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CONCENTRATED MONOTONE MEASURES WITH NON-UNIQUE TANGENTIAL BEHAVIOR IN $\mathbb{R}^3$

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Abstract. We show that for every $\varepsilon > 0$ there is a set $A \subset \mathbb{R}^3$ such that $\mathcal{H}^1 \llcorner A$ is a monotone measure, the corresponding tangent measures at the origin are non-conical and non-unique and $\mathcal{H}^1 \llcorner A$ has the 1-dimensional density between 1 and $2+\varepsilon$ everywhere in the support.

Keywords: monotone measure, monotonicity formula, tangent measure

MSC 2010: 53A10, 49Q15, 28A75

1. Introduction

This paper is devoted to a construction of a monotone measure with bad tangential behavior satisfying some additional density assumptions natural for minimal surfaces.

Definition 1.1. Let μ be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that μ is *k-monotone* if the function $r \mapsto \mu B(z,r)/r^k$ is non-decreasing on $(0,\infty)$ for every $z \in \mathbb{R}^n$.

The tangent measures of μ at $z \in \mathbb{R}^n$ were introduced by Preiss in [5]. They are defined by blowing up μ by sequences of expansive homotheties around z, normalizing suitably and taking the vague limits. The mapping $T_{z,r}$ that blows up $B(z,r)$ to $B(0,1)$ is given by

$$T_{z,r}(x) = \frac{x-z}{r}.$$

Since every k-monotone measure μ, where $k \leqslant n$, has a finite k-dimensional density defined by $\theta^k_z \mu = \lim\limits_{r\to 0} \mu B(z,r)/\omega_k r^k$ at every point $z \in \mathbb{R}^n$, where ω_k is the volume of the unit ball in $\mathbb{R}^k$, it is natural to normalize the blow-up by $1/r^k$.

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Definition 1.2. Let μ be a Radon measure on $\mathbb{R}^n$, $z \in \operatorname{spt}\mu$ and $k \leqslant n$. We say that ν is a *k-tangent measure of μ at z* ($\nu \in \operatorname{Tan}^k_z \mu$) if ν is a non-zero Radon measure on $\mathbb{R}^n$ and if there is a sequence $\{r_j\}_{j=1}^{\infty}$, $r_j > 0$, $r_j \to 0$ as $j \to \infty$, such that

$$\frac{1}{r_j^k} T_{z,r_j}(\mu) \to \nu \quad \text{vaguely as } j \to \infty,$$

i.e. for every continuous function φ on $\mathbb{R}^n$ with a compact support we have

$$\lim_{j\to\infty} \frac{1}{r_j^k} \int \varphi\Big(\frac{x-z}{r_j}\Big)\,\mathrm{d}\mu(x) = \int \varphi(x)\,\mathrm{d}\nu(x).$$

Definition 1.3. Let ν be a Radon measure on $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that ν is *k-conical* if $\nu(\lambda A) = \lambda^k \cdot \nu(A)$ for every Borel set $A \subset \mathbb{R}^n$ and every $\lambda > 0$.

Instead of 1-monotone, 1-tangent and 1-conical we simply write *monotone*, *tangent* and *conical*, respectively.

The first monotone measure with bad tangential behavior was given by the second author in [3]. He constructed a measure in $\mathbb{R}^2$ which does not have a unique tangent measure at the origin, and observing that the monotonicity can be sometimes obtained by adding a suitable monotone measure he added a measure absolutely continuous with respect to the Lebesgue measure to obtain the desired result.

Immediately, a new question arises whether there exists a monotone measure with similarly bad tangential behavior and more related to minimal surfaces. It is a well-known open problem whether for each fixed $\varepsilon > 0$ a k-monotone Radon measure μ in $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, can be found, with non-unique k-tangent measures at the origin $0 \in \operatorname{spt}\mu$ and with the density properties

(1) $\quad \theta^k_z \mu \geqslant 1 \quad$ for every $z \in \operatorname{spt}\mu \quad$ (then μ is called a concentrated measure),

(2) $$\theta^k_z \mu = 1 \quad \text{for every } z \in \operatorname{spt}\mu \setminus \{0\}$$

and

(3) $$\theta^k_0 \mu \leqslant 1 + \varepsilon.$$

There are several partial results. A well-known result is an unpublished example by Kirchheim. He studied the so called logarithmic spirals in $\mathbb{R}^2$ and proved the "local monotonicity" (there is $r_0 > 0$ depending on $|z|$ such that $r \mapsto \mu B(z,r)/r$ is non-decreasing on $(0, r_0)$) for $\mathcal{H}^1$ restricted to these curves. Then it is not difficult to show that there is a finite number of lines passing through the origin such that $\mathcal{H}^1$ restricted to the union of these lines and the symmetrical pair of logarithmic spirals is monotone (and the spirals ensure the bad tangential behavior).

An improved version of Kirchheim's result can be found in [2]. The authors gave a different proof of the "local monotonicity" which enables estimating r_0. In fact, the estimate of r_0 was accurate enough to show that for some types of logarithmic spirals only two suitably chosen lines are a sufficient compensation for the monotonicity (hence there is a monotone measure in $\mathbb{R}^2$ with bad tangential behavior satisfying (1), (2) up to the points of intersection of the spirals and the lines, and the version of (3) with $\theta_0^1\mu \leqslant 3+\varepsilon$).

Motivated by the above result, in this paper we give a version of logarithmic spirals in $\mathbb{R}^3$ such that only one line provides a sufficient compensation to the monotonicity. Thus the final measure satisfies $\theta_0^1\mu \leqslant 2+\varepsilon$, which becomes the best achieved result.

For $a, c > 0$ fixed we define a version of an anti-symmetrical pair of logarithmic spirals in $\mathbb{R}^3$ by

$$\Gamma_{a,c}^+(t) = (c\exp(at)\cos t, c\exp(at)\sin t, \exp(at)), \quad t \in \mathbb{R},$$
$$\Gamma_{a,c}^-(t) = -\Gamma_{a,c}^+(t) = (-c\exp(at)\cos t, -c\exp(at)\sin t, -\exp(at)), \quad t \in \mathbb{R}.$$

Further, we set

$$\mu_{a,c} = \mathcal{H}^1 \llcorner ([\Gamma_{a,c}^+] \cup [\Gamma_{a,c}^-]),$$

where $[\Gamma_{a,c}^+] = \{\Gamma_{a,c}^+(t) \colon t \in \mathbb{R}\}$ and $[\Gamma_{a,c}^-] = \{\Gamma_{a,c}^-(t) \colon t \in \mathbb{R}\}$.

Next, we define $L = \{(0,0,t) \colon t \in \mathbb{R}\}$. This is the line we use as a compensation for the monotonicity. Now, we can state or main result.

Theorem 1.4. *Let $\varepsilon > 0$. Then there is $K = K(\varepsilon) > 0$ such that for every $a > K$ and $c = a^{-3}$ the measure $\mu_{a,c} + \mathcal{H}^1 \llcorner L$ satisfies:*

$$\mu_{a,c} + \mathcal{H}^1 \llcorner L \quad \text{is monotone},$$
$$\mu_{a,c} + \mathcal{H}^1 \llcorner L \quad \text{does not have a unique tangent measure at the origin},$$
$$\text{the tangent measures at the origin are not conical},$$
$$\theta_z^1(\mu_{a,c} + \mathcal{H}^1 \llcorner L) = 1 \quad \text{for every } z \in (\operatorname{spt} \mu_{a,c} \cup L) \setminus \{0\},$$

and

$$\theta_0^1(\mu_{a,c} + \mathcal{H}^1 \llcorner L) < 2 + \varepsilon.$$

We refer to [4], [5] and [6] for further information concerning the geometry of measures and the Monotonicity Formula.

The paper is organized in a way similar to [2]. In the third section we study the tangential behavior. The last two sections are devoted to the proof of the monotonicity, which is the most difficult part of the proof of Theorem 1.4. We prove the monotonicity showing that the lower derivative of $r \mapsto (\mu_{a,c} + \mathcal{H}^1 \llcorner L)B(z,r)/r$ is non-negative for every pair (z,r), $z \in \mathbb{R}^3$, $r > 0$. When checking this pointwise property, we distinguish several cases. In the fourth section we consider the cases concerning z and r such that the proof of the non-negativity of the lower derivative is just a straightforward computation. The fifth section is devoted to very small radii. It was the most difficult and challenging part of our work to obtain a reasonably short proof of the monotonicity at (z,r) for very small r. Let us note that the planar method from [2, Section 5] cannot be used in our case. Our proof is finally based on a method from [1].

2. Preliminaries

Notation. The *scalar product* of $x, y \in \mathbb{R}^3$ is denoted by $x \cdot y$. The Euclidean *norm* of x is $|x|$, x_1, x_2 and x_3 are the first, the second and the third coordinates of x. We use the following notation for a ball, a sphere, a northern hemisphere and for an equator:

$$\begin{aligned} B(z,r) &= \{x \in \mathbb{R}^3 \colon \ |x - z| \leqslant r\}, \\ S(z,r) &= \{x \in \mathbb{R}^3 \colon \ |x - z| = r\}, \\ S^+(z,r) &= S(z,r) \cap \{x \in \mathbb{R}^3 \colon \ x_3 > z_3\}, \\ E(z,r) &= S(z,r) \cap \{x \in \mathbb{R}^3 \colon \ x_3 = z_3\}. \end{aligned}$$

When $z = 0$, we simply write $B(r)$, $S(r)$, $S^+(r)$ and $E(r)$.

For fixed $z \in S^+(1) \cup E(1)$ and $c > 0$ we define the function

$$\Psi_z(t) = cz_1 \cos t + cz_2 \sin t + z_3, \quad t \in \mathbb{R}.$$

Further, if $z \in S^+(1)$ and $c > 0$, let

$$s = s(z,c) = \frac{\sqrt{z_1^2 + z_2^2}}{cz_3} = \frac{\sqrt{z_1^2 + z_2^2}}{c\sqrt{1 - z_1^2 - z_2^2}}.$$

Hence

$$z_3 = \frac{1}{\sqrt{1 + s^2c^2}} \quad \text{and} \quad \sqrt{z_1^2 + z_2^2} = \frac{sc}{\sqrt{1 + s^2c^2}}. \tag{4}$$

The 1-*dimensional Hausdorff measure* is denoted by $\mathcal{H}^1$. If $A \subset \mathbb{R}^3$ is a Borel set and μ is a Radon measure, then $\mu \llcorner A$ is the restriction of μ to A, i.e. $(\mu \llcorner A)(M) = \mu(M \cap A)$. If $I \subset \mathbb{R}$ is an interval and $\Gamma \colon \ I \mapsto \mathbb{R}^3$ is a curve then $[\Gamma] = \{\Gamma(t) \colon \ t \in I\}$.

