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EXPONENTS FOR THREE-DIMENSIONAL SIMULTANEOUS DIOPHANTINE APPROXIMATIONS

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Abstract. Let $\Theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Suppose that $1, \theta_1, \theta_2, \theta_3$ are linearly independent over $\mathbb{Z}$. For Diophantine exponents

$$\alpha(\Theta) = \sup\{\gamma > 0\colon \limsup_{t\to+\infty} t^{\gamma}\psi_{\Theta}(t) < +\infty\},$$
$$\beta(\Theta) = \sup\{\gamma > 0\colon \liminf_{t\to+\infty} t^{\gamma}\psi_{\Theta}(t) < +\infty\}$$

we prove

$$\beta(\Theta) \geqslant \frac{1}{2}\left(\frac{\alpha(\Theta)}{1-\alpha(\Theta)} + \sqrt{\left(\frac{\alpha(\Theta)}{1-\alpha(\Theta)}\right)^2 + \frac{4\alpha(\Theta)}{1-\alpha(\Theta)}}\right)\alpha(\Theta).$$

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1. Diophantine exponents

Let $\Theta = (\theta_1, \ldots, \theta_n)$ be a real vector. We deal with the function

$$\psi_{\Theta}(t) = \min_{x\leqslant t}\ \max_{1\leqslant i\leqslant n} \|\theta_i x\|.$$

Here the minimum is taken over positive integers x and $\|\cdot\|$ stands for the distance to the nearest integer.

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Suppose that at least one of the numbers $\theta_1, \dots, \theta_n$ is irrational. Then $\psi_\Theta(t) > 0$ for all $t \geqslant 1$. The *uniform* Diophantine exponent $\alpha(\Theta)$ is defined as the supremum of the set

$$\{\gamma > 0\colon \limsup_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty\}.$$

It is a well-known fact that for all Θ one has

$$\frac{1}{n} \leqslant \alpha(\Theta) \leqslant 1.$$

The *ordinary* Diophantine exponent $\beta(\Theta)$ is defined as the supremum of the set

$$\{\gamma > 0\colon \liminf_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty\}.$$

Obviously

$$\beta(\Theta) \geqslant \alpha(\Theta). \tag{1}$$

2. Functions

For each $\alpha \in \left[\frac{1}{3}, 1\right)$, define

$$g_1(\alpha) = \frac{\alpha}{1-\alpha}$$

and

$$g_2(\alpha) = \frac{\alpha(1-\alpha) + \sqrt{\alpha(\alpha^3 + 6\alpha^2 - 7\alpha + 4)}}{2(2\alpha^2 - 2\alpha + 1)}.$$

The value $g_2(\alpha)$ is the largest root of the equation

$$(2\alpha^2 - 2\alpha + 1)x^2 + \alpha(\alpha - 1)x - \alpha = 0.$$

Note that

$$g_2(1/3) = g_2(1) = 1,$$

and for $1/3 < \alpha < 1$ one has $g_2(\alpha) > 1$. Let α_0 be the unique real root of the equation

$$x^3 - x^2 + 2x - 1 = 0.$$

In the interval $1/3 < \alpha < \alpha_0$ one has

$$g_2(\alpha) > \max(1, g_1(\alpha)). \tag{2}$$

In the interval $\alpha_0 \leqslant \alpha < 1$ we see that

$$g_2(\alpha) \leqslant g_1(\alpha).$$

We define one more function. Put

$$g_3(\alpha) = \frac{1}{2}\left(\frac{\alpha}{1-\alpha} + \sqrt{\left(\frac{\alpha}{1-\alpha}\right)^2 + \frac{4\alpha}{1-\alpha}}\right). \tag{3}$$

Simple calculation shows that

$$g_3(\alpha) > \max(g_1(\alpha), g_2(\alpha)) \quad \forall\, \alpha \in \left(\tfrac{1}{3}, 1\right). \tag{4}$$

3. Jarník's result

In a fundamental paper [1] V. Jarník proved the following theorem.

