
&\leqslant \|g\circ\varphi\|_{\beta_\nu} \sup_{\|f\|=1}\sup_{|\varphi(z)|\leqslant r} |f(r_n\varphi(z)) - f(\varphi(z))| \\
&\quad + C\sup_{|\varphi(z)|>r} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} \\
&\quad + C\sup_{|\varphi(z)|>r} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(r_n|\varphi(z)|)(1-r_n|\varphi(z)|^2)^{1/p}}.
\end{aligned}
$$

We consider

$$
I_n = \sup_{\|f\|=1}\sup_{|\varphi(z)|\leqslant r} |f(r_n\varphi(z)) - f(\varphi(z))|.
$$

By using the mean value theorem and the subharmonicity of the derivative of f and Lemma 2.2 we have

$$
\begin{aligned}
(3.6)\qquad I_n &\leqslant \sup_{\|f\|=1}\sup_{|\varphi(z)|\leqslant r} (1-r_n)|\varphi(z)| \sup_{|w|\leqslant r} |f'(w)| \\
&\leqslant \frac{C(1-r_n)}{\min\{u(t)\colon\ 0\leqslant t\leqslant r\}(1-r^2)^{1/p+1}}.
\end{aligned}
$$

Letting $n\to\infty$ in (3.6), we obtain that $I_n \to 0$ as $n\to\infty$. Since

$$
\|J_{g,\varphi}\|_{e,A^p_u\to\beta_\nu} \leqslant \|J_{g,r_n\varphi} - J_{g,\varphi}\|,
$$

by inequality (3.5) we get that

$$
\begin{aligned}
\|J_{g,\varphi}\|_{e,A^p_u\to\beta_\nu} &\leqslant \|g\circ\varphi\|_{\beta_\nu} I_n + C\sup_{|\varphi(z)|>r} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} \\
&\quad + C\sup_{|\varphi(z)|>r} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(r_n|\varphi(z)|)(1-r_n|\varphi(z)|^2)^{1/p}},
\end{aligned}
$$

from which, letting $n\to\infty$, we obtain that

$$
(3.7)\qquad \|J_{g,\varphi}\|_{e,A^p_u\to\beta_\nu} \leqslant 2C \sup_{|\varphi(z)|>r} \frac{\nu(z)|g'(\varphi(z))||\varphi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}}.
$$

From inequalities (3.4) and (3.7), the asymptotic relation in (3.1) is obtained. The result (ii) in this theorem can be proved similarly and thus its proof is omitted. $\square$

Remark 3.2. From Theorem 3.1, we can obtain Theorem 2.4 for the case $1 \leqslant p < \infty$. However, on behalf of readers and completeness we have formulated and proved it.

4. An appendix

In this appendix we show that the examples in the introduction are normal weight functions.

Since $\alpha > 0$, we can choose s such that $0 < s < \alpha$. For this fixed s, because $0 < s < \alpha$, it is clear that $\lim\limits_{r\to 1^-} u(r)/(1-r)^s = \lim\limits_{r\to 1^-} (1-r^2)^\alpha/(1-r)^s = 0$. Set $f(r) = (1-r^2)^\alpha/(1-r)^s$. Then

$$f'(r) = (1+r)^{\alpha-1}(1-r)^{\alpha-s-1}[(s-2\alpha)r+s].$$

Take $\delta = s/(2\alpha - s)$. Because $0 < s < \alpha$, it follows that $0 < \delta < 1$. By the expression of $f'(r)$, we can see that $f'(r) \leqslant 0$ on $[\delta, 1)$. This shows that $f(r)$ is decreasing on $[\delta, 1)$.

On the other hand, for an arbitrary number t with $t > \alpha$, it is obvious that $\lim\limits_{r\to 1^-} u(r)/(1-r)^t = \lim\limits_{r\to 1^-} (1-r^2)^\alpha/(1-r)^t = \infty$. If $f(r) = (1-r^2)^\alpha/(1-r)^t$, then since $2t \geqslant (\alpha - t)r$ for all $r \in [\delta, 1)$ implies that $f'(r) \geqslant 0$, it follows that $f(r)$ is increasing on $[\delta, 1)$. By the definition of the normal weight, we conclude that $u(r) = (1-r^2)^\alpha$ with $\alpha > 0$ is normal.