For $z \in \mathbb{R}^3$ and $r > 0$ such that $\operatorname{spt} \mu_{a,c} \cap S(z,r) \neq \emptyset$, our estimates often deal with the points of intersection with maximal and minimal third coordinates. Let

$$\xi \in \operatorname{spt} \mu_{a,c} \cap S(z,r) \quad \text{be such that } \xi_3 \geqslant \theta_3 \text{ whenever } \theta \in \operatorname{spt} \mu_{a,c} \cap S(z,r)$$

and let

$$\eta \in \operatorname{spt} \mu_{a,c} \cap S(z,r) \quad \text{be such that } \eta_3 \leqslant \theta_3 \text{ whenever } \theta \in \operatorname{spt} \mu_{a,c} \cap S(z,r).$$

Properties of $\operatorname{spt} \mu_{a,c}$**.** We denote $\alpha = \alpha(c) = \sqrt{1+c^2}$ and

$$\begin{aligned}
\beta = \beta(a,c) &= \frac{|\dot{\Gamma}^+_{a,c}(t)|}{\mathrm{d}(\Gamma^+_{a,c}(t))_3/\mathrm{d}t} \\
&= \frac{\sqrt{\big(\mathrm{d}(\Gamma^+_{a,c}(t))_1/\mathrm{d}t\big)^2 + \big(\mathrm{d}(\Gamma^+_{a,c}(t))_2/\mathrm{d}t\big)^2 + \big(\mathrm{d}(\Gamma^+_{a,c}(t))_3/\mathrm{d}t\big)^2}}{\mathrm{d}(\Gamma^+_{a,c}(t))_3/\mathrm{d}t} \\
&= \frac{\sqrt{(ca\mathrm{e}^{at}\cos t - c\mathrm{e}^{at}\sin t)^2 + (ca\mathrm{e}^{at}\sin t + c\mathrm{e}^{at}\cos t)^2 + (a\mathrm{e}^{at})^2}}{a\mathrm{e}^{at}} \\
&= \sqrt{1 + c^2 + \frac{c^2}{a^2}}.
\end{aligned}$$

Using the symmetry between $\Gamma^+_{a,c}$ and $\Gamma^-_{a,c}$ we obtain from the above

$$\mu_{a,c}(\{x \in \mathbb{R}^3 \colon\ d_1 \leqslant x_3 \leqslant d_2\}) = \beta(d_2 - d_1) \tag{5}$$

for any $d_1 \leqslant d_2$. Further, if $\zeta \in \operatorname{spt} \mu_{a,c}$, then we see that

$$\sqrt{\zeta_1^2 + \zeta_2^2} = c|\zeta_3|. \tag{6}$$

From (5) and (6) we obtain

$$\frac{\mu_{a,c}B(r)}{2r} = \frac{\mu_{a,c}\{x \in \mathbb{R}^3 \colon\ -r/\alpha \leqslant x_3 \leqslant r/\alpha\}}{2r} = \frac{\beta}{\alpha}. \tag{7}$$

Notice that $\operatorname{spt} \mu_{a,c}$ is self-similar in the sense that multiplying all three coordinates by the same constant corresponds to some rotation of the plane generated by the X and Y-axis. More precisely, if we define

$$\begin{aligned}
\Gamma^+_{a,c,t_0}(t) &= (c\mathrm{e}^{a(t-t_0)}\cos t, c\mathrm{e}^{a(t-t_0)}\sin t, \mathrm{e}^{a(t-t_0)}), \quad t \in (-\infty,\infty), \\
\Gamma^-_{a,c,t_0}(t) &= (-c\mathrm{e}^{a(t-t_0)}\cos t, -c\mathrm{e}^{a(t-t_0)}\sin t, -\mathrm{e}^{a(t-t_0)}), \quad t \in (-\infty,\infty),
\end{aligned}$$

and

$$\mu_{a,c,t_0} = \mathcal{H}^1 \llcorner ([\Gamma^+_{a,c,t_0}] \cup [\Gamma^-_{a,c,t_0}]),$$

then for every $\varrho > 0$ we have

$$\frac{1}{\varrho} T_{0,\varrho}(\mu_{a,c}) = \mu_{a,c,t_0} \quad \text{where } t_0 = \frac{\ln \varrho}{a}. \tag{8}$$

Some notes on monotonicity. Let us recall some well-known facts concerning the monotonicity of Radon measures. Let $\Gamma\colon [a,b] \mapsto \mathbb{R}^n$ be a regular C^1-curve and let $\nu = \mathcal{H}^1 \llcorner [\Gamma]$. If we want to prove that $r \mapsto \nu B(z,r)/r$ is nondecreasing on $(0,\infty)$ for some $z \in \mathbb{R}^n$, then it is enough to show that

$$\underline{\mathrm{D}}_{\mathrm{r}} \frac{\nu B(z,r)}{r} = \frac{1}{r^2}\big(r \, \underline{\mathrm{D}}_{\mathrm{r}} \, \nu B(z,r) - \nu B(z,r)\big) \tag{9}$$

is nonnegative on $(0,\infty)$. Here we use the notation $\underline{\mathrm{D}}_{\mathrm{r}} f(r) = \liminf_{\delta \to 0} (f(r+\delta) - f(r))/\delta$.

Notice that the condition $\underline{\mathrm{D}}_{\mathrm{r}}(\nu B(z,r)/r) \geqslant 0$ is satisfied when $\nu B(z,r) \leqslant 2r$ and $\Gamma(a), \Gamma(b) \notin B(z,r)$ (if $\nu B(z,r) = 0$ then the proof is trivial and if $0 < \nu B(z,r) \leqslant 2$, then there are at least two points of the intersection $S(z,r) \cap \Gamma((a,b))$ and the contribution of each of them to $\underline{\mathrm{D}}_{\mathrm{r}} \, \nu B(z,r)$ is at least 1). We use this criterion very often.

We say that a measure ν is *monotone at* (z,r) if $\underline{\mathrm{D}}_{\mathrm{r}}(\nu B(z,r)/r) \geqslant 0$. The super-additivity of the lower derivative $\underline{\mathrm{D}}_{\mathrm{r}}$ implies that a sum of monotone measures at (z,r) is again monotone at (z,r).

We need the following criterion for the monotonicity at (z,r) (see [1, Proposition 3.1] and use a suitable rescaling of the coordinates).

Proposition 2.1. *Let* $d > 0$, $\varepsilon \in (0, \frac{1}{20}]$, $\delta \in (0, \frac{1}{20} d^{-1}]$ *and let* $f, \varphi \in C^2((-\delta,\delta),\mathbb{R})$ *be functions satisfying*

$$f(0) = f'(0) = \varphi(0) = \varphi'(0) = 0,$$
$$|f''(x) - d| \leqslant \varepsilon d \quad \text{and} \quad |\varphi''(x)| \leqslant \varepsilon d \quad \text{on } (-\delta,\delta).$$

Set $\gamma(x) = (x, f(x), \varphi(x))$ *for* $x \in (-\delta,\delta)$ *and* $\mu_\gamma = \mathcal{H}^1 \llcorner (\{\gamma(x)\colon x \in (-\delta,\delta)\})$. *Then* $r \mapsto \mu_\gamma B((0,h,g),r)/r$ *is non-decreasing on* $(0,\delta)$ *for every* $h \in \mathbb{R}$ *and every* $g \in \mathbb{R}$.

We also make use of the following easy lemma.

Lemma 2.2. *Let* $f\colon (0,\infty) \mapsto [0,\infty)$ *be a non-negative non-decreasing function. If* $\underline{\mathrm{D}}_{\mathrm{r}}(f(r)/r) \geqslant 0$ *on* $(0,1) \cup (1,\infty)$, *then* $r \mapsto f(r)/r$ *is non-decreasing on* $(0,\infty)$.

P r o o f. In a standard way, it can be shown that $f(r)/r$ is non-decreasing on $(0,1)$ and on $(1,\infty)$. Hence there exist

$$A=\lim_{r\to 1_-}\frac{f(r)}{r}\quad\text{and}\quad B=\lim_{r\to 1_+}\frac{f(r)}{r}.$$

We want to show that $A\leqslant f(1)/1\leqslant B$. Let us prove the first inequality. For contradiction, suppose $A-f(1)=2\varepsilon>0$. There is $\delta\in(0,1)$ small enough so that

$$\frac{f(1-\delta)}{1-\delta}>A-\varepsilon\quad\text{and}\quad f(1)+\varepsilon>\frac{f(1)}{1-\delta}.$$

Thus, as f is non-decreasing we obtain

$$\frac{f(1)}{1-\delta}\geqslant\frac{f(1-\delta)}{1-\delta}>A-\varepsilon=f(1)+\varepsilon>\frac{f(1)}{1-\delta}.$$

This is a contradiction, hence $A\leqslant f(1)$. The inequality $f(1)\leqslant B$ is proved in the same way. □

Finally, let us show that the measure $\mathcal{H}^1\llcorner L$ is abundant in monotonicity. If $z\in S(1)$ and $r>\sqrt{z_1^2+z_2^2}$, then we have

$$\text{(10)}\qquad \underline{\mathrm{D}}_{\mathrm{r}}\frac{(\mathcal{H}^1\llcorner L)B(z,r)}{r}=\frac{\mathrm{d}}{\mathrm{d}r}\frac{2\sqrt{r^2-z_1^2-z_2^2}}{r}=2\frac{z_1^2+z_2^2}{r^2\sqrt{r^2-z_1^2-z_2^2}}\geqslant 2\frac{z_1^2+z_2^2}{r^3}.$$

If $r\in(0,\sqrt{z_1^2+z_2^2}]$, then $(\mathcal{H}^1\llcorner L)B(z,r)=0$. Hence $\underline{\mathrm{D}}_{\mathrm{r}}((\mathcal{H}^1\llcorner L)B(z,r)/r)\geqslant 0$ and thus

$$\text{(11)}\qquad \underline{\mathrm{D}}_{\mathrm{r}}\frac{(\mathcal{H}^1\llcorner L)B(z,r)}{r}\geqslant 0\quad\text{for every } r\in(0,\infty).$$

3. Tangential behavior

In this section we show that the measure $\mu_{a,c}+\mathcal{H}^1\llcorner L$ has infinitely many tangent measures at the origin. Moreover, these tangent measures are non-conical.