Theorem 1. *Let $\psi(t)$ be a continuous function in t, decreasing to zero as $t \to +\infty$. Suppose that the function $t\psi(t)$ increases to infinity as $t \to +\infty$. Let $\varrho(t)$ be the inverse function to the function $t\psi(t)$. Put*

$$\varphi^{[\psi]}(t) = \psi\left(\varrho\left(\frac{1}{6\psi(t)}\right)\right).$$

Suppose that $n \geqslant 2$ and among numbers $\theta_1, \ldots, \theta_n$ there exist at least two numbers which, together with 1, are linearly independent over $\mathbb{Z}$. Suppose that

$$\psi_\Theta(t) \leqslant \psi(t)$$

for all t large enough. Then there exist infinitely many integers x such that

$$\max_{1 \leqslant j \leqslant n} \|x\theta_j\| \leqslant \varphi^{[\psi]}(x).$$

The next Jarník's result on Diophantine exponents is an obvious corollary of Theorem 1.

Theorem 2. *Suppose that $n \geqslant 2$ and among numbers $\theta_1, \ldots, \theta_n$ there exist at least two numbers which, together with* 1, *are linearly independent over $\mathbb{Z}$. Then*

$$\beta(\Theta) \geqslant \alpha(\theta) g_1(\alpha(\Theta)).$$

To obtain Theorem 2 from Theorem 1 one takes $\psi(t) = t^{-\alpha}$ with $\alpha < \alpha(\Theta)$.

On the other hand, V. Jarník [1] proved that there exists a collection of numbers $\Theta = (\theta_1, \ldots, \theta_n)$ such that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$ and

$$\beta(\Theta) < \frac{\alpha(\Theta)}{1 - \alpha(\Theta)}.$$

In the case $n = 2$ the lower bound in Jarník's Theorem 2 is optimal. The following result was proved by M. Laurent [2].

Theorem 3. *For any $\alpha, \beta > 0$ satisfying*

$$\frac{1}{2} \leqslant \alpha \leqslant 1, \quad \beta \geqslant \alpha g_1(\alpha)$$

there exists a vector $\Theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ such that

$$\alpha(\Theta) = \alpha, \quad \beta(\Theta) = \beta.$$

This result is a corollary of a general theorem concerning four two-dimensional Diophantine exponents.

Note that in the case $n \geqslant 3$ the bound in Theorem 2 in the range $1/n \leqslant \alpha < \frac{1}{2}$ is weaker than the trivial bound (1).

N. Moshchevitin [3] (see also [4], Section 5.2) improved Jarník's result in the case $n = 3$ and for $\alpha \in (\frac{1}{3}, \alpha_0)$. He obtained

Theorem 4. *Suppose that $m = 1$, $n = 3$ and the collection $\Theta = (\theta_1, \theta_2, \theta_3)$ consists of numbers which, together with* 1, *are linearly independent over $\mathbb{Z}$. Then*

$$\beta(\Theta) \geqslant \alpha(\Theta) g_2(\alpha(\Theta)).$$

In the case $n = 3$, Theorems 2 and 4 together give an estimate which is better than the trivial estimate (1) for all admissible values of $\alpha(\Theta)$.

4. New result

In this paper we give a new lower bound for $\beta(\Theta)$ in terms of $\alpha(\Theta)$. From (4) it follows that this bound is better than all the previous bounds (Theorems 2 and 4) for all admissible values of $\alpha(\Theta)$.

Theorem 5. *Suppose that $m = 1$, $n = 3$ and the vector $\Theta = (\theta_1, \theta_2, \theta_3)$ consists of numbers linearly independent, together with 1, over $\mathbb{Z}$. Then*

$$\beta(\Theta) \geqslant \alpha(\Theta) g_3(\alpha(\Theta)).$$

Sections 5, 6, 7 below contain auxiliary results. Theorem 5 is proved in Section 8.

