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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 12 (1971), No. 1, 39--49

Persistent URL: http://dml.cz/dmlcz/142258

T-quasigroups

Part I.

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Received 17 September 1970

In this paper and others that are to appear either in AUC or CMUC we are studying a certain class of quasigroups very closely related to Abelian groups. An impulse to our investigation was Toyoda's theorem:

Be $Q(o)$ an Abelian quasigroup, e. g. a quasigroup satisfying the identity

$$(a \, o \, b) \, o \, (c \, o \, d) = (a \, o \, c) \, o \, (b \, o \, d).$$

Then there is an Abelian group $Q(+)$, its automorphisms φ, ψ and $g \in Q$ such that $\varphi\psi = \psi\varphi$ and for every $x, y \in Q$,

$$x \, o \, y = \varphi(x) + \psi(y) + g. \qquad (\cdot)$$

This theorem was proved by Toyoda, Murdoch and Bruck in the early 40's. (See: K. Toyoda, On axioms of linear functions, Proc. Imp. Acad. Tokyo, 1941, 17; D. G. Murdoch, Structure of Abelian quasigroups, Trans. Amer. Math. Soc., 1941, 43; R. H. Bruck, Some results in the theory of quasigroups, Trans. Amer. Math. Soc., 1944, 55.) The proof of Toyoda's theorem can be also found in the book of V. D. Belousov: Osnovy teorii kvazigrupp i lup (Russian). As the requirement of commutativity of the automorphisms φ, ψ seems to have no principial significance in the algebraic properties (in relation to Abelian groups) of quasigroups satisfying $(\cdot)$, we came to the definition of T – quasigroup. Our aim is to clear up the relation between the algebraic properties of T – quasigroups and Abelian groups and to determine to what extent the algebraic properties of T – quasigroups are similar to those of Abelian groups. In this first part we investigate some basic properties of T – quasigroups and give several examples. First we shall make several arrangements concerning notation. A groupoid with the underlying set G and the operation o we shall denote by $G(o)$, e. g. the symbol of operation we put into brackets behind the symbol of the underlying set. Instead of $G(.)$, $a \, . \, b$, $(a \, . \, b) \, . \, (c \, . \, d)$ we shall write G, ab, $ab \, . \, cd$ respectively e.t.c. We shall always denote an Abelian group with the underlying set G by $G(+)$ or $G(o)$, O being zero in $G(+)$, e zero in $G(o)$, — subtraction in $G(+)$, $\cdot$ subtraction in $G(o)$. If φ is a mapping of the set A into the set B and ψ a mapping of the set B into the set C then the composite mapping of the set A into the set C we shall denote by $\psi\varphi$. If X_i, $i \in I$ is a system of groupoids and φ_i is an endomorphism of X_i, then the symbol $\prod_{i \in I} X_i$ means a Cartesian product of groupoids X_i, $\prod_{i \in I} \varphi_i$ is an endomorphism of $\prod_{i \in I} X_i$ such that its restriction on X_i, $\prod_{i \in I} \varphi_i | X_i$, is equal to φ_i, $i \in I$, and $< x_i >$ means an element of $\prod_{i \in I} X_i$ such that its components are $x_i \in X_i, i \in I$. Let $Q(\cdot)$ be a groupoid, $H \subseteq Q$. The subgroupoid generated in $Q(\cdot)$ by the set H we shall denote by $\{H\}^{\cdot}$. The left (right) translation by element a we shall denote by $L_a^{\cdot}$ $(R_a^{\cdot})$. Let Q be a quasigroup. The group of all permutations of the set Q we shall denote by S_Q, the multiplication (associated) group of Q by G_Q, the group of automorphisms of Q by $Aut\ Q$. A right (left) local unit belonging to the element a we shall denote by $e(a)$ or e_a $(f(a), f_a)$. Thus $f(a) . a = a . e(a) = a$. A mapping $\lambda : Q \to Q$ is called left (right) regular if there is a mapping $\lambda^{\cdot} : Q \to Q$ such that for every $x, y \in Q$,

$$\lambda(xy) = \lambda^{\cdot}(x) . y \ (\lambda(xy) = x . \lambda^{\cdot}(y)).$$

A mapping $\varphi : Q \to Q$ is called central regular if there is a mapping $\varphi^{\cdot}$ such that $\varphi(x) . y = x . \varphi^{\cdot}(y)$.

Our definition of left and right regular mappings is slightly more general than the usual one. Some results of the theory of regular mappings can be found in the paper of T. Kepka: "Regular mappings of groupoids".

1° Introduction

Definition 1: Let Q be a quasigroup. A tetrad $(Q(+), \varphi, \psi, g)$, $Q(+)$ being an Abelian group, φ and ψ its automorphisms and $g \in Q$, is called a T – form of the quasigroup Q if for every $x, y \in Q$,

$$xy = \varphi(x) + \psi(y) + g. \tag{1}$$

The group $Q(+)$ is called a T – group of the quasigroup Q.

Definition 2: A quasigroup Q is called a T – quasigroup if there exists at least one T – form of Q.

The following Lemma is obvious from (1).

Lemma 1: Let Q be a T – quasigroup and $Q(+)$ its T – group. Then the group $Q(+)$ is a principal isotope of Q.

Theorem 1: Let $Q(+)$ and $Q(o)$ be two T – groups of a T – quasigroup Q. Then the groups $Q(+)$ and $Q(o)$ are isomorphic.