Proposition 3.1. *Let $a,c>0$. Then*

$$\mathrm{Tan}_0^1(\mu_{a,c}+\mathcal{H}^1\llcorner L)=\{\mu_{a,c,t_0}+\mathcal{H}^1\llcorner L\colon\ 0\leqslant t_0<2\pi\}.$$

P r o o f. Let us fix $0 \leqslant t_0 < 2\pi$. Since we plainly have

$$\mu_{a,c,t_0} = \mu_{a,c,t_0+2k\pi} \quad \text{for all } k \in \mathbb{Z}, \tag{12}$$

considering the sequence $\varrho_j = \exp(a(t_0 - 2j\pi))$, $j \in \mathbb{N}$, we obtain from (8)

$$\frac{1}{\varrho_j} T_{0,\varrho_j}(\mu_{a,c}) = \mu_{a,c,t_0-2j\pi} = \mu_{a,c,t_0}.$$

Further, we obviously have for any $\varrho > 0$

$$\frac{1}{\varrho} T_{0,\varrho}(\mathcal{H}^1 \llcorner L) = \mathcal{H}^1 \llcorner L. \tag{13}$$

Hence

$$\frac{1}{\varrho_j} T_{0,\varrho_j}(\mu_a + \mathcal{H}^1 \llcorner L) = \mu_{a,c,t_0} + \mathcal{H}^1 \llcorner L \quad \text{vaguely converges to } \mu_{a,c,t_0} + \mathcal{H}^1 \llcorner L$$

and thus

$$\operatorname{Tan}_0^1(\mu_{a,c} + \mathcal{H}^1 \llcorner L) \supset \{\mu_{a,c,t_0} + \mathcal{H}^1 \llcorner L \colon\ 0 \leqslant t_0 < 2\pi\}.$$

The reverse inclusion is obtained by a suitable choice of a test function. Assume that $\varrho_j > 0$ for $j \in \mathbb{N}$, $\varrho_j \to 0$ and $(1/\varrho_j) T_{0,\varrho_j}(\mu_{a,c} + \mathcal{H}^1 \llcorner L)$ vaguely converges. Set $t_j = \ln \varrho_j / a$. Hence from (8) and (13) we see that μ_{a,c,t_j} vaguely converges. Let $\psi\colon \mathbb{R} \mapsto [0,\infty)$ be a continuous function with a compact support satisfying $\psi(t) = 0$ for $t \leqslant 0$ and $\int_0^\infty \psi^2(t)\,\mathrm{d}t = 1$. We define on $\mathbb{R}^3$ a continuous function with a compact support by $\varphi_1(0,0,0) = 0$ and

$$\varphi_1(x) = \psi\Big(\frac{|x|}{\alpha}\Big)\psi(x_3)\Big(\frac{x_1}{c|x|}\cos\Big(\frac{\ln|x|}{a}\Big) + \frac{x_2}{c|x|}\sin\Big(\frac{\ln|x|}{a}\Big)\Big) \quad \text{for } |x| > 0.$$

For $x \in \operatorname{spt}\mu_{a,c,t_j} \setminus [\Gamma^+_{a,c,t_j}]$ we have $\varphi_1(x) = 0$ and if $x = \Gamma^+_{a,c,t_j}(t)$ for some $t \in \mathbb{R}$, then we have

$$\varphi_1(x) = \psi\Big(\frac{|x|}{\alpha}\Big)\psi(x_3)(\cos t\cos(t - t_j) + \sin t\sin(t - t_j)) = \psi^2(x_3)\cos t_j.$$

Hence we obtain from (5)

$$\int_{\mathbb{R}^3} \varphi_1\,\mathrm{d}\mu_{a,c,t_j} = \int_0^\infty \beta\psi^2(t)\cos t_j\,\mathrm{d}t = \beta\cos t_j.$$

Therefore $\cos t_j$ converges. If $\cos t_j \to 1$, then from (12) we see that $\mu_{a,c,t_j} \to \mu_{a,c,0} = \mu_{a,c}$ vaguely. Similarly, if $\cos t_j \to -1$, then $\mu_{a,c,t_j} \to \mu_{a,c,\pi}$ vaguely.

Finally, if $\cos t_j \to d \in (-1, 1)$, then there is $t_0 \in (0, \pi)$ such that $\cos t_j \to \cos t_0 = \cos(2\pi - t_0)$. Let us set $\varphi_2(0, 0, 0) = 0$ and

$$\varphi_2(x) = \psi\Big(\frac{|x|}{\alpha}\Big)\psi(x_3)\Big(\frac{x_1}{c|x|}\cos\Big(\frac{\ln|x|}{a} + t_0\Big) + \frac{x_2}{c|x|}\sin\Big(\frac{\ln|x|}{a} + t_0\Big)\Big) \quad \text{for } |x| > 0,$$

where the function ψ is the same as above. This time we obtain for $x = \Gamma^+_{a,c,t_j}(t)$

$$\varphi_2(x) = \psi\Big(\frac{|x|}{\alpha}\Big)\psi(x_3)(\cos t\cos(t - t_j + t_0) + \sin t\sin(t - t_j + t_0)) = \psi^2(x_3)\cos(t_j - t_0).$$

The vague convergence implies in the same way as above that $\cos(t_j - t_0)$ converges. If $\cos(t_j - t_0) \to 1$, then (12) implies $\mu_{a,c,t_j} \to \mu_{a,c,t_0}$ vaguely. Otherwise, since $\cos(t_j - t_0) \to b \neq 1$ and $\cos t_j \to \cos(2\pi - t_0)$, using (12) we obtain $\mu_{a,c,t_j} \to \mu_{a,c,2\pi-t_0}$ vaguely. Hence we have the remaining inclusion

$$\operatorname{Tan}^1_0(\mu_{a,c} + \mathcal{H}^1 \llcorner L) \subset \{\mu_{a,c,t_0} + \mathcal{H}^1 \llcorner L\colon\ 0 \leqslant t_0 < 2\pi\}.$$

□

4. Large radii: monotonicity by compensation

Because of the self-similarity of our logarithmic spirals it is enough to prove monotonicity at (z, r) only for $z \in S^+(1) \cup E(1) \cup \{0\}$ and $r > 0$. In the case of large radii, we carefully estimate each term on the right-hand side of (9) for $\nu = \mu_{a,c}$.

Proposition 4.1. *There is $K_1 > 0$ with the following property: If $a \geqslant K_1$, $c = a^{-3}$ and if one of the following conditions is satisfied*

(i) $z = 0$ *and* $r \in (0, \infty)$,
(ii) $z \in E(1)$ *and* $r \in (0, \infty) \setminus \{1\}$,
(iii) $z \in S^+(1)$, *with* $s(z) \notin (\frac{1}{2}, 2)$, *and* $r \in (0, \infty) \setminus \{1\}$,
(iv) $z \in S^+(1)$, *with* $s(z) \in (\frac{1}{2}, 2)$, *and* $r \in (4c, \infty) \setminus \{1\}$,

then the measure $\mu_{a,c} + \mathcal{H}^1 \llcorner L$ is monotone at (z, r).

Note that Proposition 4.1 is in fact satisfied with $K_1 = 1000$. Therefore we prefer to state and prove our auxiliary lemmata with the assumption $a \geqslant 1000$ rather then always warn the reader that we pass to a sufficiently large.

The monotonicity of $\mu_{a,c}$ at (z, r) is easily obtained when the center z is far enough from $\operatorname{spt} \mu_{a,c}$.

Lemma 4.2. *Let $a \geqslant 1000$, $c = a^{-3}$ and $z \in S^+(1) \cup E(1)$.*

(i) *If $\sqrt{z_1^2 + z_2^2} \in (\frac{1}{2}c, \frac{3}{2}c)$ and $r \in (0, \sqrt{z_1^2 + z_2^2} + 2c]$, then $r < 4c$.*

(ii) *If $\sqrt{z_1^2 + z_2^2} \notin (\frac{1}{2}c, \frac{3}{2}c)$ and $r \in (0, \sqrt{z_1^2 + z_2^2} + 2c]$, then $\mu_{a,c}$ is monotone at (z, r).*

(iii) *If $\sqrt{z_1^2 + z_2^2} \geqslant 4c$ and $r \in (\sqrt{z_1^2 + z_2^2} + 2c, \frac{11}{10}]$, then $\mu_{a,c}$ is monotone at (z, r).*

Proof. Assumptions of (i) imply

$$r \leqslant \sqrt{z_1^2 + z_2^2} + 2c \leqslant \frac{3}{2}c + 2c < 4c.$$

Thus, we have proved (i) and it remains to prove (ii) and (iii).

We can suppose that $\mu_{a,c}B(z, r) > 0$ (otherwise the proof easily follows from (9)). Therefore the points ξ and η (see Preliminaries for the definition) are well defined, $\underline{\mathrm{D}}_{\mathrm{r}}\, \mu_{a,c}B(z, r) \geqslant 2$ and, in addition, (6) implies

$$\sqrt{\eta_1^2 + \eta_2^2} = c|\eta_3| \leqslant c(|z_3| + r) \leqslant c(|z| + r) = c(1 + r). \tag{14}$$

Our next step is to prove the estimate

$$\left|\sqrt{z_1^2 + z_2^2} - \sqrt{\eta_1^2 + \eta_2^2}\right| \geqslant \frac{4}{3}cr. \tag{15}$$

We distinguish four cases. If $\sqrt{z_1^2 + z_2^2} \in [0, \frac{1}{2}c]$ and $r \in \big(0, \sqrt{z_1^2 + z_2^2} + 2c\big]$, then we have $r \leqslant 3c$,

$$z_3 = \sqrt{1 - z_1^2 - z_2^2} \geqslant \sqrt{1 - (\tfrac{1}{2}c)^2} \geqslant 1 - c$$

and $\eta_3 \geqslant z_3 - r \geqslant 1 - 4c$. Hence from (6) we obtain

$$\sqrt{\eta_1^2 + \eta_2^2} = c|\eta_3| \geqslant c(1 - 4c) \geqslant \frac{3}{4}c.$$

Therefore

$$\sqrt{\eta_1^2 + \eta_2^2} - \sqrt{z_1^2 + z_2^2} \geqslant \frac{3}{4}c - \frac{1}{2}c = \frac{1}{4}c \geqslant \frac{1}{4}c \cdot \frac{r}{3c} = \frac{1}{12}r \geqslant \frac{4}{3}cr.$$

If $\sqrt{z_1^2 + z_2^2} \in [\frac{3}{2}c, \frac{1}{5}]$ and $r \in \big(0, \sqrt{z_1^2 + z_2^2} + 2c\big]$, then we obtain from (14)

$$\sqrt{\eta_1^2 + \eta_2^2} \leqslant c(1 + r) \leqslant c\Big(1 + \frac{1}{5} + 2c\Big) \leqslant \frac{7}{5}c$$

and thus

$$\sqrt{z_1^2 + z_2^2} - \sqrt{\eta_1^2 + \eta_2^2} \geqslant \frac{3}{2}c - \frac{7}{5}c = \frac{1}{10}c \geqslant \frac{1}{10}c \cdot \frac{r}{\frac{1}{5} + 2c} \geqslant \frac{4}{10}r \geqslant \frac{4}{3}cr.$$

If $\sqrt{z_1^2+z_2^2} \in [\frac{1}{5}, 1]$ and $r \in (0, \sqrt{z_1^2+z_2^2}+2c]$, then (14) implies $\sqrt{\eta_1^2+\eta_2^2} \leqslant 3c$. Thus

$$\sqrt{z_1^2+z_2^2} - \sqrt{\eta_1^2+\eta_2^2} \geqslant \frac{1}{5} - 3c \geqslant \frac{1}{6} \geqslant \frac{1}{6}\frac{r}{1+2c} \geqslant \frac{1}{7}r \geqslant \frac{4}{3}cr.$$