5. Best approximations

For each integer x, put

$$\zeta(x) = \max_{1 \leqslant j \leqslant n} \|\theta_j x\|.$$

A positive integer x is said to be a *best approximation* if

$$\zeta(x) = \min_{x'} \zeta(x'),$$

where the minimum is taken over all $x' \in \mathbb{Z}$ such that

$$0 < x' \leqslant x.$$

Consider the case when all numbers 1 and θ_j, $1 \leqslant j \leqslant n$ are linearly independent over $\mathbb{Z}$. Then all best approximations lead to sequences

$$x_1 < x_2 < \ldots < x_\nu < x_{\nu+1} < \ldots,$$
$$\zeta(x_1) > \zeta(x_2) > \ldots > \zeta(x_\nu) > \zeta(x_{\nu+1}) > \ldots.$$

We use the notation

$$\zeta_\nu = \zeta(x_\nu).$$

Choose $y_{1,\nu}, \ldots, y_{n,\nu} \in \mathbb{Z}$ such that

$$\|\theta_j x_\nu\| = |\theta_j \mathbf{x}_\nu - y_{j,\nu}|.$$

We define

$$\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, \ldots, y_{n,\nu}) \in \mathbb{Z}^{n+1}.$$

If $\psi(t)$ is a continuous function decreasing to 0 as $t \to \infty$, with

$$\psi_\Theta(t) \leqslant \psi(t),$$

then one easily sees that

$$\zeta_\nu \leqslant \psi(x_{\nu+1}). \tag{5}$$

Some useful fact about best approximations can be found in [4].

6. Two-dimensional subspaces

Lemma 1. *Suppose that all vectors of the best approximations $\mathbf{z}_l$, $\nu \leqslant l \leqslant k$ lie in a certain two-dimensional linear subspace $\pi \subset \mathbb{R}^4$. Consider the two-dimensional lattice $\Lambda = \pi \cap \mathbb{Z}^4$ with the two-dimensional fundamental volume* $\det \Lambda$. *Then for all l from the interval $\nu \leqslant l \leqslant k-1$ one has*

$$C_1 \det \Lambda \leqslant \zeta_l x_{l+1} \leqslant 2 \det \Lambda \tag{6}$$

where $C_1 = \left(2\sqrt{3(1 + (|\theta_1| + \frac{1}{2})^2 + (|\theta_2| + \frac{1}{2})^2 + (|\theta_3| + \frac{1}{2})^2)}\right)^{-1}$. *In particular,*

$$\det \Lambda \geqslant \frac{\min(\zeta_\nu x_{\nu+1}, \zeta_{k-1} x_k)}{2}. \tag{7}$$

P r o o f. The parallelepiped

$$\Omega_l = \left\{ \mathbf{z} = (x, y_1, y_2, y_3) \colon \ |x| < x_{l+1}, \ \max_{1 \leqslant j \leqslant 3} |\theta_j x - y_j| < \zeta_l \right\}$$

has no non-zero integer points inside for every l. Consider the two-dimensional $\mathbf{0}$-symmetric convex body

$$\Xi_l = \Omega_l \cap \pi.$$

One can see that the two-dimensional Lebesgue measure $\mu(\Xi_l)$ of Ξ_l admits the following lower and upper bounds:

$$2\zeta_l x_{l+1} \leqslant \mu(\Xi_l) \leqslant 4\sqrt{3\left(1 + \left(|\theta_1| + \frac{1}{2}\right)^2 + \left(|\theta_2| + \frac{1}{2}\right)^2 + \left(|\theta_3| + \frac{1}{2}\right)^2\right)}\ \zeta_l x_{l+1}. \tag{8}$$

We see that there is no non-zero point of Λ inside Ξ_l and that there are two linearly independent points $\mathbf{z}_l, \mathbf{z}_{l+1} \in \Lambda$ on the boundary of Ξ_l. So obviously

$$2 \det\Lambda \leqslant \mu(\Xi_l). \tag{9}$$

From the Minkowski convex body theorem it follows that

$$\mu(\Xi_l) \leqslant 4 \det\Lambda. \tag{10}$$

Now (6) follows from (8, 9, 10). Lemma is proved. □

7. Three-dimensional subspaces

Consider three consecutive best approximation vectors $\mathbf{z}_{l-1}$, $\mathbf{z}_l$, $\mathbf{z}_{l+1}$. Suppose that these vectors are linearly independent. Consider the three-dimensional linear subspace

$$\Pi_l = \operatorname{span}(\mathbf{z}_{l-1}, \mathbf{z}_l, \mathbf{z}_{l+1}).$$