Proof: Groups $Q(+)$ and $Q(o)$ are, by Lemma 1, isotopes of the same quasigroup Q. Hence they are isotopic. According to Albert's theorem, they are isomorphic.

Lemma 2: Let $Q(+)$ be an Abelian group, φ and ψ its automorphisms, $Q(o)$ a loop with unit e. Be α, β permutations of the set Q and g an element of Q such that for every $x, y \in Q$,

$$\alpha(x) \ o \ \beta(y) = \varphi(x) + \psi(y) + g \tag{2}$$

Then $Q(o)$ is the Abelian group and there exist its automorphisms φ_1, ψ_1 such that for every $x, y \in Q$,

$$\alpha(x) = \varphi_1(x) \, o \, \alpha(e), \ \beta(x) = \psi_1(e) \, o \, \beta(e). \tag{3}$$

Proof: Define permutations α_1, β_1 as follows:

$$\alpha_1(x) = \varphi\alpha^{-1}(x), \ \beta_1(x) = \psi\beta^{-1}(x) + g \ \text{ for every } \ x \in Q.$$

From (2) follows for every $x \in Q$,

$$x \, o \, y = \alpha_1(x) + \beta_1(y). \tag{4}$$

Thus the loop $Q(o)$ is the principal isotope of the Abelian group $Q(+)$ and henceforth $Q(o)$ is an Abelian group.
By (4), we can write

$$x + y = \alpha_1^{-1}(x) \, o \, \beta_1^{-1}(y). \tag{5}$$

Thus $\alpha_1^{-1}(x) = x \cdot \beta_1^{-1}(O)$, $\beta_1^{-1}(y) = y \cdot \alpha_1^{-1}(O)$. Therefore

$$x + y = x \, o \, y \cdot a, \ \text{where} \ a = \alpha_1^{-1}(O) \, o \, \beta_1^{-1}(O). \tag{6}$$

Define a mapping γ, $\gamma(x) = x \cdot a$ for every $x \in Q$. Since $\gamma(x + y) = (x + y) \cdot a = x \, o \, y \cdot a \cdot a = x \cdot a \, o \, y \cdot a = \gamma(x) \, o \gamma(y)$, γ is an isomorphism of $Q(+)$ onto $Q(o)$.
Evidently, a mapping $\varphi_1 = \gamma\varphi\gamma^{-1}$ is an automorphism of the group $Q(o)$.
From (2) we get $\varphi(x) = \alpha(x) \, o \, b$ where $b = \beta\psi^{-1}(-g)$. Hence $\varphi_1(x) = \gamma\varphi\gamma^{-1}(x) = \varphi\gamma^{-1}(x) \cdot a = \alpha\gamma^{-1}(x) \, o \, b \cdot a = \alpha(x \, o \, a) \, o \, c$, where $c = b \cdot a$. Thus for every $x \in Q$,

$$\alpha(x) = \varphi_1 (x \cdot a) \cdot c = \varphi_1(x) \, o \, \varphi_1(\cdot a) \cdot c = \varphi_1(x) \, o \, \alpha(e).$$

Similarly we can prove the existence of an automorphism ψ_1 of the group $Q(o)$ such that $\beta(x) = \psi_1(x) \, o \, \beta(e)$.
This completes the proof of (3).

Lemma 3: Let Q be a T – quasigroup and $a, b \in Q$. Define $Q(o) = Q^{(R_a^{-1}, L_b^{-1}, 1)}$. Then $Q(o)$ is an Abelian group. Put for every $x \in Q$, $\varphi_1(x) = R_a(x) \cdot [ba \, . \, a]$, $\psi_1(x) = L_b(x) \cdot [b \, . \, ba]$. Then φ_1, ψ_1 are automorphisms of the group $Q(o)$ and the tetrad $(Q(o), \varphi_1, \psi_1, ba \, . \, ba)$ is a T – form of the T – quasigroup Q.

Proof: It is well known, that for every quasigroup and for every $a, b \in Q$ the quasigroup $Q^{(R_a^{-1}, L_b^{-1} 1)}$ is a loop with unit $e = ba$. Since $Q(o) = Q^{(R_a^{-1}, L_b^{-1} 1)}$, $Q = Q(o)^{(R_a, L_b, 1)}$. Hence for every $x, y \in Q$, $xy = R_a(x) \, o \, L_b(y)$.
Be $(Q(+), \varphi, \psi, g)$ any T – form of the T – quasigroup Q. Then we have for every $x, y \in Q$,

$$xy = \varphi(x) + \psi(y) + g = R_a(x) \, o \, L_b(y)$$

In view of Lemma 2, $Q(o)$ is an Abelian group and permutations φ_1, ψ_1, where $\varphi_1(x) = R_a(x) \cdot R_a(e) = R_a(x) \cdot [ba \, . \, a]$, $\psi_1(x) = L_b(x) \cdot L_b(e) = L_b(x) \cdot [b \, . \, ba]$, are its automorphismus.
For every $x, y \in Q$ we can write

$$xy = R_a(x) \, o \, L_b(y) = \varphi_1(x) \, o \, \psi_1(y) \, o \, [(ba \, . \, a) \, o \, (b \, . \, ba)].$$

Further, $(ba\,.\,a)\;o\;(b\,.\,ba) = R_a^{-1}\,(ba\,.\,a)\,.\,L_b^{-1}\,(b\,.\,ba) = ba\,.\,ba$. Thus $(Q(o), \varphi_1, \psi_1, ba\,.\,ba)$ is a T – form of the T – quasigroup Q.

