Ivo Marek

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A Reduction Method for Approximate Solving Large Elliptic Systems

I. MAREK
Faculty of Mathematics and Physics, Charles University, Prague

A method is presented for reducing a linear elliptic system with N-component solution-vector to solving an approximate system consisting of $2n$, $n < N$, unknowns. Some well known procedures are obtained as particular cases, e.g. coarse mesh methods and various averaging procedures appearing in reactor physics calculations are included into the scheme shown.

I. Introduction. Formulation of the Problem

Let $\boldsymbol{H}_j^{(k)}$ and $\boldsymbol{K}_j^{(k)}$, $j = 1, ..., N$, $k = 1,2$, be complex Hilbert spaces such that $\boldsymbol{H}_j^{(k)} \subset \boldsymbol{K}_j^{(k)}$ and the embedding be compact. Let $\boldsymbol{H}^{(k)} = \boldsymbol{H}_1^{(k)} \times ... \times \boldsymbol{H}_N^{(k)}$ and $\boldsymbol{K}^{(k)} = \boldsymbol{K}_1^{(k)} \times ... \times \boldsymbol{K}_N^{(k)}$. The inner product on $\boldsymbol{H}_j^{(k)}$ and $\boldsymbol{K}_j^{(k)}$ is denoted by $(u_j, v_j)_{\boldsymbol{H}_j^{(k)}}$ and $(u_j, v_j)_{\boldsymbol{K}_j^{(k)}}$ respectively; as usual $(U, V)_{\boldsymbol{H}^{(k)}} = \sum_{j=1}^{N} (u_j, v_j)_{\boldsymbol{H}_j^{(k)}}$ and $(U, V)_{\boldsymbol{K}^{(k)}} = \sum_{j=1}^{N} (u_j, v_j)_{\boldsymbol{K}_j^{(k)}}$, $U = (u_1, ..., u_N)^T$, $V = (v_1, ..., v_N)^T$. Further $\|u_j\|_{\boldsymbol{H}^{(k)}} = (u_j, u_j)_{\boldsymbol{H}^{(k)}}$ and $\|U\|_{\boldsymbol{H}^{(k)}} = (U, U)_{\boldsymbol{H}^{(k)}}$ etc.

Let B and C be bilinear forms on $\boldsymbol{H}^{(1)} \times \boldsymbol{H}^{(2)}$ and $\boldsymbol{K}^{(1)} \times \boldsymbol{K}^{(2)}$ respectively and let B and C have the following properties

(i) $$|B(U, V)| \leqq c_1 \|U\|_{\boldsymbol{H}^{(1)}} \|V\|_{\boldsymbol{H}^{(2)}},$$

(ii) $$\inf_{\substack{U \in \boldsymbol{H}^{(1)} \\ \|U\|_{\boldsymbol{H}^{(1)}} = 1}} \sup_{\substack{V \in \boldsymbol{H}^{(2)} \\ \|V\|_{\boldsymbol{H}^{(2)}} \leqq 1}} |B(U, V)| \geqslant C_2 > 0,$$

(iii) $$\sup_{U \in \boldsymbol{H}^{(1)}} |B(U, V)| > 0 \quad \forall\, V \in \boldsymbol{H}^{(2)},\ V \neq 0,$$

(iv) $$|C(U, V)| \leqq c_3 \|U\|_{\boldsymbol{K}^{(1)}} \|V\|_{\boldsymbol{K}^{(2)}},$$

where c_1, c_2, c_3 are positive constants.

We consider the following eigenvalue problem. To find a parameter λ and a nonzero element $U \in \boldsymbol{H}^{(1)}$ such that

$$B(U, V) = \frac{1}{\lambda} C(U, V) \quad \forall\, V \in \boldsymbol{H}^{(2)}. \tag{1}$$

Remark. In applications to reactor physics problems which we are mainly concerned with the required parameter λ is maximal in the modulus in the set of all eigenvalues of (1). We thus restrict ourselves to finding the eigenvectors corresponding to the eigenvalue with maximum modulus.

Since the dimension N of the forms B and C is too large, the system of algebraic equations resulting from (1) by discretization contains a large number of unknowns, and this number exceeds the capacity of computer memory as a rule. This is the case if an essentially threedimensional problem is solved. Thus, we are concerned with the problem of reducing the number N in a way to obtain an effective computing scheme for determining the approximate eigenelements.