Consider the lattice

$$\Gamma_l = \Pi_l \cap \mathbb{Z}^4$$

with the fundamental volume $\det \Gamma_l$. Let Δ be the three-dimensional volume of the three-dimensional simplex $\mathcal{S}$ with vertices $\mathbf{0}$, $\mathbf{z}_{l-1}$, $\mathbf{z}_l$, $\mathbf{z}_{l+1}$. We see that

$$\Delta \geqslant \frac{\det \Gamma_l}{6}. \tag{11}$$

Consider determinants

$$\Delta_1 = -\begin{vmatrix} x_{l-1} & y_{2,l-1} & y_{3,l-1} \\ x_l & y_{2,l} & y_{3,l} \\ x_{l+1} & y_{2,l+1} & y_{3,l+1} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} x_{l-1} & y_{1,l-1} & y_{3,l-1} \\ x_l & y_{1,l} & y_{3,l} \\ x_{l+1} & y_{1,l+1} & y_{3,l+1} \end{vmatrix}, \tag{12}$$

$$\Delta_3 = -\begin{vmatrix} x_{l-1} & y_{1,l-1} & y_{2,l-1} \\ x_l & y_{1,l} & y_{2,l} \\ x_{l+1} & y_{1,l+1} & y_{2,l+1} \end{vmatrix}.$$

The absolute values of these determinants are equal to the three-dimensional volumes of the projections of the simplex $\mathcal{S}$ onto the three-dimensional coordinate subspaces ($\{y_1 = 0\}$, $\{y_2 = 0\}$ and $\{y_3 = 0\}$ respectively) multiplied by 6.

Note that for $j = 1, 2, 3$ one has

$$|\Delta_j| \leqslant 6\zeta_{l-1}\zeta_l x_{l+1}. \tag{13}$$

Lemma 2. *Among determinants* (12) *there exists a determinant with absolute value* $\geqslant C_2\Delta$, *where* $C_2 = 2/\bigl(2 + \max\limits_{1\leqslant i\leqslant 3} |\theta_i|\bigr)$.

Proof. Consider the determinant

$$\Delta_0 = \begin{vmatrix} y_{1,l-1} & y_{2,l-1} & y_{3,l-1} \\ y_{1,l} & y_{2,l} & y_{3,l} \\ y_{1,l+1} & y_{2,l+1} & y_{3,l+1} \end{vmatrix}$$

and the vector

$$\mathbf{w} = (\Delta_0, \Delta_1, \Delta_2, \Delta_3) \in \mathbb{Z}^4.$$

We see that $\mathbf{w}$ is orthogonal to the subspace Π_l, that is

$$\Delta_0 x_j + \Delta_1 y_{1,j} + \Delta_2 y_{2,j} + \Delta_3 y_{3,j} = 0, \quad j = l-1, l, l+1.$$

So

$$\Delta_0 = -\sum_{i=1}^{3} \Delta_i \frac{y_{i,l}}{x_l} = -\sum_{i=1}^{3} \Delta_i \Big(\frac{y_{i,l}}{x_l} - \theta_i\Big) - \sum_{i=1}^{3} \Delta_i \theta_i.$$

As $|y_{i,l}/x_l - \theta_i| \leqslant 1$ we see that

$$|\Delta_0| \leqslant \big(1 + \max_{1\leqslant i\leqslant 3} |\theta_i|\big)(|\Delta_1| + |\Delta_2| + |\Delta_3|). \tag{14}$$

However,

$$36\Delta^2 = \Delta_0^2 + \Delta_1^2 + \Delta_2^2 + \Delta_3^2. \tag{15}$$

From (14), (15) we deduce the inequality

$$\Delta \leqslant \frac{1}{6}\big(2 + \max_{1\leqslant i\leqslant 3} |\theta_i|\big)(|\Delta_1| + |\Delta_2| + |\Delta_3|),$$

and the lemma follows. □

8. Proof of Theorem 5

Take $\alpha < \alpha(\Theta)$. Then

$$\zeta_l \leqslant x_{l+1}^{-\alpha} \tag{16}$$

for all l large enough.