Lemma 4: Let Q be a T – quasigroup and a loop $Q(o)$ be a principal isotope of the quasigroup Q, $Q(o) = Q^{(\alpha,\beta,1)}$. Then $Q(o)$ is an Abelian group and the mappings φ_1, ψ_1, where $\varphi_1(x) = \alpha^{-1}(x)\cdot\alpha^{-1}(e)$, $\psi_1(x) = \beta^{-1}(x)\cdot\beta^{-1}(e)$ are its automorphisms. The tetrad $(Q(o), \varphi_1, \psi_1, \alpha^{-1}(e)\;o\;\beta^{-1}(e))$ is a T – form of Q.

Proof: Since $Q(o)$ is the principal isotope of Q, there exist elements a, b of Q such that $\alpha = R_a^{-1}$, $\beta = L_b^{-1}$. Further, $e = ba$. Now we can use Lemma 3.

Theorem 2: Let Q be a T – quasigroup and $Q(o)$ be a loop. Then, $Q(o)$ is a T – group of the T – quasigroup Q if and only if $Q(o)$ is a principal isotope of Q.

Proof: If $Q(o)$ is a T – group of Q, then, by Lemma 1, $Q(o)$ is a principal isotope of Q. Conversely, if $Q(o)$ is a principal isotope of Q, then, by Lemma 4, $Q(o)$ is a T – group of Q.

Lemma 5: Let $(Q(+), \varphi, \psi, g)$ be an arbitrary T – form of a T – quasigroup Q. Then $\varphi = R_{e(o)}$, $\varphi = L_{f(o)}$, $g = O\,.\,O$.

Proof: From (1) follows $\varphi(x) = x\,.\,\psi^{-1}(-g)$ for every $x \in Q$.
But $O\,.\,\psi^{-1}(-g) = \varphi(O) + \psi\psi^{-1}(-g) + g = O$. Thus $\psi^{-1}(-g) = e(O)$ and $\varphi = R_{e(o)}$. Similarly $\psi = L_{f(o)}$. Further, $O\,.\,O = \varphi(O) + \psi(O) + g = g$.

Lemma 6: Be Q a T – quasigroup, u an arbitrary element of Q. Then there exists a T – group $Q(+)$ of the T – quasigroup Q such that u is zero in $Q(+)$.

Proof: Put $Q(+) = Q^{(R_u{}^{-1}, L_{f(u)}{}^{-1}, 1)}$. $Q(+)$ is a loop having unit O, $O = f(u)\,.\,u = u$. By Theorem 2, $Q(+)$ is the T – group of Q.

Lemma 7: Let Q be a T – quasigroup and v be an arbitrary element of Q. Then there is $z \in Q$ such that $zz = v$ if and only if there exists a T – form $(Q(+), \varphi, \psi, v)$ of the T – quasigroup Q.

Proof: 1) Be $z \in Q$ such that $zz = v$. With regard to Lemma 6 there is a T – form $(Q(o), \varphi, \psi, g)$ of Q such that z is zero in $Q(o)$. By Lemma 5, $g = zz = v$.
2) Be $(Q(+), \varphi, \psi, v)$ a T – form of Q. Then $v = O\,.\,O$.

Lemma 8: Let Q be a T – quasigroup and u, v be arbitrary elements of Q. Then there exists a T – form $(Q(+), R_u, L_v, g)$ of Q if and only if there is $z \in Q$ such that $vz = zu = z$.

Proof: Let $z \in Q$ be such that $v = f(z)$, $u = e(z)$. Select a T – form $(Q(o), \varphi, \psi, h)$ such that z is zero in $Q(o)$. Then, by Lemma 5, $\varphi = R_{e(z)} = R_u$, $\psi = L_{f(z)} = L_v$. If, on the contrary, $(Q(+), R_u, L_v, g)$ is a T – form of Q, then R_u, L_v are certain automorphisms of $Q(+)$. Hence $R_u(O) = O = O\,.\,u$, $L_v(O) = O = v\,.\,O$.

Lemma 9: Let Q be a T – quasigroup and $(Q(+), \varphi, \psi, g)$, $(Q(o), \xi, \varrho, h)$ two its T – forms. Let the groups $Q(+)$ and $Q(o)$ have the same zero. Then $Q(+) = Q(o)$, $\varphi = \xi$, $\psi = \varrho$, $g = h$.

Proof: By Lemma 5, $g = O\,.\,O = h$, $\varphi = R_{e(o)} = \eta$, $\psi = L_{f(o)} = \varrho$.
Further, $xy = \varphi(x) + \psi(y) + g = \xi(x)\;o\;\varrho(y)\;o\;h = \varphi(x)\;o\;\psi(y)\;o\;g$.

As φ, ψ are permutations, we have for every $x, y \in Q, x + y + g = x \circ y \circ g$. Let u, v be arbitrary elements of Q. We can write

$$u + v = u + (v - g) + g = u \circ (v - g) \circ g = u \circ (O \circ (v - g) \circ g) =$$
$$= u \circ (O + v - g + g) = u \circ v$$

Thus $Q(+) = Q(o)$.

Definition 3: Let Q be a T – quasigroup, P its subquasigroup. A T – form $(Q(+), \varphi, \psi, g)$ of the T – quasigroup Q is called P – canonic if $P(+)$ is subgroup in $Q(+)$, $\varphi|P$ and $\psi|P$ are automorphisms of $P(+)$ and $g \in P$.