We propose a method which can be considered as a two step finite element method. It is based on an idea of KANTOROVICH and KRYLOV [4] and some ideas typical for the finite element method. The resulting special averaging procedure implied by the method to the diffusion coefficients is suitable namely if an already homogenized problem [2] is solved. In this case the averaging procedure is easily to be performed. Some reactor criticality calculation methods which are special cases of our method were described in [5].

2. Group Reducing Method

Let $n < N$ and $p_1, \ldots, p_n$ be positive integers such that $N = \sum_{j=1}^{n} p_j$. Let P_j be an orthogonal projection of the Euclidean space $\boldsymbol{R}^N$ onto $\boldsymbol{R}^N_{\hat{p}_j} \subset \boldsymbol{R}_N$, $j = 1, \ldots, n$, and let $\sum_{j=1}^{n} P_j = I_N$, where I_N is the identity operator on $\boldsymbol{R}^N$, dim $\boldsymbol{R}^N_{\hat{p}_j} = p_j$. We let $U^{(N)} \in \boldsymbol{R}^N$, $U^{(N)}_{\hat{p}_k} \in \boldsymbol{R}^N_{\hat{p}_k}$

$$U_{\hat{p}_k} = \{u_{\hat{p}_k}, P_k\, U^{(N)}\} \stackrel{\text{df}}{=} u_{\hat{p}_k}\, U^{(N)}_{\hat{p}_k}, \qquad k = 1, \ldots, n, \tag{2}$$

where $u_{\hat{p}_k}$ is an element of a Hilbert space $\boldsymbol{W}^{(1)}_{\hat{p}_k}$; here p_k characterizes the set of independent components of $P_k \boldsymbol{R}^N$; we write $\sum_{j \in \hat{p}_k}$ to denote the fact that the summation is to be taken over linearly independent vectors belonging to $P_k R^N$. According to (2) we can write U approximately as

$$\hat{U} = \sum_{k=1}^{n} u_{\hat{p}_k}\, U^{(N)}_{\hat{p}_k}, \qquad u_{\hat{p}_k} \in \boldsymbol{W}^{(1)}_{\hat{p}_k}, \qquad U^{(N)}_{\hat{p}_k} \in \boldsymbol{R}^N_{\hat{p}_k}$$

$$\hat{V} = \sum_{k=1}^{n} v_{\hat{p}_k}\, V^{(N)}_{\hat{p}_k}, \qquad v_{\hat{p}_k} \in \boldsymbol{W}^{(2)}_{\hat{p}_k}, \qquad V^{(N)}_{\hat{p}_k} \in \boldsymbol{R}^N_{\hat{p}_k}. \tag{3}$$

We require $\hat{U} \in \boldsymbol{H}^{(1)}$ and this implies the existence of a projection $\pi_{\hat{p}_k}$ of $\boldsymbol{H}^{(1)}$ onto $\boldsymbol{W}^{(1)}_{\hat{p}_k}$ such that $\hat{U} = \pi_{\hat{p}_k}\, U$, $\pi_{\hat{p}_k}\, U = u_{\hat{p}_k} U^{(N)}_{\hat{p}_k}$; similarly for $\hat{V} \in \boldsymbol{H}^{(2)}$.

Let us consider problem (1) in the subspaces generated by the elements of the type (3). We let $\hat{\boldsymbol{H}}^{(k)} = \boldsymbol{W}_{\hat{p}_1}^{(k)} \times \ldots \times \boldsymbol{W}_{\hat{p}_n}^{(k)}$, $k = 1,2$. We then solve the system

$$B(\hat{U}, \hat{V}) = \frac{1}{\lambda} C(\hat{U}, \hat{V}) \quad \forall\, V \in \boldsymbol{H}^{(2)} . \tag{4}$$

Let us assume that the vectors $U_{\hat{p}_k}^{(N)}$ and $V_{\hat{p}_k}^{(N)}$, $k = 1, \ldots, n$, are known. To find the conditions which must be satisfied by still unknown $u_{\hat{p}_k}$ we substitute (3) into (4); we obtain

$$\sum_{k=1}^{n} B(u_{\hat{p}_k} U_{\hat{p}_k}^{(N)}, \; v_{\hat{p}_j} V_{\hat{p}_j}^{(N)}) = \frac{1}{\lambda} \sum_{k=1}^{n} C(u_{\hat{p}_k} U_{\hat{p}_k}^{(N)}, \; v_{\hat{p}_j} V_{\hat{p}_j}^{(N)}) . \tag{5}$$