Finally, if $\sqrt{z_1^2+z_2^2} \geqslant 4c$ and $r \in \left(\sqrt{z_1^2+z_2^2}+2c, \frac{11}{10}\right]$, then using (14) we arrive at $\sqrt{\eta_1^2+\eta_2^2} \leqslant \frac{5}{2}c$. Therefore

$$\sqrt{z_1^2+z_2^2} - \sqrt{\eta_1^2+\eta_2^2} \geqslant 4c - \frac{5}{2}c = \frac{3}{2}c \geqslant \frac{3}{2}c \cdot \frac{r}{\frac{11}{10}} \geqslant \frac{4}{3}cr.$$

Since our four cases cover the assumptions of (ii) and (iii), we have proved (15). Consequently,

$$\begin{aligned}|\eta_3 - z_3| &\leqslant \sqrt{r^2 - \left|\sqrt{\eta_1^2+\eta_2^2} - \sqrt{z_1^2+z_2^2}\right|^2} \\ &\leqslant \sqrt{r^2 - \left(\frac{4}{3}cr\right)^2} = r\sqrt{1 - \frac{16}{9}c^2} < \frac{r}{\beta}.\end{aligned}$$

In the same way we obtain $|\xi_3 - z_3| < r/\beta$. Hence $|\xi_3 - \eta_3| < 2r/\beta$ and thus from (9), (5), and $\underline{\mathrm{D}}_{\mathrm{r}}\, \mu_{a,c}B(z,r) \geqslant 2$ we conclude

$$\underline{\mathrm{D}}_{\mathrm{r}}\, \frac{\mu_{a,c}B(z,r)}{r} \geqslant \frac{1}{r^2}\left(2r - 2\beta\frac{2r}{\beta}\right) \geqslant 0.$$

□

The rest of this section is devoted to the difficult case when the center z is not so far from $\operatorname{spt}\mu_{a,c}$.

Let us briefly outline our strategy. Since there are always at least two points of the intersection $B(z,r) \cap \operatorname{spt}\mu_{a,c}$ in our case (i.e. ξ and η are well defined), from (5) and (9) we obtain (28). Next, we estimate all the terms on the right-hand side of (28) using the identities from Lemma 4.4. Notice that when estimating $\mathrm{d}\xi_3/\mathrm{d}r$ (and similarly $\mathrm{d}(-\eta_3)/\mathrm{d}r$), we do not use the explicit formula (23) (which is not convenient to work with), but we proceed in the following way. First, we obtain a rough estimate (see Lemma 4.5). Then we use formula (22) together with this rough estimate on the right-hand side (where $\mathrm{d}\xi_3/\mathrm{d}r$ is multiplied by a^{-1}, which can be made very small).

We start with some estimates concerning the function $\Psi_z(t)$.

Lemma 4.3. *Assume* $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1) \cup E(1)$, $r \in \big[\sqrt{z_1^2 + z_2^2} + 2c, \infty\big)$ *and* $t \in \mathbb{R}$. *Then we have*

$$|\Psi_z(t)| \leqslant \alpha, \tag{16}$$

$$|\Psi_z'(t)| \leqslant c\sqrt{z_1^2 + z_2^2}, \tag{17}$$

$$\sqrt{\Psi_z^2(t) + \alpha^2(r^2 - 1)} \leqslant \alpha r, \tag{18}$$

$$\sqrt{\Psi_z^2(t) + \alpha^2(r^2 - 1)} - \frac{1}{a}|\Psi_z'(t)| > 0. \tag{19}$$

If moreover either $r \geqslant 4c$ *and* $z \in S^+(1)$ *with* $0 \leqslant s \leqslant 2$ *or* $r \geqslant \frac{11}{10}$, *then*

$$\sqrt{\Psi_z^2(t) + \alpha^2(r^2 - 1)} \geqslant \frac{1}{3}\alpha r. \tag{20}$$

P r o o f. By the Schwartz inequality for the scalar product we have

$$\begin{aligned}\max_{\substack{z\in S^+(1)\\ t\in\mathbb{R}}} |\Psi_z(t)| &= \max_{\substack{z\in S^+(1)\\ t\in\mathbb{R}}} |c(z_1, z_2) \cdot (\cos t, \sin t) + z_3| \\ &= \max_{z\in S(1)} \Big|c\sqrt{z_1^2 + z_2^2} + z_3\Big| = \max_{z\in S(1)} \Big|(c, 1) \cdot \Big(\sqrt{z_1^2 + z_2^2}, z_3\Big)\Big| \\ &= \sqrt{1 + c^2} = \alpha.\end{aligned}$$

Hence we have proved (16), and (18) follows. Further, we obtain (17) from

$$|\Psi_z'(t)| = c|(z_1, z_2) \cdot (-\sin t, \cos t)| \leqslant c\sqrt{z_1^2 + z_2^2}.$$

Let us prove (19). First, for $z \in E(1)$ we have $r \geqslant 1 + 2c$ and thus (17) implies

$$\begin{aligned}\sqrt{\Psi_z^2(t) + \alpha^2(r^2 - 1)} &\geqslant \sqrt{\alpha^2(r^2 - 1)} \\ &\geqslant \sqrt{\alpha^2((1 + 2c)^2 - 1)} \geqslant 2\alpha c > \frac{c}{a} \geqslant \frac{1}{a}|\Psi_z'(t)|\end{aligned}$$

and we are done. In the case $z \in S^+(1)$ we have by (4)

$$\Psi_z^2(t) \geqslant \Big(z_3 - c\sqrt{z_1^2 + z_2^2}\Big)^2 = \Big(\frac{1}{\sqrt{1 + s^2c^2}} - \frac{sc^2}{\sqrt{1 + s^2c^2}}\Big)^2 = \frac{(1 - sc^2)^2}{1 + s^2c^2}.$$

Further, from (4) and

$$(s + 2)^2c^2(1 + c^2) = (s + 2)^2c^4 + (s + 2)^2c^2 \geqslant \frac{s^2c^4}{a^2} + (s + 1)^2c^2$$

we see that

$$r^2 \geqslant \Big(\sqrt{z_1^2+z_2^2}+2c\Big)^2 = \Big(\frac{sc}{\sqrt{1+s^2c^2}}+2c\Big)^2$$
$$\geqslant \frac{(s+2)^2c^2}{1+s^2c^2} > \frac{s^2c^4/a^2+(s+1)^2c^2}{(1+c^2)(1+s^2c^2)}.$$

Hence, as $\alpha^2 = 1+c^2$, we obtain from (4), (17) and from the above estimates

$$\frac{1}{a^2}\Psi_z'^2(t) \leqslant \frac{s^2c^4}{a^2(1+s^2c^2)} = \frac{(1-sc^2)^2}{1+s^2c^2} + \alpha^2\Big(\frac{s^2c^4/a^2+(s+1)^2c^2}{\alpha^2(1+s^2c^2)} - 1\Big)$$
$$< \Psi_z^2(t) + \alpha^2(r^2-1).$$

This proves (19). Finally, let us prove the last assertion. If $r \geqslant \frac{11}{10}$, then

$$\sqrt{\Psi_z^2(t)+\alpha^2(r^2-1)} \geqslant \sqrt{\alpha^2(r^2-1)} = \alpha r\sqrt{1-\frac{1}{r^2}} \geqslant \alpha r\sqrt{1-\Big(\frac{10}{11}\Big)^2} > \frac{1}{3}\alpha r.$$

If $r \geqslant 4c$ and $z \in S^+(1)$ with $s \leqslant 2$, then we have from (4)

$$\frac{\Psi_z^2(t)}{\alpha^2} \geqslant \frac{1}{\alpha^2}\Big(z_3 - c\sqrt{z_1^2+z_2^2}\Big)^2 = \frac{1}{\alpha^2}\Big(\frac{1}{\sqrt{1+s^2c^2}} - \frac{sc^2}{\sqrt{1+s^2c^2}}\Big)^2$$
$$= \frac{(1-sc^2)^2}{(1+c^2)(1+s^2c^2)} \geqslant \frac{(1-2c^2)^2}{(1+c^2)(1+4c^2)} > 1-10c^2.$$

Therefore

$$\sqrt{\Psi_z^2(t)+\alpha^2(r^2-1)} \geqslant \alpha\sqrt{1-10c^2+r^2-1}$$
$$= \alpha r\sqrt{1-\frac{10c^2}{r^2}} \geqslant \alpha r\sqrt{1-\frac{10c^2}{16c^2}} > \frac{1}{3}\alpha r.$$

□

The following lemma enables us to use the parameterization of our logarithmic spirals when estimating the right-hand side of (9).

Lemma 4.4. *Assume $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1)\cup E(1)$. If $r > \sqrt{z_1^2+z_2^2}+2c$, then $S(z,r)\cap \operatorname{spt}\mu_{a,c} \neq \emptyset$ and thus the points ξ and η are well defined. The function $r \mapsto \xi_3$ is continuously differentiable on $\big(\sqrt{z_1^2+z_2^2}+2c, \infty\big)$ and satisfies*

$$\xi_3 = \frac{1}{\alpha^2}\Big(\Psi_z(\tau) + \sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}\Big), \tag{21}$$

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}r} = \frac{a^{-1}\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r + r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}, \tag{22}$$

and

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}r} = \frac{r}{\sqrt{\Psi_z^2(\tau) + \alpha^2(r^2-1)} - a^{-1}\Psi_z'(\tau)}, \tag{23}$$

where $\tau \in \mathbb{R}$ *is such that* $\xi = \Gamma_{a,c}^+(\tau)$. *The function* $r \mapsto \eta_3$ *is continuously differentiable on* $\big(\sqrt{z_1^2+z_2^2}+2c, \infty\big) \setminus \{1\}$ *and satisfies*

$$\eta_3 = \frac{1}{\alpha^2}\big(\Psi_z(\sigma) - \sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}\big), \tag{24}$$

$$\frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r} = \frac{-a^{-1}\Psi_z'(\sigma)\,\mathrm{d}(-\eta_3)/\mathrm{d}r + r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}, \tag{25}$$

and

$$\frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r} = \frac{r}{\sqrt{\Psi_z^2(\sigma) + \alpha^2(r^2-1)} + a^{-1}\Psi_z'(\sigma)}, \tag{26}$$

where $\sigma \in \mathbb{R}$ *is such that* $\eta = \Gamma_{a,c}^+(\sigma)$ *provided* $r \in (0,1) \cap \big(\sqrt{z_1^2+z_2^2}+2c, \infty\big)$ *and* $\eta = \Gamma_{a,c}^-(\sigma)$ *provided* $r \in (1,\infty) \cap \big(\sqrt{z_1^2+z_2^2}+2c, \infty\big)$.