Lemma 10: Let P be a subquasigroup of a T – quasigroup Q. Let a, b be any elements of P. Set $Q(+) = Q^{(R_a^{-1}, L_b^{-1}, 1)}$. Then the corresponding T – form $(Q(+), \varphi, \psi, g)$ is P – canonic.

Proof: T – form $(Q(+), \varphi, \psi, g)$ exists according to Lemma 3. The element ba is zero in $Q(+)$. As P is a quasigroup, $ba = O \in P$, hence $O \,.\, O = g \in P$. Moreover, for every $x, y \in P$ is $R_a^{-1}(x)$, $L_b^{-1}(y) \in P$. Thus $z = R_a^{-1}(x) \,.\, L_b^{-1}(y) = x + y$ is element of P. If further $x + u = y$, then $y = R_a^{-1}(x) \,.\, L_b^{-1}(y)$ so that $L_b^{-1}(u) \in P$, and hence $L_b L_b^{-1}(u) \in P$. But $L_b L_b^{-1}(u) = u$. We have proved that P is a subgroup in $Q(+)$. By Lemma 5, $\varphi = R_{e(o)}$, $\psi = L_{f(o)}$. But $e(O), f(O) \in P$, therefore $\varphi|P$, $\psi|P$ are permutations of the set P, hence automorphisms of $P(+)$.

Theorem 3: Every subquasigroup of a T – quasigroup Q is a T – quasigroup.

Proof.: Let P be a subquasigroup of Q and a be an arbitrary element of P. Put $Q(+) = Q^{(R_a^{-1}, L_{f(a)}^{-1}, 1)}$. By Lemma 10 and Lemma 3, $(Q(+), R_{e(a)}, L_{f(a)}, a \,.\, a)$ is *a* P–canonic T – form of Q. Therefore $(P(+), R_{e(a)}|P, L_{f(a)}|P, a \,.\, a)$ is a T – form of P. Thus P is T – quasigroup.

Lemma 11: Let P be a subquasigroup of a T – quasigroup Q. A T – form $(Q(+), \varphi, \psi, g)$ of Q is P – canonic if and only if $O \in P$.

Proof: If $(Q(+), \varphi, \psi, g)$ is P – canonic, then $P(+)$ is a subgroup of $Q(+)$, so that $O \in P$.

On the contrary, be $O \in P$. Define $Q(o) = Q^{(R_o^{-1}, L_{f(o)}^{-1}, 1)}$. By Lemma 3, there exists a T – form $(Q(o), \xi, \varrho, h)$ of the T – quasigroup Q. Since $O \in P$, $f(O) \in P$. Hence, by Lemma 10, the T – form $(Q(o), \xi, \varrho, h)$ is P – canonic. As $f(O) \,.\, O = O$ is zero in $Q(o)$, from Lemma 9 follows that $Q(+) = Q(o)$, $\varphi = \xi$, $\psi = \varrho, g = h$. Thus $(Q(+), \varphi, \psi, g)$ is a P – canonic T – form of Q.

Lemma 12: Let Q and P be two T – quasigroups. Let a mapping ξ, $\xi : Q \to P$, be an epimorphism and $(Q(+), \varphi, \psi, g)$ be an arbitrary T – form of Q. There are $a, b \in Q$ such that $Q(+) = Q^{(R_a^{-1}, L_b^{-1}, 1)}$. Define $P(o) = P^{(R_{\xi(a)}^{-1}, L_{\xi(b)}^{-1}, 1)}$. Then $\xi : Q(+) \to P(o)$ is an epimorphism, hence $P(o)$ is an Abelian group. If x, y are arbitrary elements of Q such that $\xi(x) = \xi(y)$, then $\xi\varphi(x) = \xi\varphi(y)$ and $\xi\psi(x) = \xi\psi(y)$. Define mappings φ_1, ψ_1 as follows: For every $p \in P$, $\varphi_1(p) = \xi\varphi(x)$, $\psi_1(p) = \xi\psi(x)$, where $x \in Q$ such that $\xi(x) = p$. Then φ_1, ψ_1 are automorphisms of the group $P(o)$ and $(P(o), \varphi_1, \psi_1, \xi(y))$ is a T – form of the quasigroup P.

Proof: For every $x, y \in Q$ we have
$\xi(x+y) = \xi(R_a^{-1}(x) \,.\, L_b^{-1}(y)) = R_{\xi(a)}^{-1}\, \xi(x) \,.\, L_{\xi(b)}^{-1}\, \xi(y) = \xi(x) \,o\, \xi(y)$
That is, ξ is a homomorphism of the group $Q(+)$ onto $P(o)$.
Further, $\xi(x) \,.\, \xi(y) = \xi(xy) = \xi(\varphi(x) + \psi(y) + g) = \xi\varphi(x) \,o\, \xi\psi(y) \,o\, \xi(g)$.
Obviously $\xi(O) = e$, where e is the zero of the group $P(o)$. Let $x, y \in Q$ and $\xi(x) = \xi(y)$. Then $\xi(x) \,.\, e = \xi(x) \,.\, \xi(O) = \xi(y) \,.\, \xi(O) = \xi(y) \,.\, e$. Hence $\xi\varphi(x) \,o\, \xi(g) = \xi\varphi(y) \,o\, \xi(g)$, so that $\xi\varphi(x) = \xi\varphi(y)$. Similarly we can prove that $\xi\psi(x) = \xi\psi(y)$. Thus, the definition of mappings φ_1, ψ_1 is correct.
Be p, s arbitrary elements of P. There are $x, y \in Q$ such that $\xi(x) = p$, $\xi(y) = s$. We have $\varphi_1(p) \,o\, \varphi_1(s) = \xi\varphi(x) \,o\, \xi\varphi(y) = \xi(\varphi(x) + \varphi(y)) = \xi\varphi(x+y) = \varphi_1(p \,o\, s)$.
Therefore φ_1 is an endomorphism of the group $P(o)$. Similarly ψ_1 is an endomorphism of $P(o)$. Moreover, $ps = \xi(x) \,.\, \xi(y) = \xi\varphi(x) \,o\, \xi\psi(y) \,o\, \xi(g) = \varphi_1(p) \,o\, \psi_1(s) \,o\, \xi(g)$. Therefore φ_1, ψ_1 are automorphisms of $P(o)$ and $(P(o), \varphi_1, \psi_1, \xi(g))$ is a T – form of the quasigroup P.
The following theorem is an easy consequence of Lemma 12.