Let β_{jk} and γ_{jk} be bilinear forms on $\boldsymbol{W}_{\hat{p}_k}^{(1)} \times \boldsymbol{W}_{\hat{p}_j}^{(2)}$ and b_{jk} and c_{jk} bilinear forms on $\boldsymbol{R}_{\hat{p}_k}^{N} \times \boldsymbol{R}_{\hat{p}_j}^{N}$ such that the following holds

(v)
$$B(u_{\hat{p}_k} U_{\hat{p}_k}^{(N)}, v_{\hat{p}_j} V_{\hat{p}_j}^{(N)}) = \beta_{jk}(u_{\hat{p}_k}, v_{\hat{p}_j})\, b_{jk}(U_{\hat{p}_k}^{(N)}, V_{\hat{p}_j}^{(N)}) ,$$
$$C(u_{\hat{p}_k} U_{\hat{p}_k}^{(N)}, v_{\hat{p}_j} V_{\hat{p}_j}^{(N)}) = \gamma_{jk}(u_{\hat{p}_k}, v_{\hat{p}_j})\, c_{jk}(U_{\hat{p}_k}^{(N)}, V_{\hat{p}_j}^{(N)}) .$$

It follows from (5) that $u_{\hat{p}_k}$ must satisfy

$$\sum_{k=1}^{n} b_{jk} \beta_{jk} (\hat{u}_{\hat{p}_k}, \hat{v}_{\hat{p}_j}) = \frac{1}{\lambda} \sum_{k=1}^{n} c_{jk} \gamma_{jk} (\hat{u}_{\hat{p}_k}, \hat{v}_{\hat{p}_j}) \qquad \forall\, \hat{v}_{p_j} \in \boldsymbol{W}_{\hat{p}_j}^{(2)} \tag{6}$$

where

$$b_{jk} = b_{jk} \left(\frac{U_{\hat{p}_k}^{(N)}}{|||U_{\hat{p}_k}^{(N)}|||} , \; \frac{V_{\hat{p}_j}^{(N)}}{|||V_{\hat{p}_j}^{(N)}|||} \right), \quad c_{jk} = c_{jk} \left(\frac{U_{\hat{p}_k}^{(N)}}{|||U_{\hat{p}_j}^{(N)}|||} , \; \frac{V_{\hat{p}_j}^{(N)}}{V_{\hat{p}_j}^{(N)}} \right) \tag{7}$$

and

$$\hat{u}_{\hat{p}_k} = |||U_{\hat{p}_k}^{(N)}||| , \quad \hat{v}_{\hat{p}_j} = |||V_{\hat{p}_j}^{(N)}|||\, v_{\hat{p}_j}$$

and where $|||U_{\hat{p}_k}^{(N)}|||$ is an appropriate norm on $\boldsymbol{R}_{\hat{p}_k}^{N}$. The importance of introducing the norms $|||\,.\,|||$ will be elucidated below.

If the vectors $u_{\hat{p}_k}$ and $V_{\hat{p}_k}$ in (3) are known then $U_{\hat{p}_k}^{(N)}$ must satisfy

$$\sum_{k=1}^{n} \beta_{jk} b_{jk} (\hat{U}_{\hat{p}_k}^{(N)}, \; \hat{V}_{\hat{p}_j}^{(N)}) = \frac{1}{\lambda} \sum_{k=1}^{n} \gamma_{jk} c_{jk} (\hat{U}_{\hat{p}_k}^{(N)}, \; \hat{V}_{\hat{p}_j}^{(N)}) \quad \forall\, \hat{V}_{\hat{p}_j}^{(N)} \in \boldsymbol{R}_{\hat{p}_j}^{(N)} , \tag{8}$$

where

$$\beta_{jk} = \beta_{jk} \left(\frac{u_{\hat{p}_k}}{|||u_{\hat{p}_k}|||^{(1)}} , \; \frac{v_{\hat{p}_j}}{|||v_{\hat{p}_j}|||^{(2)}} \right) ,$$

$$\gamma_{jk} = \gamma_{jk} \left(\frac{u_{\hat{p}_k}}{|||u_{\hat{p}_k}|||^{(1)}} , \; \frac{v_{\hat{p}_j}}{|||v_{\hat{p}_j}|||^{(2)}} \right) \tag{9}$$

and

$$\hat{U}_{\hat{p}_k}^{(N)} = |||u_{\hat{p}_k}|||^{(1)}\, U_{\hat{p}_k}^{(N)} , \quad \hat{V}_{\hat{p}_j}^{(N)} = |||v_{\hat{p}_j}|||^{(2)}\, V_{\hat{p}_j}^{(N)} .$$

Let us note that the systems (6) and (8) have the same spectra.