Consider best approximation vectors $\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, y_{2,\nu}, y_{3,\nu})$. From the condition that the numbers $1, \theta_1, \theta_2, \theta_3$ are linearly independent over $\mathbb{Z}$ we see that there exist infinitely many pairs of indices $\nu < k, \nu \to +\infty$ such that

• both the triples

$$\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}; \quad \mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1}$$

consist of linearly independent vectors;

• there exists a two-dimensional linear subspace π such that

$$\mathbf{z}_l \in \pi, \quad \nu \leqslant l \leqslant k; \quad \mathbf{z}_{\nu-1} \notin \pi, \quad \mathbf{z}_{k+1} \notin \pi;$$

- the vectors

$$\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_k, \mathbf{z}_{k+1}$$

are linearly independent.

Consider the two-dimensional lattice

$$\Lambda = \pi \cap \mathbb{Z}^4.$$

By Lemma 1, its two-dimensional fundamental volume $\det \Lambda$ satisfies

$$\det \Lambda \asymp_\Theta \zeta_\nu x_{\nu+1} \asymp_\Theta \zeta_{k-1} x_k. \tag{17}$$

Consider the two dimensional orthogonal complement $\pi^\perp$ to π and the lattice

$$\Lambda^\perp = \pi^\perp \cap \mathbb{Z}^4.$$

It is well-known that

$$\det \Lambda^\perp = \det \Lambda. \tag{18}$$

Consider the lattices

$$\Gamma_\nu = (\operatorname{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1})) \cap \mathbb{Z}^4, \quad \Gamma_k = (\operatorname{span}(\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1})) \cap \mathbb{Z}^4$$

and primitive integer vectors $\mathbf{w}_\nu, \mathbf{w}_k \in \mathbb{Z}^4$ which are orthogonal to $\Pi_\nu = \operatorname{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1})$, $\Pi_k = \operatorname{span}(\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1})$ respectively. Obviously

$$\mathbf{w}_\nu, \mathbf{w}_k \in \Lambda^\perp.$$

Put

$$b = \frac{1}{2}\left(-\frac{\alpha}{1-\alpha} + \sqrt{\left(\frac{\alpha}{1-\alpha}\right)^2 + \frac{4\alpha}{1-\alpha}}\right) \in (0,1), \quad a = 1-b,$$

so

$$\frac{\alpha}{1-\alpha} + b = g_3(\alpha).$$

Then

$$\det \Lambda^\perp \leqslant |w_\nu| \cdot |w_k|,$$

where $|\cdot|$ stands for the Euclidean norm, and so we obtain that either

$$\det \Gamma_\nu = |\mathbf{w}_\nu| \geqslant (\det \Lambda^\perp)^a = (\det \Lambda)^a \tag{19}$$

or

$$\det \Gamma_k = |\mathbf{w}_k| \geqslant (\det \Lambda^{\perp})^b = (\det \Lambda)^b \tag{20}$$

(using (18)).

If (19) holds then by Lemma 2, (13), (11) and (17) we see that

$$\zeta_{\nu-1}\zeta_\nu x_{\nu+1} \gg |\Delta_j| \gg_\Theta \det \Gamma_\nu \gg_\Theta (\det \Lambda)^a \gg (\zeta_\nu x_{\nu+1})^a$$

(here Δ_j is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_\nu$). From the definition of a and (16) we see that

$$x_{\nu+1} \gg_\Theta x_\nu^{g_3(\alpha)}.$$

We apply (16) again to obtain

$$\zeta_\nu \ll_\Theta x_\nu^{-\alpha g_3(\alpha)}.$$

If (20) holds then by Lemma 2, (13), (11) and (17) we see that

$$\zeta_{k-1}\zeta_k x_{k+1} \gg |\Delta_{j'}| \gg_\Theta \det \Gamma_k \gg_\Theta (\det \Lambda)^b \gg (\zeta_{k-1} x_k)^b$$

(here $\Delta_{j'}$ is the determinant from Lemma 2 applied to the lattice $\Gamma = \Gamma_k$). From the definition of b and (16) we see that

$$x_{k+1} \gg_\Theta x_k^{g_3(\alpha)}.$$

We apply (16) again to obtain

$$\zeta_k \ll_\Theta x_k^{-\alpha g_3(\alpha)}.$$

Theorem 5 is proved. □

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$$H(U \cap V)H(U+V) \ll_n H(U)H(V).$$

To prove our Theorem 5 one can use this inequality for

$$U = \operatorname{span}(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}), \quad V = \operatorname{span}(\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1}).$$

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