Theorem 4: If a quasigroup P is a homomorphic image of a T – quasigroup Q, then P is a T – quasigroup.

Theorem 5: A cartesian product of any non – empty system of T – quasigroups is a T – quasigroup.

Proof.: Let $\{Q_i, i \in I\}$ be a given system of T – quasigroups. For every $i \in I$ be $(Q_i(+), \varphi_i, \psi_i, g_i)$ arbitrary T – form of the T – quasigroup Q_i. Define $Q(+) = \prod_{i \in I} Q_i(+)$, $\varphi = \prod_{i \in I} \varphi_i$, $\psi = \prod_{i \in I} \psi_i$. Then $(Q(+), \varphi, \psi, < g_i >)$ is obviously a T – form of the quasigroup $\prod_{i \in I} Q_i$.
Now we can formulate:

Theorem 6: All T – quasigroups form a primitive class in the class of all quasigroups.

2° Abelian groups whose every isotope is a T – quasigroup

Lemma 13: Let $Q(+)$ be an Abelian group. Every quasigroup isotopic to $Q(+)$ is a T – quasigroup if and only if the symmetric group S_Q of the set Q is generated by all translations and automorphisms of the group $Q(+)$.

Proof: Denote by G the group generated by all translations and automorphisms of the group $Q(+)$ in the group S_Q. As for every $a, b \in Q(+)$ and every $\varphi \in Aut\ Q(+)$ is $L_a = R_a$, $L_a L_b = L_{a+b}$, $L_a^{-1} = L_{(-a)}$, $\varphi L_a = L_{\varphi(a)}\varphi$, it is possible to express every element of G in the form $L_u\psi$, where $\psi \in Aut\ Q(+)$ and $u \in Q(+)$ are convenient elements.
Let $G = S_Q$ and $Q(\cdot)$ be any isotope of $Q(+)$, $Q(\cdot) = Q(+)^{(\alpha,\beta,\gamma)}$. We can suppose that $\gamma = 1$ ($Q(\cdot)^{(\gamma^{-1},\gamma^{-1},\gamma^{-1})} = Q(+)^{(\alpha\gamma^{-1},\beta\gamma^{-1},1)}$ and quasigroups $Q(\cdot)$ and $Q(\cdot)^{(\gamma^{-1}\,\gamma^{-1},\gamma^{-1})}$ are isomorphic). There are $\varphi, \psi \in Aut\ Q(+)$ and $a, b \in Q(+)$ such that $\alpha = L_a\varphi$, $\beta = L_b\psi$. Put $g = a + b$. Then for every $x, y \in Q(+)$ we have

$$x \cdot y = \alpha(x) + \beta(y) = \varphi(x) + a + b + \psi(y) = \varphi(x) + \psi(y) + g.$$

Thus $Q(\cdot)$ is a T – quasigroup.

On the contrary, let every isotope of the group $Q(+)$ be a T – quasigroup. Be $\alpha \in S_Q$ an arbitrary permutation. Put $Q(\cdot) = Q(+)^{(\alpha,1,1)}$. $Q(\cdot)$ is a T – quasigroup, so that, by Lemma 4, there is $\varphi \in Aut\ Q(+)$ such that $\alpha = L_{\alpha(o)}\varphi$. Therefore $\alpha \in G$. Thus $G = S_Q$.

Lemma 14: Let $Q(+)$ be an Abelian group satisfying one of following two conditions:

(i) $Q(+)$ has at most three elements.

(ii) $Q(+)$ has four elements and each of them has order 2.

Then every quasigroup isotopic to $Q(+)$ is a T – quasigroup.

Proof: In view of Lemma 13, we shall prove that the group S_Q is generated by all translations and automorphisms of the group $Q(+)$.

If the group $Q(+)$ has at most two elements, then every element of the group S_Q is a translation of the group $Q(+)$.

Let $Q(+)$ have three elements. Then the group $G_{Q(+)}$ of all translations of $Q(+)$ has three elements. The group $Aut\ Q(+)$ has only two elements. The intersection $G_{Q(+)} \cap Aut\ Q(+)$ is, evidently, the unit subgroup. As the union $Aut\ Q(+) \cup G_{Q(+)}$ has four elements, the group G generated in S_Q by all translations and automorphisms of $Q(+)$ has at least four elements. By Lagrange's theorem, $S_Q = G$.