P r o o f. If $z \in S^+(1)$, let us set $\theta = \log(|z_3|)/a$. Then $\theta \leqslant 0$, $(\Gamma_{a,c}^+(\theta))_3 = z_3$ and for any $r \in \big(\sqrt{z_1^2+z_2^2}+2c, \infty\big)$ we have by (6)

$$|\Gamma_{a,c}^+(\theta) - z| \leqslant \sqrt{(\Gamma_{a,c}^+(\theta))_1^2 + (\Gamma_{a,c}^+(\theta))_2^2} + \sqrt{z_1^2+z_2^2} = cz_3 + \sqrt{z_1^2+z_2^2} < r.$$

Hence $\operatorname{spt}\mu_{a,c} \cap S(z,r)$ contains at least two points and thus ξ and η are well defined. Similarly for $z \in E(1)$.

Using $\xi = \Gamma_{a,c}^+(\tau) = (c\xi_3\cos\tau, c\xi_3\sin\tau, \xi_3)$ we set

$$\begin{aligned} F(r,\tau) = |\xi - z|^2 - r^2 &= (c\xi_3\cos\tau - z_1)^2 + (c\xi_3\sin\tau - z_2)^2 + (\xi_3 - z_3)^2 - r^2 \\ &= \alpha^2\xi_3^2 - 2\xi_3(cz_1\cos\tau + cz_2\sin\tau + z_3) + 1 - r^2 \\ &= \alpha^2\xi_3^2 - 2\xi_3\Psi_z(\tau) + 1 - r^2. \end{aligned}$$

Solving the equation $F(r,\tau) = 0$ with respect to ξ_3 we obtain that ξ_3 satisfies either (21) or

$$\xi_3 = \alpha^{-2}\big(\Psi_z(\tau) - \sqrt{\Psi_z^2(\tau) + \alpha^2(r^2-1)}\big). \tag{27}$$

Let us show that formula (27) cannot be satisfied. Recall that $\Gamma_{a,c}^+(\theta) \in B(z,r)$, thus $\xi_3 \geqslant z_3$. Our aim is to show that formula (27) implies $\xi_3 < z_3$. We distinguish three cases. If $r > 1$, we observe that (27) implies $\xi_3 < 0 \leqslant z_3$, hence (27) cannot be satisfied. If $r \in \left(\sqrt{z_1^2+z_2^2}+2c, 1\right]$ and $\Psi_z(\tau) \leqslant z_3$, then from (27) we obtain $\xi_3 \leqslant \alpha^{-2} z_3 < z_3$. Finally, if $r \in \left(\sqrt{z_1^2+z_2^2}+2c, 1\right]$ and $\Psi_z(\tau) > z_3$, then (27) implies

$$\begin{aligned}\xi_3 &= \frac{1-r^2}{\Psi_z(\tau)+\sqrt{\Psi_z^2(\tau)-\alpha^2(r^2-1)}}\\ &\leqslant \frac{1-\left(\sqrt{z_1^2+z_2^2}+2c\right)^2}{\Psi_z(\tau)} < \frac{1-z_1^2-z_2^2}{z_3} = z_3.\end{aligned}$$

Hence (27) cannot be satisfied. Thus we have proved (21).

The smoothness, (22) and (23) follow from the Implicit Function Theorem. Indeed, $\mathrm{d}\xi_3/\mathrm{d}\tau = \mathrm{d}(\exp(a\tau))/\mathrm{d}\tau = a\exp(a\tau) = a\xi_3$ implies

$$\frac{\partial F}{\partial \tau} = 2\alpha^2 a\xi_3^2 - 2a\xi_3\Psi_z(\tau) - 2\xi_3\Psi_z'(\tau) = 2a\xi_3\Big(\alpha^2\xi_3 - \Psi_z(\tau) - \frac{1}{a}\Psi_z'(\tau)\Big).$$

Hence applying (19) and (21) we obtain

$$\frac{\partial F}{\partial \tau} = 2a\xi_3\Big(\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)} - \frac{1}{a}\Psi_z'(\tau)\Big) > 0.$$

Further, $\partial F/\partial r = -2r$, $\mathrm{d}\xi_3/\mathrm{d}\tau = a\xi_3$ and the above formula for $\partial F/\partial \tau$ imply

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}r} = \frac{\mathrm{d}\xi_3}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}r} = \frac{\mathrm{d}\xi_3}{\mathrm{d}\tau}(-1)\frac{\partial F/\partial r}{\partial F/\partial \tau} = a\xi_3\frac{2r}{2a\xi_3\left(\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)} - a^{-1}\Psi_z'(\tau)\right)}$$

and we have proved (23). As $\alpha = \beta/(\gamma+\delta)$ is equivalent to $\alpha = (\beta-\delta\alpha)/\gamma$ provided $\gamma+\delta \neq 0 \neq \gamma$, (22) follows from (23).

For the point of intersection η, the proof is similar. Because of the symmetry between $\Gamma_{a,c}^+$ and $\Gamma_{a,c}^-$ we obtain the same formulae in both cases $\eta \in S(z,r)\cap[\Gamma_{a,c}^+]$ and $\eta \in S(z,r)\cap[\Gamma_{a,c}^-]$. Since the Implicit Function Theorem requires $\eta_3 \neq 0$, the additional assumption $r \neq 1$ occurs. Notice that the assumption $r > \sqrt{z_1^2+z_2^2}+2c$ ensures that we do not consider any case with $S(z,r)\cap[\Gamma_{a,c}^-] \neq \emptyset$ for $r<1$. $\square$

Hence if $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1)\cup E(1)$ and $r \in \left(\sqrt{z_1^2+z_2^2}+2c, \infty\right)\setminus\{1\}$, then (5), (21), (22), (24) and (25) imply

(28) $$\begin{aligned}\underline{\mathrm{D}}_{\mathrm{r}}\frac{\mu_{a,c}B(z,r)}{r} &\geqslant \frac{\beta}{r^2}\Big(\Big(\frac{\mathrm{d}\xi_3}{\mathrm{d}r}+\frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r}\Big)r - (\xi_3-\eta_3)\Big)\\ &= \frac{\beta}{\alpha^2 r^2}(U_1+U_2+U_3+V_1+V_2+V_3),\end{aligned}$$

where

$$\begin{aligned}
U_1 &= \frac{r}{a}\frac{\alpha^2\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}},\\
U_2 &= -\Psi_z(\tau)+z_3,\\
U_3 &= \frac{\alpha^2-\Psi_z^2(\tau)}{\sqrt{\Psi_z^2(\tau)^2+\alpha^2(r^2-1)}},\\
V_1 &= \frac{r}{a}\frac{-\alpha^2\Psi_z'(\sigma)\,\mathrm{d}(-\eta_3)/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}},\\
V_2 &= \Psi_z(\sigma)-z_3,\\
V_3 &= \frac{\alpha^2-\Psi_z^2(\sigma)}{\sqrt{\Psi_z^2(\sigma)+\alpha^2\,(r^2-1)}}.
\end{aligned}$$

Next, we need to obtain suitable estimates of U_1, U_2, U_3, V_1, V_2 and V_3.

Lemma 4.5. *Let* $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1) \cup E(1)$ *and* $r \in \left(\sqrt{z_1^2+z_2^2}+2c, \infty\right) \setminus \{1\}$. *Then*

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}r} \geqslant \frac{1}{\beta} \quad \text{and} \quad \frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r} \geqslant \frac{1}{\beta}.$$

If, moreover, $\mathrm{d}\xi_3/\mathrm{d}r \geqslant 2$ *or* $\mathrm{d}(-\eta_3)/\mathrm{d}r \geqslant 2$, *then* $\mu_{a,c}$ *is monotone at* (z,r).

Proof. By Lemma 4.4 we observe that ξ_3, η_3, $\mathrm{d}\xi_3/\mathrm{d}r$ and $\mathrm{d}(-\eta_3)/\mathrm{d}r$ as functions with respect to r are well defined on $\left(\sqrt{z_1^2+z_2^2}+2c,\infty\right)\setminus\{1\}$. The estimates of $\mathrm{d}\xi_3/\mathrm{d}r$ and $\mathrm{d}(-\eta_3)/\mathrm{d}r$ plainly follow from (5) and from the fact that each point of the intersection $\operatorname{spt}\mu_{a,c}\cap B(z,r)$ contributes to $\mathrm{d}\mu_{a,c}B(z,r)/\mathrm{d}r$ at least by 1.

Further, if $\mathrm{d}\xi_3/\mathrm{d}r \geqslant 2$ or $\mathrm{d}(-\eta_3)/\mathrm{d}r \geqslant 2$, then we have from the first part of the lemma that $\mathrm{d}\xi_3/\mathrm{d}r + \mathrm{d}(-\eta_3)/\mathrm{d}r \geqslant 2 + 1/\beta$ and thus (28) gives

$$\begin{aligned}
\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_{a,c}B(z,r)}{r} &\geqslant \frac{\beta}{r^2}\Bigl(\Bigl(\frac{\mathrm{d}\xi_3}{\mathrm{d}r}+\frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r}\Bigr)r-(\xi_3-\eta_3)\Bigr)\\
&\geqslant \frac{\beta}{r^2}\Bigl(\Bigl(2+\frac{1}{\beta}\Bigr)r-2r\Bigr)>0.
\end{aligned}$$

□

Lemma 4.6. *Let* $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1) \cup E(1)$ *and* $r \in \left(\sqrt{z_1^2+z_2^2}+2c, \infty\right) \setminus \{1\}$. *Then*

$$U_3 \geqslant 0 \quad \text{and} \quad V_3 \geqslant 0.$$

Proof. The proof follows from (16) and from the definition of U_3 and V_3. □

Lemma 4.7. *Assume* $a \geqslant 1000$, $c = a^{-3}$, $z \in S^{+}(1)$, *with* $s \leqslant 5$, *and* $r \in \left(\sqrt{z_1^2+z_2^2}+2c, \frac{11}{10}\right] \setminus \{1\}$. *Then*

$$U_2+U_3+V_2+V_3 \geqslant \frac{(s-1)^2c^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}} + \frac{(s-1)^2c^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\sigma)^2+\alpha^2(r^2-1)}}.$$

Proof. First, in view of the Schwartz inequality for the scalar product and of (4) we can write

$$\Psi_z(\tau)-z_3 = cz_1\cos\tau + cz_2\sin\tau = c(z_1,z_2)\cdot(\cos\tau,\sin\tau) = \frac{sc^2u}{\sqrt{1+s^2c^2}}$$

for some $u \in [-1,1]$. Therefore

$$U_2+U_3 \geqslant \min_{u\in[-1,1]}\left(\frac{-sc^2u}{\sqrt{1+s^2c^2}} + \frac{\alpha^2-\left(1/\sqrt{1+s^2c^2}+sc^2u/\sqrt{1+s^2c^2}\right)^2}{\sqrt{\Psi_z^2(\tau)-\alpha^2(r^2-1)}}\right).$$