An appropriate choice of the norms $|||U^{(N)}_{\hat{p}_k}|||$ is as follows

$$|||U^{(N)}_{\hat{p}_k}||| = \sum_{l \in \hat{p}_k} \psi_k(U^{(N)}_{\hat{p}_k}, l)\,,$$

where ψ_k is a norm on $\boldsymbol{R}^N_{\hat{p}_k}$ and $U^{(N)}_{\hat{p}_k,l}$ denotes the l-th component of $U^{(N)}_{\hat{p}_k}$ in a suitable basis of $\boldsymbol{R}^{(N)}_{\hat{p}_k}$. We then have that

$$\hat{u}_{\hat{p}_k} = \sum_{l \in \hat{p}_k} \psi_k(U^{(N)}_{\hat{p}_k,l})\, u_{\hat{p}_k} \text{ and } \hat{U}^{(N)}_{\hat{p}_k} = |||u_{\hat{p}_k}|||^{(1)}\, U^{(N)}_{\hat{p}_k}\,.$$

It follows that

$$u_{\hat{p}_k} U_{\hat{p}_k,j} = \hat{u}_{\hat{p}_k} \frac{\hat{U}_{\hat{p}_k,j}}{\sum\limits_{l \in \hat{p}_k} |||U^{(N)}_{\hat{p}_k,l}|||} = \frac{\hat{u}_{\hat{p}_k}}{|||\hat{u}_{\hat{p}_k}|||^{(1)}}\, U_{\hat{p}_k,j}\,,$$

and we see that we obtained a formula for the j-th component of the required solution expressed in terms of averages over the groups $\hat{p}_1, \ldots, \hat{p}_n$ respectively.

Remark. If we choose $v_{\hat{p}_j} V^{(N)}_{\hat{p}_j}$ in (3) in a special way we obtain by (7) and (9) some averaging procedures which are well known in the reactor physics literature, e.g.
(a) $v_{\hat{p}_j} V^N_{\hat{p}_j} = u^*_{\hat{p}_j} (U^N_{\hat{p}_j})^*$, where the asterisk means the Hermite conjugation, gives Marchouk's method [6, chapter 16];
(b) the choice $v_{\hat{p}_j} V^N_{\hat{p}_j} = u_{\hat{p}_j} U^{(N)}_{\hat{p}_j}$ is identical with the Galerkin type procedure;
(c) if $v_{\hat{p}_j} V^{(N)}_{\hat{p}_j} = E^{(N)}_{\hat{p}_j}$, where $v_{\hat{p}_j} \equiv 1$ and for the components e_t of $E^{(N)}_{\hat{p}_j}$ we have that $e_t = 1$ if $t \in \hat{p}_j$ and $e_t = 0$ if $t \notin \hat{p}_j$, we obtain the so called neutron rebalancing method [3].

Concerning the convergence and error estimates, the method described above can easily be included formally into the general framework of the finite element method; in particular, all of the theorems of Sextion 10.5 in [1] apply.

By proceedings as described above we reduce the initial problem with N component vector-solution to the systems (6) and (8) both determining n-component vectors.

As a rule the system (8) is algebraic and (6) differential. Hence, the finite element method can be used to get approximate solutions. This is the secondary use of the finite element method in solving problem (1).

3. Determining the Self-consistent System

We see that the coefficients in (6) depend nonlinearly on the solution of (8) and vice versa. A new problem arizes, how to solve this nonlinear system. We propose a simple iterative procedure which is very effective in concrete calculations.