Let $Q(+)$ have four elements and let every (nonzero) element of $Q(+)$ have order 2. The group of translations $G_{Q(+)}$ has four elements, the group $Aut\ Q(+)$ has six elements and $G_{Q(+)} \cap Aut\ Q(+) = 1$. Denote by G a group generated in S_Q by the set $Aut\ Q(+) \cup G_{Q(+)}$. Let $\alpha \in G_{Q(+)}$, $\alpha \neq 1$. If $\varphi \in Aut\ Q(+)$, $\varphi \neq 1$, then $\alpha\varphi \notin Aut\ Q(+)$, $\alpha\varphi \notin G_{Q(+)}$, but $\alpha\varphi \in G$. Therefore G has at least fourteen elements. Since S_Q has 24 elements, it must be $G = S_Q$.

Lemma 15: Let $Q(+)$ be an Abelian group having at least four elements. Let in $Q(+)$ exist a nonzero element g of order $O(g) \neq 2$. Then there exists an isotope $Q(\cdot)$ of $Q(+)$ which is not a T – quasigroup.

Proof: We have $g \neq O$, $O(g) \neq 2$. Define a permutation α of the set Q such that $\alpha(O) = g$, $\alpha(g) = O$ and $\alpha(x) = x$ for each other $x \in Q$. Consider the principal isotope $Q(\cdot) = Q(+)^{(\alpha,1,1)}$ of the group $Q(+)$. Suppose $Q(\cdot)$ to be a T – quasigroup. Then, by Lemma 4, the mapping φ, $\varphi(x) = \alpha(x) - \alpha(O) = \alpha(x) - g$ for every $x \in Q(+)$, is an automorphism of the group $Q(+)$. Especially we have $\varphi(x - g) = = \varphi(x) - \varphi(g) = \varphi(x) - \alpha(g) + \alpha(O) = \varphi(x) + g$ and $\varphi(x - g) = \alpha(x - g) - g = = \varphi(x) + g = \alpha(x)$. Thus $\alpha(x - g) = \alpha(x) + g$. (7)

Since $Q(+)$ has at least four elements, there exists $y \in Q(+)$ such that $y \neq O, y \neq g$, $y \neq 2g$. For such an element y is $y - g \neq O$, $y - g \neq g$ and thus $\alpha(y - g) = = y - g$. From (7) we have $\alpha(y - g) = y - g = \alpha(y) + g = y + g$. Hence $y - g = = y + g$ and hence, $O(g) = 2$, which is a contradiction.

Lemma 16: Let $Q(+)$ be an Abelian group having at least five elements. Let for every $g \in Q(+)$ be $O(g) = 2$. Then there exists a quasigroup $Q(\cdot)$ isotopic to $Q(+)$ such that $Q(\cdot)$ is not a T – quasigroup.

Proof: Select $g_1, g_2 \in Q(+)$ such that $g_1 \neq O, g_2 \neq O, g_1 \neq g_2$. Define a permutation α of the set Q such that $\alpha(O) = g_1$, $\alpha(g_1) = g_2$, $\alpha(g_2) = O$ and $\alpha(x) = x$ for each other $x \in Q(+)$. Suppose that $Q(\cdot) = Q(+)^{(\alpha,1,1)}$ is a T – quasigroup. Then the mapping φ, $\varphi(x) = \alpha(x) - \alpha(O)$ is an automorphism of the group $Q(+)$. There exists $y \in Q(+)$ such that $y \neq O, y \neq g_1, y \neq g_2, y \neq g_1 + g_2$. Since every element of $Q(+)$ has order 2, we have $y + g_2 \neq O, y + g_2 \neq g_1, y + g_2 \neq g_2$. Thus $\alpha(y) = y$, $\alpha(y + g_2) = y + g_2$.

Further, $\varphi(y + g_2) = \alpha(y + g_2) - \alpha(O) = y + g_2 - g_1 = y + g_2 + g_1 = \varphi(y) + \varphi(g_2) = \alpha(y) - \alpha(O) - \alpha(O) = \alpha(y) = y$. Thus $y + g_1 + g_2 = y$. Hence $g_1 + g_2 = O$, and hence, $g_1 = g_1 + g_1 + g_2 = g_2$, which is a contradiction.

Theorem 7: Let $Q(+)$ be an Abelian group. Then every quasigroup isotopic to the group $Q(+)$ is a T – quasigroup if and only if $Q(+)$ satisfies one of the following conditions:

(i) $Q(+)$ has at most three elements.

(i i) $Q(+)$ has four elements and each of them has order 2.

Proof: This theorem follows directly from Lemmas 14, 15, 16.

3° Several examples of T – quasigroups

Theorem 8: Every Abelian group is a T – quasigroup.

Proof: Be $Q(+)$ any Abelian group. Then evidently $(Q(+), 1, 1, O)$ is a T – form of the group $Q(+)$.

Definition 4: Let n be a positive integer, $n \geqslant 2$. Quasigroup Q is called an α_n – quasigroup if for every $x_1, \ldots, x_n \in Q$ and for every $y_1, \ldots, y_n \in Q$,

$$\begin{aligned}&(x_1 (x_2 (\ldots (x_{n-1} x_n)))) ((((y_n y_{n-1}) \ldots) y_2) y_1) = \\ &= (x_1 (x_2 (\ldots (x_{n-1} y_n)))) ((((x_n y_{n-1}) \ldots) y_2) y_1).\end{aligned} \tag{8}$$

Q is called a β_n – quasigroup if

$$\begin{aligned}&((((x_1 x_2) x_3) \ldots) x_n) (y_n (\ldots (y_3(y_2 y_1)))) = \\ &= ((((y_1 x_2) x_3) \ldots) x_n) (y_n (\ldots (y_3 (y_2 x_1)))).\end{aligned} \tag{9}$$

Theorem 9: Let n be a positive integer, $n \geqslant 2$. Let Q be an α_n – quasigroup or β_n – quasigroup. Then Q is a T – quasigroup.