The minimum is attained for $u=1$, hence (recall $\alpha^2 = 1+c^2$)

$$\begin{aligned}(29)\qquad U_2+U_3 &\geqslant \frac{-sc^2}{\sqrt{1+s^2c^2}} + \frac{\alpha^2-(1+sc^2)^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\tau)-\alpha^2(r^2-1)}}\\ &= \frac{-sc^2}{\sqrt{1+s^2c^2}} + \frac{(s-1)^2c^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\tau)-\alpha^2(r^2-1)}}.\end{aligned}$$

The estimate of V_2+V_3 is similar, though we have to estimate the minimum a bit more carefully. Indeed, first we observe $\alpha r\sqrt{1+s^2c^2} < 2$ and thus from (18) we obtain

$$(30)\qquad \frac{sc^2}{\sqrt{1+s^2c^2}} \leqslant \frac{2sc^2/(1+s^2c^2)}{\alpha r} \leqslant \frac{2sc^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}.$$

Next, as above we can write

$$\Psi_z(\sigma)-z_3 = cz_1\cos\sigma + cz_2\sin\sigma = \frac{sc^2u}{\sqrt{1+s^2c^2}},$$

where $u\in[-1,1]$. Hence we have

$$V_2+V_3 \geqslant \min_{u\in[-1,1]}\left(\frac{sc^2u}{\sqrt{1+s^2c^2}} + \frac{\alpha^2-\left(1/\sqrt{1+s^2c^2}+sc^2u/\sqrt{1+s^2c^2}\right)^2}{\sqrt{\Psi_z^2(\tau)-\alpha^2(r^2-1)}}\right).$$

By (30) we can see that the minimum is attained for $u=1$ again and we obtain

$$\begin{aligned}(31)\qquad V_2+V_3 &\geqslant \frac{sc^2}{\sqrt{1+s^2c^2}} + \frac{\alpha^2-(1+sc^2)^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\sigma)-\alpha^2(r^2-1)}}\\ &= \frac{sc^2}{\sqrt{1+s^2c^2}} + \frac{(s-1)^2c^2/(1+s^2c^2)}{\sqrt{\Psi_z^2(\sigma)-\alpha^2(r^2-1)}}.\end{aligned}$$

Now, summing estimates (29) and (31) up we obtain the assertion. □

Lemma 4.8. *Assume* $a \geqslant 1000$, $c = a^{-3}$, $z \in S^+(1) \cup E(1)$, $r \geqslant \frac{11}{10}$, $\mathrm{d}\xi_3/\mathrm{d}r \leqslant 2$ *and* $\mathrm{d}(-\eta_3)/\mathrm{d}r \leqslant 2$. *Then*

$$U_1 + V_1 \geqslant -\frac{c\sqrt{z_1^2 + z_2^2}}{a(r-1)}$$

and

$$U_2 + V_2 \geqslant -\frac{4c\sqrt{z_1^2 + z_2^2}}{a(r-1)}.$$

P r o o f. Notice that for $r \geqslant \frac{11}{10}$ we can use Lemma 4.4. Since $r > 1$ and $z \in S(1)$, we have $\xi_3, |\eta_3| \in [r-1, r+1]$. Set $\delta = |\tau - \sigma|$ (recall $\xi = \Gamma^+_{a,c}(\tau)$, $\eta = \Gamma^-_{a,c}(\sigma)$). We obtain

$$\begin{aligned} (32) \qquad 0 \leqslant \delta &= \Big|\frac{1}{a}\ln \xi_3 - \frac{1}{a}\ln|\eta_3|\Big| \\ &\leqslant \frac{1}{a}\ln\Big(\frac{r+1}{r-1}\Big) = \frac{1}{a}\ln\Big(1 + \frac{2}{r-1}\Big) \leqslant \frac{2}{a(r-1)}. \end{aligned}$$

Further, we plainly have

$$\begin{aligned} (33) \qquad |\Psi_z(\tau) - \Psi_z(\sigma)| &= |cz_1(\cos\tau - \cos\sigma) + cz_2(\sin\tau - \sin\sigma)| \\ &\leqslant 2c\sqrt{z_1^2 + z_2^2}\,\delta, \\ (34) \qquad |\Psi_z'(\tau) - \Psi_z'(\sigma)| &= |cz_1(-\sin\tau + \sin\sigma) + cz_2(\cos\tau - \cos\sigma)| \\ &\leqslant 2c\sqrt{z_1^2 + z_2^2}\,\delta. \end{aligned}$$

Since for $a, b \in [B, \infty]$ we have $|1/a - 1/b| = |b^2 - a^2|/(ab(a+b)) \leqslant |b^2 - a^2|/2B^3$, we obtain from (16), (20), (34) and (35)

$$\begin{aligned} (35) \qquad \Big|\frac{1}{\sqrt{\Psi_z^2(\tau) + \alpha^2(r^2-1)}} - \frac{1}{\sqrt{\Psi_z^2(\sigma) + \alpha^2(r^2-1)}}\Big| &\leqslant \frac{|\Psi_z^2(\sigma) - \Psi_z^2(\tau)|}{\frac{2}{27}\alpha^3 r^3} \\ &\leqslant \frac{|\Psi_z(\sigma) + \Psi_z(\tau)||\Psi_z(\sigma) - \Psi_z(\tau)|}{\frac{2}{27}\alpha^3 r^3} \\ &\leqslant \frac{2\alpha 2c\sqrt{z_1^2 + z_2^2}\,\delta}{\frac{2}{27}\alpha^3 r^3} < \frac{c}{2r(r-1)}. \end{aligned}$$

Now, (17), (20), (22), (25), (34), (36) and (37) imply

$$(38)\qquad \Big|\frac{\mathrm{d}\xi_3}{\mathrm{d}r}-\frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r}\Big|$$

$$=\Big|\Big(\frac{r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}-\frac{r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big)$$

$$+\frac{1}{a}\Big(\Big(\frac{\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}-\frac{\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big)$$

$$+\frac{(\Psi_z'(\tau)-\Psi_z'(\sigma))\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}+\frac{\Psi_z'(\sigma)(\mathrm{d}\xi_3/\mathrm{d}r+\mathrm{d}(-\eta_3)/\mathrm{d}r)}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big)\Big|$$

$$\leqslant\frac{c}{2(r-1)}+\frac{1}{a}\Big(c\sqrt{z_1^2+z_2^2}\cdot 2\,\frac{c}{2r(r-1)}$$

$$+\frac{2c\sqrt{z_1^2+z_2^2}\,\delta\,2}{\frac13\alpha r}+\frac{c\sqrt{z_1^2+z_2^2}\,4}{\frac13\alpha r}\Big)\leqslant\frac{c}{r-1}.$$

Next, using (17), (20), (34), (36), (37) and (38) we obtain

$$|U_1+V_1|=\frac{r}{a}\Big|\frac{\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}-\frac{\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}$$

$$+\frac{(\Psi_z'(\tau)-\Psi_z'(\sigma))\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}+\frac{\Psi_z'(\sigma)\big(\mathrm{d}\xi_3/\mathrm{d}r-\mathrm{d}(-\eta_3)/\mathrm{d}r\big)}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big|$$

$$\leqslant\frac{r}{a}\Big(c\sqrt{z_1^2+z_2^2}\cdot 2\,\frac{c}{2r(r-1)}+\frac{2c\sqrt{z_1^2+z_2^2}\,\delta\,2}{\frac13\alpha r}+\frac{c\sqrt{z_1^2+z_2^2}c/(r-1)}{\frac13\alpha r}\Big)$$

$$\leqslant\frac{c\sqrt{z_1^2+z_2^2}}{a(r-1)}.$$

Hence we have proved (32). Finally, as $U_2+V_2=\Psi_z(\sigma)-\Psi_z(\tau)$, the estimate (33) follows from (34) and (35). □

P r o o f of Proposition 4.1. We distinguish four cases concerning (z,r) when showing that $\mu_{a,c}+\mathcal{H}^1\llcorner L$ is monotone at (z,r).

Case 1: $z=0$ and $r\in(0,\infty)$.

Using (11) and (7) we obtain

$$\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{\mu_{a,c}B(z,r)}{r}+\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{(\mathcal{H}^1\llcorner L)B(z,r)}{r}\geqslant\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{2(\beta/\alpha)r}{r}+0=0.$$

Case 2: $z\in S^+(1)\cup E(1)$ and $r\in(0,\sqrt{z_1^2+z_2^2}+2c]$.

If $\sqrt{z_1^2+z_2^2}\notin(\frac12c,\frac32c)$, then the proof follows from Lemma 4.2 (ii) and (11). Conversely, if $\sqrt{z_1^2+z_2^2}\in(\frac12c,\frac32c)$, then $r<4c$ by Lemma 4.2 (i) and we need not care about this case (see the statement of Proposition 4.1 (iv) and notice that $\sqrt{z_1^2+z_2^2} \in (\frac{1}{2}c, \frac{3}{2}c)$ implies that $s(z) \in (\frac{1}{2}, 2)$ for c small enough).

In the remaining cases, we have $r \in \big[\sqrt{z_1^2+z_2^2}+2c, \infty\big) \setminus \{1\}$. Hence we can use the estimate (28). We can also suppose that

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}r} \leqslant 2 \quad \text{and} \quad \frac{\mathrm{d}(-\eta_3)}{\mathrm{d}r} \leqslant 2, \tag{37}$$

otherwise the proof follows from Lemma 4.5 and (11).

Case 3: $z \in S^+(1) \cup E(1)$ and $r \in \big[\sqrt{z_1^2+z_2^2}+2c, \frac{11}{10}\big] \setminus \{1\}$.

If $z \in S^+(1)$ with $s \in [0, \frac{1}{2}] \cup [2, 5]$, then from (4), (17) and (39) we conclude

$$\begin{aligned} |U_1+V_1| &\leqslant \Big|\frac{r}{a}\frac{\alpha^2\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}\Big| + \Big|\frac{r}{a}\frac{-\alpha^2\Psi_z'(\sigma)\,\mathrm{d}(-\eta_3)/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big| \\ &\leqslant \frac{3c\sqrt{z_1^2+z_2^2}}{a\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}} + \frac{3c\sqrt{z_1^2+z_2^2}}{a\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}} \\ &\leqslant \frac{3sc^2}{a\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}} + \frac{3sc^2}{a\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}. \end{aligned}$$

Further, Lemma 4.7 gives

$$U_2+U_3+V_2+V_3 \geqslant \frac{\frac{1}{4}c^2/\sqrt{1+25c^2}}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}} + \frac{\frac{1}{4}c^2/\sqrt{1+25c^2}}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}} \geqslant |U_1+V_1|$$

and thus (11) and (28) imply $\underline{\mathrm{D}}_{\mathrm{r}}((\mu_{a,c}+\mathcal{H}^1\llcorner L)B(z,r)/r) \geqslant 0$.