We let

$$\sum_{k=1}^{n} b_{jk}^{(m)}\ \beta_{jk}(\hat{u}_{\hat{p}k}^{(m)}, \hat{v}_{\hat{p}j}) = \frac{1}{\nu^{(m)}} \sum_{k=1}^{n} c_{jk}^{(m)}\ \gamma_{jk}(\hat{u}_{\hat{p}k}^{(m)}, \hat{v}_{\hat{p}j}) \quad \forall\ \hat{v}_{\hat{p}j} \in \boldsymbol{W}_{\hat{p}j}^{(2)} \tag{10}$$

and

$$\sum_{k=1}^{n} \beta_{jk}^{(m+1)}\ b_{jk}(\hat{U}_{\hat{p}k}^{(N),(m+1)}, \hat{V}_{\hat{p}j}^{(N)}) =$$

$$= \frac{1}{\mu^{(m+1)}} \sum_{k=1}^{n} \gamma_{jk}^{(m+1)}\ c_{jk}(\hat{U}_{\hat{p}k}^{(N),(m+1)}, \hat{V}_{\hat{p}k}^{(N)}) \quad \forall\ V_{\hat{p}j}^{(N)} \in \boldsymbol{R}_{\hat{p}j}^{(N)}\,, \tag{11}$$

where

$$b_{jk}^{(m)} = b_{jk}(\hat{U}_{\hat{p}k}^{(N),(m)}, V_{\hat{p}j}^{(N)}), \quad c_{jk}^{(m)} = c_{jk}(\hat{U}_{\hat{p}k}^{(N),(m)}, V_{\hat{p}j}^{(N)})\,,$$

and

$$\beta_{jk}^{(m+1)} = \beta_{jk}(\hat{u}_{\hat{p}k}^{(m)}, \hat{v}_{\hat{p}j}),\ \gamma_{jk}^{(m+1)} = \gamma_{jk}(\hat{u}_{\hat{p}k}^{(m)}, \hat{v}_{\hat{p}j})\,,$$

$\hat{U}^{(0)} = \sum_{k=1}^{n} \hat{U}_{\hat{p}k}^{(0)}$ being an in a sense arbitrary element.

Besides the already mentioned assumptions concerning B and C let us assume further that

(vi) The eigenspace $\mathfrak{N}$ of (6) and the eigenspace $\mathfrak{M}$ of (8) corresponding to the eigenvalue with maximal modulus are one-dimensional for every $u \in \mathscr{V} \in \hat{\boldsymbol{H}}^{(1)}$ and $U^{(N)} \in \mathscr{V}^{(N)} \subset \boldsymbol{R}^{(N)}$, where $\mathscr{V}$ and $\mathscr{V}^{(N)}$ are suitable convex sets.

It follows from the assumptions (i) — (vi) that the iteration process (10) — (11) is convergent $\hat{u}_{\hat{p}k}^{(m)} \to u_{\hat{p}k}^{0}$, $U_{\hat{p}k}^{(N),(m)} \to U_{\hat{p}j}^{(N),0}$, $\nu^{(m)} \to \bar{\lambda}$, $\mu^{(m)} \to \bar{\lambda}$ and it holds

$$\sum_{k=1}^{n} \bar{b}_{jk}\beta_{jk}(\hat{u}_{\hat{p}k}^{0}, \hat{v}_{\hat{p}j}^{0}) = \frac{1}{\bar{\lambda}} \sum_{k=1}^{n} \bar{c}_{jk}\gamma_{jk}(\hat{u}_{\hat{p}k}^{0}, \hat{v}_{\hat{p}j}) \quad \forall\ \hat{v}_{\hat{p}j} \in \boldsymbol{W}_{\hat{p}j}^{(2)}$$

and

$$\sum_{k=1}^{n} \bar{\beta}_{jk}b_{jk}(\hat{U}_{\hat{p}k}^{(N),0}, \hat{V}_{\hat{p}j}^{(N)}) = \frac{1}{\bar{\lambda}} \sum_{k=1}^{n} \bar{\gamma}_{jk}c_{jk}(\hat{U}_{\hat{p}k}^{(N),0}, \hat{V}_{\hat{p}j}^{(N)}) \quad \forall\ \hat{V}_{\hat{p}j}^{(N)} \in \boldsymbol{R}_{\hat{p}j}^{N}\,,$$

where

$$\bar{b}_{jk} = b_{jk}(\hat{U}_{\hat{p}k}^{(N),0}, \hat{V}_{\hat{p}j}^{(N)}), \bar{c}_{jk} = c_{jk}(\hat{U}_{\hat{p}k}^{(N),0}, \hat{V}_{\hat{p}j}^{(N)})$$

and

$$\bar{\beta}_{jk} = \beta_{jk}(\hat{u}_{\hat{p}k}^{0}, \hat{v}_{\hat{p}j}),\ \bar{\gamma}_{jk} = \gamma_{jk}(\hat{u}_{\hat{p}k}^{0}, \hat{v}_{\hat{p}j})\,.$$

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