Proof: Assume that Q is an α_n – quasigroup.

1) Be $x_1, \ldots, x_n, y_1, \ldots, y_n$ arbitrary elements of Q.

Define $a_1 = x_2 (x_3 (\ldots (x_{n-1} x_n)))$, $a_2 = x_2(x_3 (\ldots (x_{n-1} y_n)))$, $b_1 = (((y_n y_{n-1}) \ldots) y_3) y_2$, $b_2 = (((x_n y_{n-1}) \ldots) y_3) y_2$.

Let $x, y \in Q$. In view of (8), we can write

$$R_{a_1} (x) \,.\, L_{b_1} (y) = R_{a_2} (x) \,.\, L_{b_2} (y).$$

Thus $R_{a_1} R_{a_2}^{-1}(x) . y = x . L_{b_2} L_{b_1}^{-1}(y)$. Hence $R_{a_1} R_{a_2}^{-1}$ is a central regular mapping of Q. Let u, v be arbitrary elements of Q. There is $z \in Q$ such that $uz = v$. Since $R_{e(u)}(u) = u$, $v = R_z R_{e(u)}^{-1}(u)$. We can select the elements $x_2, \ldots x_n, y_2, \ldots, y_n$ such that $a_1 = z$, $a_2 = e(u)$. Then $R_{a_1} R_{a_2}^{-1}(u) = v$. We have proved that Q is a transitive quasigroup.

2) Be $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ fixed elements of Q. For every $x, y \in Q$ define

$$x \circ y = (x_1(x_2 (\ldots (x_{n-1} x)))) ((((yy_{n-1}) \ldots) y_2) y_1).$$

According to (8), $Q(\circ)$ is a commutative groupoid. Moreover, $Q(\circ)$ is a principal isotope of Q.

3) From 1) and 2) follows that Q is a principal isotope of some Abelian group $Q(+)$, $Q = Q(+)^{(\alpha,\beta,1)}$. By (8), we have

$$\begin{aligned}&\alpha(\alpha(x_1) + \beta(\alpha(x_2) + \beta (\ldots + \beta(\alpha(x_{n-1}) + \beta(x_n))))) + \\ &+ \beta(\beta(y_1) + \alpha(\beta(y_2) + \alpha (\ldots + \alpha(\beta(y_{n-1}) + \alpha(y_n))))) = \\ &= \alpha(\alpha(x_1) + \beta(\alpha(x_2) + \beta (\ldots + \beta(\alpha(x_{n-1}) + \beta(y_n))))) + \\ &+ \beta(\beta(y_1) + \alpha(\beta(y_2) + \alpha (\ldots + \alpha(\beta(y_{n-1}) + \alpha(x_n))))).\end{aligned} \tag{10}$$

Since α, β are permutations of the set Q, from (10) easily follows

$$\begin{aligned}&\alpha(x_1 + \beta(x_2 + \beta(\ldots + \beta(x_{n-1} + \beta (x_n))))) + \\ &+ \beta(y_1 + \alpha(y_2 + \alpha (\ldots + \alpha(y_{n-1} + \alpha(y_n))))) = \\ &= \alpha(x_1 + \beta (x_2 + \beta (\ldots + \beta (x_{n-1} + \beta (y_n))))) + \\ &+ \beta(y_1 + \alpha(y_2 + \alpha (\ldots + \alpha(y_{n-1} + \alpha(x_n))))).\end{aligned}$$

If we put $x_2 = x_3 = \ldots = x_{n-1} = y_1 = y_2 = \ldots = y_n = O$, we get for every $x_1, x_n \in Q$, $\alpha(x_1 + \beta^{n-1}(x_n)) + \beta^{n-1}(O) = \alpha(x_1 + \beta^{n-1}(O)) + \beta\alpha^{n-1}(x_n)$. Hence there are permutations γ, δ of the set Q such that for every $a, b \in Q$, $\alpha(a + b) = \gamma(a) + \delta(b)$. That is, α is a quasiautomorphism of the group $Q(+)$ and henceforth there exists an automorphism φ of the group $Q(+)$ and an element g_1 of Q such that for every $x \in Q$, $\alpha(x) = \varphi(x) + g_1$. Similarly we can prove the existence of $\psi \in Aut\ Q(+)$ and $g_2 \in Q$ such that for every $y \in Q$, $\beta(y) = \psi(y) + g_2$. Denote $g = g_1 + g_2$. Then $xy = \alpha(x) + \beta(y) = \varphi(x) + \psi(y) + g$. Thus Q is a T – quasigroup.

The proof for β_n – quasigroups is similar and of no principal difficulties.

Theorem 10: Let n be a positive integer, $n \geqslant 2$. Let Q be a T – quasigroup. Then Q is an α_n – quasigroup if and only if for any (and then for every) T – form $(Q(+), \varphi, \psi, g)$ of the quasigroup Q is $\varphi\psi^{n-1} = \psi\varphi^{n-1}$.