If $z \in S^+(1)$ is such that $s \in [\frac{1}{2}, 2]$, by Lemma 4.7 we have $U_2+U_3+V_2+V_3 \geqslant 0$. Moreover, the estimates (17), (20) and (39) imply

$$\begin{aligned} |U_1+V_1| &\leqslant \Big|\frac{r}{a}\frac{\alpha^2\Psi_z'(\tau)\,\mathrm{d}\xi_3/\mathrm{d}r}{\sqrt{\Psi_z^2(\tau)+\alpha^2(r^2-1)}}\Big| + \Big|\frac{r}{a}\frac{-\alpha^2\Psi_z'(\sigma)\,\mathrm{d}(-\eta_3)/\mathrm{d}r}{\sqrt{\Psi_z^2(\sigma)+\alpha^2(r^2-1)}}\Big| \\ &\leqslant 2\frac{r}{a}\frac{\alpha^2c\sqrt{z_1^2+z_2^2}\,2}{\frac{1}{3}\alpha r} \leqslant \frac{13c\sqrt{z_1^2+z_2^2}}{a}. \end{aligned} \tag{38}$$

Further, from (4) we have

$$2\frac{\sqrt{z_1^2+z_2^2}}{r} \geqslant 2\frac{sc/\sqrt{1+s^2c^2}}{\frac{11}{10}} \geqslant sc \geqslant \frac{1}{2}c > \frac{\beta}{\alpha^2}\frac{13c}{a},$$

and thus from (10), (28) and (40) we obtain

$$\underline{\mathrm{D}}_{\mathrm{r}}\frac{(\mu_{a,c}+\mathcal{H}^1\llcorner L)B(z,r)}{r} \geqslant 2\,\frac{z_1^2+z_2^2}{r^3} - \frac{\beta}{\alpha^2r^2}\,\frac{13c\sqrt{z_1^2+z_2^2}}{a} \geqslant 0.$$

If $\sqrt{z_1^2+z_2^2} \geqslant 4c$, then the proof follows from Lemma 4.2 (iii) and from (11). Since

$$S^+(1) \cup E(1) = \big\{z \in S^+(1) \cup E(1)\colon\ \sqrt{z_1^2+z_2^2} \geqslant 4c\big\} \cup \{z \in S^+(1)\colon\ s \leqslant 5\}$$

for c small enough, the monotonicity at (z,r) in the third case is proved.

Case 4: $z \in S^+(1) \cup E(1)$ and $r \in [\frac{11}{10}, \infty)$.

If $z \in S^+(1)$ is such that $s \leqslant \frac{1}{2}$, then by Lemma 4.7 and the estimate (18) we have

$$U_2 + U_3 + V_2 + V_3 \geqslant 2\frac{\frac{1}{4}c^2/(1+\frac{1}{4}c^2)}{\alpha r} \geqslant \frac{1}{4}\frac{c^2}{r}.$$

Thus, from (4) and (32) we obtain

$$|U_1 + V_1| \leqslant \frac{c\sqrt{z_1^2+z_2^2}}{a(r-1)} \leqslant \frac{csc/\sqrt{1+s^2c^2}}{ar/11} \leqslant \frac{11c^2}{ar} < \frac{1}{4}\frac{c^2}{r} \leqslant U_2 + U_3 + V_2 + V_3.$$

Hence (11) and (28) imply $\underline{\mathrm{D}}_{\mathrm{r}}((\mu_{a,c} + \mathcal{H}^1 \llcorner L)B(z,r)/r) \geqslant 0$.

Finally, if either $z \in E(1)$ or $z \in S^+(1)$ is such that $s \geqslant \frac{1}{2}$, then by Lemma 4.6 we have $U_3 + V_3 \geqslant 0$. Moreover, Lemma 4.8 gives

$$|U_1 + U_2 + V_1 + V_2| \leqslant \frac{c\sqrt{z_1^2+z_2^2}}{a(r-1)} + \frac{4c\sqrt{z_1^2+z_2^2}}{a(r-1)} = \frac{5c\sqrt{z_1^2+z_2^2}}{a(r-1)}.$$

Hence from (4), (10) and (28) we obtain

$$\begin{aligned}
\underline{\mathrm{D}}_{\mathrm{r}}\,\frac{(\mu_{a,c} + \mathcal{H}^1 \llcorner L)B(z,r)}{r} &\geqslant 2\,\frac{z_1^2+z_2^2}{r^3} - \frac{\beta}{\alpha^2 r^2}\,\frac{5c\sqrt{z_1^2+z_2^2}}{a(r-1)} \\
&\geqslant \frac{\sqrt{z_1^2+z_2^2}}{r^3}\Big(2\sqrt{z_1^2+z_2^2} - \frac{6c}{a(r-1)/r}\Big) \\
&\geqslant \frac{\sqrt{z_1^2+z_2^2}}{r^3}\left(2\frac{\frac{1}{2}c}{\sqrt{1+\frac{1}{4}c^2}} - \frac{6c}{a\frac{1}{11}}\right) \geqslant 0.
\end{aligned}$$

This is the monotonicity at (z,r) in the fourth case.

Finally, it can be easily seen that our Cases 1–4 cover the assumptions of Proposition 4.1 (i)–(iv), and thus we are done. □

5. Small radii

In this section we deal with the case when the center z is very close to $\operatorname{spt}\mu_{a,c}$. Our strategy is the following. We use the self-similarity of our logarithmic spirals to pass to the case of a ball such that $\operatorname{spt}\mu_{a,c}$ on a neighborhood of the ball can be parametrized as the graph of a suitable multivalued function to which we can apply Proposition 2.1.

Proposition 5.1. *There is $K_2 > 0$ with the following property:*
If $a \geqslant K_2$, $c = a^{-3}$, $z \in S^+(1)$ with $s(z) \in [0,2]$ and $r \in (0,4c]$, then $\mu_{a,c}$ is monotone at (z,r).

Because of the self similarity of $\operatorname{spt}\mu_{a,c}$ we can use the following idea: having any fixed plane not passing through the origin and using a suitable transformation, we see that the monotonicity at (z,r) in the case of the ball $B(z,r)$ centered on $S(1)$ is equivalent to the monotonicity at $(\tilde{z},\tilde{r})$ in the case of the ball $B(\tilde{z},\tilde{r})$ centered at the proper point on the above mentioned plane. This can be done in such a way that $B(\tilde{z},\tilde{r})$ is "not far away from" to the $B(z,r)$.

Let us therefore define a new orthogonal basis $\{\tilde{u},\tilde{v},\tilde{w}\}$ in $\mathbb{R}^3$ by

$$\begin{aligned} u &= \Big(\frac{1}{a^3},\frac{1}{a^4},1\Big), & \tilde{u} &= \frac{u}{|u|},\\ v &= \Big(-\frac{1}{a},1,0\Big), & \tilde{v} &= \frac{v}{|v|},\\ w &= \Big(1,\frac{1}{a},-\frac{a^2+1}{a^5}\Big), & \tilde{w} &= \frac{w}{|w|}. \end{aligned}$$

Now, it can be seen that Proposition 5.1 follows from Proposition 5.2 below because of the self similarity of $\operatorname{spt}\mu_{a,c}$. Indeed, for large a we have $\tilde{u} \sim (0,0,1)$, $\tilde{v} \sim (0,1,0)$, $\tilde{w} \sim (1,0,0)$, $\Gamma^+_{a,c}(0) = (a^{-3},0,1) \sim (0,0,1)$. Hence, the center of the "new" ball is very close to the center of the "old" one. Furthemore, we can assume the "new" radius r to be only a little larger ($5a^{-3}$ instead of $4a^{-3}$), and similarly, the distance of the center of the "new" ball from the "new reference point" $\Gamma^+_{a,c}(0)$ is "almost equal" to the distance of the center of the "old" ball from the "old reference point" $\operatorname{spt}\mu_{a,c} \cap S^+(1)$ (compare the assumptions concerning s in Proposition 5.1 and the assumptions concerning $\sqrt{t_1^2+t_2^2}$ in Proposition 5.2). Thus the assumptions of Proposition 5.2 cover the case of the "rescaled" assumptions of Proposition 5.1.

Proposition 5.2. *Let $a \geqslant 1000$ and $c = a^{-3}$. Then $\mu_{a,c}$ is monotone at (z,r) whenever $z = \Gamma^+_{a,c}(0) + t_1\tilde{v} + t_2\tilde{w}$ with $\sqrt{t_1^2+t_2^2} \leqslant 5a^{-3}$, and $r \in (0,5a^{-3})$.*

The proof of Proposition 5.2 is based on Proposition 2.1. Suppose $a \geqslant 1000$ and $c = a^{-3}$ in the sequel.

Let us consider a new coordinate system with respect to the basis $\{\tilde{u}, \tilde{v}, \tilde{w}\}$ with the new origin at $\Gamma^+_{a,c}(0)$. Then the curve $\Gamma^+_{a,c}$ turns to

$$\begin{aligned}\tilde{\gamma}(t) &= (\tilde{x}(t), \tilde{f}(t), \tilde{\varphi}(t)) \\ &= \big((\Gamma^+_{a,c}(t) - \Gamma^+_{a,c}(0)) \cdot \tilde{u}, (\Gamma^+_{a,c}(t) - \Gamma^+_{a,c}(0)) \cdot \tilde{v}, (\Gamma^+_{a,c}(t) - \Gamma^+_{a,c}(0)) \cdot \tilde{w}\big).\end{aligned}$$

Let $t \in (-6a^{-4}, 6a^{-4})$. Since $|1 - \cos t| \leqslant |t|$, $|\sin t| \leqslant |t|$ and $\mathrm{e}^{at} \in (1 - 18a^{-3}, 1 + 18a^{-3})$, from

$$\dot{\Gamma}^+_{a,c}(t) = \Big(\frac{1}{a^3}\mathrm{e}^{at}(a\cos t - \sin t), \frac{1}{a^3}\mathrm{e}^{at}(a\sin t + \cos t), a\mathrm{e}^{at}\Big)$$

and

$$\ddot{\Gamma}^+_{a,c}(t) = \Big(\frac{1}{a^3}\mathrm{e}^{at}((a^2-1)\cos t - 2a\sin t), \frac{1}{a^3}\mathrm{e}^{at}((a^2-1)\sin t + 2a\cos t), a^2\mathrm{e}^{at}\Big),$$

we obtain

$$\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{u} = \frac{\mathrm{e}^{at}}{|u|}\Big(\frac{a^2+1}{a^7}\cos t + a\Big), \qquad |\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{u}| \geqslant a - \frac{20}{a^2}, \tag{39}$$

$$|\ddot{\Gamma}^+_{a,c}(t) \cdot \tilde{u}| = \frac{\mathrm{e}^{at}}{|u|}\Big|\frac{a^2+1}{a^6}\cos t - \frac{a^2+1}{a^7}\sin t + a^2\Big| \leqslant 2a^2, \tag{40}$$

$$\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{v} = \frac{\mathrm{e}^{at}}{|v|}\Big(\frac{a^2+1}{a^4}\sin t\Big), \qquad |\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{v}| \leqslant \frac{7}{a^6}, \tag{41}$$

$$\ddot{\Gamma}^+_{a,c}(t) \cdot \tilde{v} = \frac{\mathrm{e}^{at}}{|v|}\Big(\frac{a^2+1}{a^4}\cos t + \frac{a^2+1}{a^3}\sin t\Big), \tag{42}$$

$$\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{w} = \frac{\mathrm{e}^{at}}{|w|}\frac{a^2+1}{a^4}(\cos t - 1), \qquad |\dot{\Gamma}^+_{a,c}(t) \cdot \tilde{w}| \leqslant \frac{7}{a^6}, \tag{43}$$

$$|\ddot{\Gamma}^+_{a,c}(t) \cdot \tilde{w}| = \frac{\mathrm{e}^{at}}{|w|}\Big|\frac{a^2+1}{a^3}(\cos t - 1) - \frac{a^2+1}{a^4}\sin t\Big| \leqslant \frac{7}{a^5}. \tag{44}$$