Now we show that $u(r) = (1-r^2)^\alpha\{\log 2(1-r^2)^{-1}\}^\beta$ is normal provided $\alpha \in (0,1)$ and $\beta \in [\frac{1}{2}(\alpha-1)/2\log 2, 0]$. Because $\alpha > 0$, we can choose a number s satisfying $0 < s < \alpha$. Using the L'Hospital rule, we see that $\lim\limits_{r\to 1^-} u(r)/(1-r)^s = 0$. Taking $f(r) = u(r)/(1-r)^s$, we have

$$f'(r) = \Big[(1+r)^{\alpha-1}(1-r)^{\alpha-s-1}\log^{\beta-1}\frac{2}{1-r^2}\Big]\Big[((s-2\alpha)r+s)\log\frac{2}{1-r^2} + 2\beta r\Big].$$

Since $(s-2\alpha)r + s \leqslant 0$ on $[s/(2\alpha-s), 1)$ and $2\beta r \leqslant 0$, $f'(r) \leqslant 0$ on $[s/(2\alpha-s), 1)$, we conclude that $f(r)$ is decreasing on $[s/(2\alpha-s), 1)$.

On the other hand, if we take $t = \alpha + 1$, then $\lim\limits_{r\to 1^-} u(r)/(1-r)^t = \infty$. In order to show the normality of $u(r)$, we only need to prove that $g(r) = u(r)/(1-r)^{\alpha+1}$ is increasing on $[s/(2\alpha-s), 1)$. By a simple calculation, we have

$$g'(r) = \frac{(1+r)^{\alpha-1}\{\log 2(1-r^2)^{-1}\}^{\beta-1}h(r)}{(1-r)^2},$$

where $h(r) = [(1-\alpha)r + \alpha + 1] \log 2(1-r^2)^{-1} + 2\beta r$. By calculating $h'(r)$, we see that when $\alpha \in (0,1)$ and $\beta \in [\frac{1}{2}(\alpha - 1)\log 2, 0]$, then $h'(r)$ is increasing on $[0,1)$ and $h'(0) \geqslant 0$. Hence $h'(r) \geqslant 0$ on $[0,1)$, and this implies that $h(r)$ is increasing on $[0,1)$. It is clear that $h(0) = (\alpha+1)\log 2 > 0$, so that $h(r) > 0$ on $[0,1)$. This leads to $g'(r) \geqslant 0$ on $[0,1)$. In particular, $g'(r) \geqslant 0$ on $[s/(2\alpha - s), 1)$, which implies that $g(r)$ is increasing on $[s/(2\alpha - s), 1)$.

In the end of this section we prove that $u(r) = (1-r^2)^\alpha \{\log\log \mathrm{e}^2(1-r^2)^{-1}\}^\gamma$ with $\alpha \in (0,1)$ and $\gamma \in [\frac{1}{2}(\alpha-1)\log 2, 0]$ is normal. Once again by the L'Hospital rule, the limit $\lim\limits_{r\to 1^-} u(r)/(1-r)^s = 0$ for fixed s, $0 < s < \alpha$. Let $f(r) = u(r)/(1-r)^s$. Then we have

$$f'(r) = (1-r)^{\alpha-s-1}(1+r)^{\alpha-1}\{\log\log \mathrm{e}^2(1-r^2)^{-1}\}^{\gamma-1} \times \Big\{[(s-2\alpha)r + s]\log\log \mathrm{e}^2(1-r^2)^{-1} + \frac{2\gamma r}{2-\log(1-r^2)}\Big\},$$

from which it is seen that $f'(r) \leqslant 0$ on $[s/(2\alpha - s), 1)$ and thus $f(r)$ is decreasing on this interval. Let $t = \alpha + 1$ and $h(r) = u(r)/(1-r)^{\alpha+1}$. Then $\lim\limits_{r\to 1^-} h(r) = \infty$. Similarly to the proof of $g'(r) \geqslant 0$ on $[s/(2\alpha - s), 1)$, we can prove that when α and γ satisfy conditions $\alpha \in (0,1)$ and $\gamma \in [\frac{1}{2}(\alpha-1)\log 2, 0]$, $h'(r)$ is nonnegative on $[s/(2\alpha - s), 1)$. Therefore, $h(r)$ is increasing on $[s/(2\alpha - s), 1)$. By these facts we deduce that $u(r)$ is normal. This completes the proof. □

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