Proof: Since Q is a T – quasigroup, the condition (8) is obviously equivalent to the condition

$$\begin{aligned}&\varphi^2(x_1) + \varphi(g) + \varphi\psi\varphi(x_2) + \ldots + \psi\varphi^{n-1}(x_n) + g + \\ &+ \psi^2(y_1) + \psi(g) + \psi\varphi\psi(y_2) + \ldots + \varphi\psi^{n-1}(y_n) = \\ &= \varphi^2(x_1) + \varphi(g) + \varphi\psi\varphi(x_2) + \ldots + \psi\varphi^{n-1}(y_n) + g + \\ &+ \psi^2(y_1) + \psi(g) + \psi\varphi\psi(y_2) + \ldots + \varphi\psi^{n-1}(x_n).\end{aligned}$$

But this is equivalent to $\psi\varphi^{n-1}(x_n) + \varphi\psi^{n-1}(y_n) = \psi\varphi^{n-1}(y_n) + \varphi\psi^{n-1}(x_n)$ which is equivalent to $\varphi\psi^{n-1} = \psi\varphi^{n-1}$.

Theorem 11: Let Q be a T-quasigroup and n a positive integer, $n \geqslant 2$. Then Q is a β_n - quasigroup if and only if for any (and then for every) T - form ($Q(+)$, φ, ψ, g) of Q is $\varphi^n = \psi^n$.

Proof: The proof is quite similar to that of Theorem 11.

Remark: An α_2-quasigroup is often called Abelian quasigroup.

As a corollary of Theorem 10 we get:

Theorem 12: *A* T-quasigroup Q is Abelian if and only if for any (and then for every) its T - form ($Q(+)$, φ, ψ, g) is $\varphi\psi = \psi\varphi$.

Theorem 13: Let Q be a T - quasigroup and ($Q(+)$, φ, ψ, g) its T - form such that φ, ψ have finite order in the group S_Q. Then there is a natural number n such that Q is a β_n - quasigroup.

Corollary: Every finite T-quasigroup is a β_n-quasigroup for some convenient natural number n.

Proof: Denote $O(\varphi) = k$, $O(\psi) = l$. Then $\varphi^{kl} = (\varphi^k)^l = 1 = (\psi^l)^k = \psi^{kl}$. Therefore, by Theorem 11, Q is β_{kl} - quasigroup.

Definition 5: Quasigroup Q is called

B_1 - quasigroup if $\quad x(yz) = y(xz) \qquad (11)$

B_2 - quasigroup if $\quad (xy)z = (xz)y \qquad (12)$

B_3 - quasigroup if $\quad x(yz) = z(yx) \qquad (13)$

B_4 - quasigroup if $\quad (xy)z = (zy)x \qquad (14)$

for every $x, y, z \in Q$.

Theorem 14: A T - quasigroup Q is a left (right) loop if and only if for any (and then for every) its T - form (Q +), φ, ψ, g) is $\psi = 1$ ($\varphi = 1$).

Proof: We shall prove the statement for left loops only. Let the quasigroup Q have a T - form ($Q(+)$, φ, 1, g). Then for every $x \in Q$, $\varphi^{-1}(-g)\ x = -g + x + g = x$. Hence $\varphi^{-1}(-g)$ is a left unit in Q.

On the contrary, be e a left unit in Q. Let ($Q(+)$, φ, ψ, g) be arbitrary T - form of Q. Then for every $x \in Q$, $x = ex = \varphi(e) + \psi(x) + g$. For $x = O$ we get $\varphi(e) + g = O$, hence $\psi = 1$.

Theorem 15: Let a T-quasigroup Q be a left or right loop. Then the quasigroup Q is Abelian.

Proof: By Theorem 14 and Theorem 12.

Theorem 16: Let Q be a T - quasigroup. Then Q is a left (right) loop if and only if Q is a B_1 - quasigroup (B_2 - quasigroup).

Proof: Let Q be a left loop and e be its left unit. Then, by Theorem 15, Q is an Abelian quasigroup. For every $x, y, z \in Q$ we have $ex \,.\, yz = x(yz) = ey \,.\, xz = y(xz)$. Thus Q is a B_1 - quasigroup.

Conversely, let Q be a B_1 - quasigroup and ($Q(+)$, φ, ψ, g) be a T - form of Q. Then, by (11), for every x, y, $z \in Q$, $\varphi(x) + \psi\varphi(y) + \psi^2(z) + g + \psi(g) = \varphi(y) + {}$ $+ \psi\varphi(x) + \psi^2(z) + g + \psi(g)$. Hence $\varphi(x) + \psi\varphi(y) = \varphi(y) + \psi\varphi(x)$. For $y = O$ we get $\varphi(x) = \psi\varphi(x)$. Therefore $\psi = 1$ and, by Theorem 14, Q is a left loop. The other part of the proof is similar.

Theorem 17: Every B_3 – quasigroup and every B_4 – quasigroup is an Abelian quasigroup and hence a T – quasigroup.

Proof: Be Q a quasigroup and x, y, u, v arbitrary elements of Q. If Q is a B_3 – quasigroup, then $xy \,.\, uv = v(u \,.\, xy) = v(y \,.\, xu) = xu \,.\, yv$, which is the Abelian identity. If Q is a B_4 – quasigroup, then $xy \,.\, uv = (uv \,.\, y)\, x = (yv \,.\, u)x = xu \,.\, yv$.

Theorem 18: A T – quasigroup Q is a B_4 – quasigroup (B_3 – quasigroup) if and only if for any (and then for every) its T – form $(Q(+), \varphi, \psi, g)$ is $\psi = \varphi^2$ ($\varphi = \psi^2$).

Proof: The proof is similar to that of Theorem 16.