Next, by (41) we have $\tilde{x}'(t) > a - 20a^{-2}$ on $l[-6a^{-4}, 6a^{-4}]$. Therefore we can define the inverse function $\psi = \tilde{x}^{-1}$ on $(\tilde{x}(-6a^{-4}), \tilde{x}(6a^{-4})) \supset [-5a^{-3}, 5a^{-3}]$. We set

$$\gamma(x) = (x, f(x), \varphi(x)) = (x, \tilde{f}(\psi(x)), \tilde{\varphi}(\psi(x))).$$

Hence we have

$$f'(x) = \frac{\tilde{f}'(\psi(x))}{\tilde{x}'(\psi(x))} = \frac{\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{v}}{\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u}}, \tag{45}$$

$$f''(x) = \frac{\tilde{f}''(\psi(x))}{\tilde{x}'^2(\psi(x))} - \frac{\tilde{f}'(\psi(x))\tilde{x}''(\psi(x))}{\tilde{x}'^3(\psi(x))} \tag{46}$$
$$= \frac{\ddot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{v}}{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})^2} - \frac{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{v})(\ddot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})}{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})^3},$$

$$\varphi'(x) = \frac{\tilde{\varphi}'(\psi(x))}{\tilde{x}'(\psi(x))} = \frac{\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{w}}{\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u}} \tag{47}$$

and

$$\varphi''(x) = \frac{\tilde{\varphi}''(\psi(x))}{\tilde{x}'^2(\psi(x))} - \frac{\tilde{\varphi}'(\psi(x))\tilde{x}''(\psi(x))}{\tilde{x}'^3(\psi(x))} \tag{48}$$
$$= \frac{\ddot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{w}}{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})^2} - \frac{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{w})(\ddot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})}{(\dot{\Gamma}^+_{a,c}(\psi(x)) \cdot \tilde{u})^3}.$$

Now, let us check that f and φ satisfy the assumptions of Proposition 2.1.

Lemma 5.3. *Let $a \geqslant 1000$ and $c = a^{-3}$. Then*

$$f(0) = f'(0) = \varphi(0) = \varphi'(0) = 0.$$

Proof. Since $\tilde{x}(0) = 0$ we have $\psi(0) = 0$ and thus from the definition of $\tilde{\gamma}$ we obtain $f(0) = \tilde{f}(0) = 0$ and $\varphi(0) = \tilde{\varphi}(0) = 0$. Further, (43), (45), (47), (49) and $\psi(0) = 0$ imply $f'(0) = \varphi'(0) = 0$. □

Lemma 5.4. *Let $a \geqslant 1000$ and $c = a^{-3}$ and $\psi(x) \in (-6a^{-4}, 6a^{-4})$. Then*

$$|f''(x) - f''(0)| \leqslant \frac{2}{a} f''(0) \quad \textit{and} \quad |\varphi''(x)| \leqslant \frac{2}{a} f''(0). \tag{49}$$

Proof. First, we need some estimates concerning $f''(0)$. As $\psi(0) = 0$, (41), (42), (43), (44) and (48) imply

$$f''(0) = \frac{|u|^2}{|v|} \frac{1/a^2 + 1/a^4}{(a + 1/a^5 + 1/a^7)^2} + 0 \tag{50}$$
$$= 1/a^4 \frac{|u|^2}{|v|} \frac{1 + 1/a^2}{(1 + 1/a^6 + 1/a^8)^2} = \frac{1}{a^4} \frac{\sqrt{1 + 1/a^2}}{1 + 1/a^6 + 1/a^8}.$$

Hence for every $t \in (-6a^{-4}, 6a^{-4})$ we have from $\mathrm{e}^{at} \in (1 - 18a^{-3}, 1 + 18a^{-3})$

$$\left| f''(0) - \frac{1}{a^4} \right| \leqslant \frac{1}{a} f''(0) \quad \text{and} \quad \left| f''(0) - \frac{|u|^2}{|v|} \frac{1}{a^4 \mathrm{e}^{at}} \right| \leqslant \frac{1}{a} f''(0). \tag{51}$$

Next, since $t \in (-6a^{-4}, 6a^{-4})$ implies $\mathrm{e}^{at} \in (1-18a^{-3}, 1+18a^{-3})$, from (41), (42), (45), (46), (49) and (52) we obtain

$$|\varphi''(x)| \leqslant \frac{7/a^5}{(a-20/a^2)^2} + \frac{7/a^6 \cdot 2a^2}{(a-20/a^2)^3} < \frac{8}{a^7} + \frac{15}{a^7} = \frac{23}{a^7} < \frac{2}{a} f''(0).$$

Hence we have the second inequality in (51) and it remains to prove the first. As $a \geqslant 1000$, $|t| \leqslant 6a^{-4}$, $|\sin t| \leqslant |t|$, $|1-\cos t| \leqslant |t|$, (41) and (44) give

$$\begin{aligned} &|a^4(\ddot{\Gamma}^+_{a,c}(t)\cdot v) - \mathrm{e}^{-at}(\dot{\Gamma}^+_{a,c}(t)\cdot u)^2| \\ &\qquad = \mathrm{e}^{at}\Big|(a^2+1)\cos t + (a^3+a)\sin t - \Big(a + \frac{a^2+1}{a^7}\cos t\Big)^2\Big| \leqslant 2 \end{aligned}$$

and thus from (41) we obtain

$$(52) \qquad \left|\frac{\ddot{\Gamma}^+_{a,c}(t)\cdot v/|v|}{(\dot{\Gamma}^+_{a,c}(t)\cdot u/|u|)^2} - \frac{|u|^2}{a^4|v|\mathrm{e}^{at}}\right| = \frac{1}{|v|}\frac{|a^4\ddot{\Gamma}^+_{a,c}(t)\cdot v - \mathrm{e}^{-at}(\dot{\Gamma}^+_{a,c}(t)\cdot u)^2|}{a^4(\dot{\Gamma}^+_{a,c}(t)\cdot u/|u|)^2} \leqslant \frac{3}{a^6}.$$

Finally, using (41), (42), (43), (44), (48), (53) and (54) we conclude the proof by the following estimate (where we simply write t instead of $\psi(x)$):

$$\begin{aligned} |f''(x) - f''(0)| &\leqslant \Big|f''(x) - \frac{|u|^2}{a^4|v|\mathrm{e}^{at}}\Big| + \Big|\frac{|u|^2}{a^4|v|\mathrm{e}^{at}} - f''(0)\Big| \\ &\leqslant \Big|\frac{\tilde{f}''(t)}{\tilde{x}'^2(t)} - \frac{|u|^2}{a^4|v|\mathrm{e}^{at}}\Big| + \Big|\frac{\tilde{f}'(t)\tilde{x}''(t)}{\tilde{x}'^3(t)}\Big| + \frac{1}{a}f''(0) \\ &= \Big|\frac{\ddot{\Gamma}^+_{a,c}(t)\cdot v/|v|}{(\dot{\Gamma}^+_{a,c}(t)\cdot u/|u|)^2} - \frac{|u|^2}{a^4|v|\mathrm{e}^{at}}\Big| \\ &\quad + \Big|\frac{(\dot{\Gamma}^+_{a,c}(t)\cdot v/|v|)(\ddot{\Gamma}^+_{a,c}(t)\cdot u/|u|)}{(\dot{\Gamma}^+_{a,c}(t)\cdot u/|u|)^3}\Big| + \frac{1}{a}f''(0) \\ &\leqslant \frac{3}{a^6} + \frac{7/a^6\cdot 2a^2}{(a-20/a^2)^3} + \frac{1}{a}f''(0) \leqslant \frac{2}{a}f''(0). \end{aligned}$$

□

P r o o f of Proposition 5.2. For all considered pairs (z,r) we observe that $B(z,r) \cap [\Gamma^-_{a,c}] = \emptyset$. Further, as $a \geqslant 1000$ and $\psi(x) \in (-6a^{-4}, 6a^{-4})$ for $x \in [-5a^{-3}, 5a^{-3}]$, Lemma 5.3 and Lemma 5.4 imply that all assumptions of Proposition 2.1 are satisfied (with $d = f''(0) \sim a^{-4}$, $\varepsilon = 2a^{-1}$ and $\delta = 5a^{-3} < \frac{1}{20}d^{-1}$). Thus, Proposition 2.1 completes the proof. □

Therefore we have also proved Proposition 5.1 (by the comments at the beginning of this section).

P r o o f of Theorem 1.4. First, let us prove the monotonicity. Suppose that $K \geqslant \max\{K_1, K_2\}$. Because of the self-similarity of the logarithmic spirals and lines, it is enough to check the monotonicity for $z \in S^+(1) \cup E(1) \cup \{0\}$ and $r > 0$. By Lemma 2.2 we can further suppose that $r \neq 1$. In every case but $z \in S^+(1)$ with $s \in \left(\frac{1}{2}, 2\right)$ and $r \in (0, 4c)$ we can use Proposition 4.1. Finally, Proposition 5.1 completes the proof in the remaining case.

The assertions concerning the tangential behavior follow from Proposition 3.1.

From the definitions, we easily obtain that

$$\theta^1_z(\mu_{a,c} + \mathcal{H}^1 \llcorner L) = 1 \quad \text{for every } z \in (\operatorname{spt} \mu_{a,c} \cup L) \setminus \{0\}.$$

Finally, (7) gives

$$\theta^1_0(\mu_{a,c} + \mathcal{H}^1 \llcorner L) = 1 + \frac{\beta}{\alpha} = 1 + \frac{\sqrt{1 + 1/a^6 + 1/a^8}}{\sqrt{1 + 1/a^6}} = 1 + \sqrt{1 + \frac{1}{a^8 + a^2}}.$$

Consequently, we have $\theta^1_0(\mu_{a,c} + \mathcal{H}^1 \llcorner L) < 2 + \varepsilon$ for a sufficiently large. □

Remark 5.5. Our restriction about a and c in Theorem 1.4 was made to obtain simple proofs and does not mean that the other compensated measures $\mu_{a,c} + \mathcal{H}^1 \llcorner L$ cannot be monotone.

On the other hand, using a similar method as in the last section of paper [2] it can be shown that $\mu_{a,c}$ itself is not monotone for any $a, c > 0$.

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