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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 16 (1975), No. 2, 63--69

Persistent URL: http://dml.cz/dmlcz/142369

Stable and Costable Preradicals

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Received 4 March 1974

In the last few years, the stable torsion theories were investigated by several authors (see e.g. [3], [9], [10], [11]). Our present paper is devoted to the study of stable preradicals. Among other results, the Skornjakov's criterion for stability is generalized (Theorem 2.8 below) and almost all the results are dualized to the constable preradicals. This paper can be viewed as a continuation of our previous investigations [4], [5] of the general theory of preradicals.

1. Preliminaries

In what follows R stands for an associative ring with identity and *R-Mod* denotes the category of all unitary left R-modules. The injective hull of a module M will be denoted by $E(M)$, the direct product (direct sum) by $\prod_{i \in I} M_i$ ($\coprod_{i \in I} M_i$, $M_1 \oplus M_2$). A submodule N of a module M is called essential in M if $K \cap N \neq 0$ for every nonzero submodule K of M (in this case M is said to be an envelope of N) and it is called superfluous in M if $K = M$ whenever $K + N = M$. A module M is called cocyclic if its socle is a simple essential submodule (see [8], [12]). An exact sequence $O \to K \to P \to M \to O$ is said to be
— a projective presentation of M if P is projective,
— an injective presentation of K if P is injective,
— a cover of M if K is superfluous in P,
— a projective cover of M if P is projective and K is superfluous in P (see [1] for details).
A class $\mathfrak{M}$ of modules is called
— abstract if it is closed under isomorphic images,
— hereditary if it is abstract and closed under submodules,
— cohereditary if it is abstract and closed under factormodules,
— closed under (projective) covers if $P \in \mathfrak{M}$ whenever $M \in \mathfrak{M}$ and $O \to K \to P \to M \to O$ is a (projective) cover of M.

A preradical r for *R-Mod* is any subfunctor of identity, i.e. r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $r(M)$ into $r(N)$ by restriction.

The class $\mathcal{T}_r$ of all the modules M with $r(M) = M$ is a cohereditary class closed under direct sums and the class $\mathcal{F}_r$ of all the modules M with $r(M) = O$ is a hereditary class closed under direct products. Modules from $\mathcal{T}_r$ are called r-torsion and those from $\mathcal{F}_r$ r-torsionfree. A preradical r is said to be

— idempotent if $r^2 = r$,

— hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M,

— cohereditary if $r(M/N) = (r(M) + N)/N$ for every submodule N of a module M,

— a radical if $r(M/r(M)) = 0$ for every module M. A preradical r is hereditary iff it is idempotent and $\mathcal{T}_r$ is hereditary, and r is cohereditary iff it is a radical and $\mathcal{F}_r$ is cohereditary (see [5]).

There are several preradicals associated with every preradical r. The idempotent core $\bar{r}$ is defined by $\bar{r}(M) = \Sigma K$, where K runs through all the submodules K of M with $K \in \mathcal{T}_r$, and the radical closure $\tilde{r}$ is defined by $r(M) = \bigcap L$, where L runs through all the submodules L of M for which $M/L \in \mathcal{F}_r$. Obviously, $\mathcal{T}_r = \mathcal{T}_{\bar{r}}$ and $\mathcal{F}_r = \mathcal{F}_{\tilde{r}}$. Further, the hereditary closure $h(r)$ is defined by $h(r)(M) = M \cap r(E(M))$ and the cohereditary core $ch(r)$ by $ch(r)(M) = r(R)M$. For every projective module P, $ch(r)(P) = r(P)$. The basic properties of these preradicals are studied in [4] and [5]. For two preradicals r, s we shall write $r \subseteq s$ if $r(M) \subseteq s(M)$, for all $M \in R\text{-}Mod$. We shall say that a module M splits in a preradical r if $r(M)$ is a direct summand of M. A preradical r has the cyclic splitting property (CSP) if every cyclic module splits in r.

2. Stable Preradicals

2.1. Definition: A class $\mathfrak{M}$ of modules is called stable if every module $M \in \mathfrak{M}$ has an injective presentation

$O \to M \to Q \to K \to O$ such that $Q \in \mathfrak{M}$.

2.2. Proposition: Let $\mathfrak{M}$ be an abstract class of modules closed under direct summands. Then $\mathfrak{M}$ is stable iff $\mathfrak{M}$ is closed under injective hulls.

Proof: If $M, Q \in \mathfrak{M}$, Q injective, $M \subseteq Q$, then $Q = E(M) \oplus S$. The rest is obvious.

2.3. Definition: A preradical r for R-Mod is said to be stable if every injective module splits in r.

2.4. Proposition: Let r be a preradical for R-Mod. Then

(i) if r is stable then $\mathcal{T}_r$ is stable,

(ii) if r is idempotent and $\mathcal{T}_r$ is stable then r is stable.

Proof: (i) For every $T \in \mathcal{T}_r$, $E(T) = r(E(T)) \oplus F$, r being stable. Since T is essential in $E(T)$ and $T = r(T) \subseteq r(E(T))$, we get $F = O$, and consequently $r(E(T)) = E(T)$.

(ii) If Q is injective then $Q = E(r(Q)) \oplus S$. Since r is idempotent and $\mathcal{T}_r$ is stable, $E(r(Q)) \in \mathcal{T}_r$ by 2.2. Thus $E(r(Q)) = r(Q)$.

2.5. Proposition: Every stable hereditary preradical is a radical.

Proof: Let r be a stable hereditary preradical. First, we shall prove that $\mathcal{T}_r$ is closed under extensions. For, let $O \to A \to B \to C \to O$ be an exact sequence with $A, C \in \mathcal{T}_r$.

It is an easy exercise to prove the existence of a monomorphism f such that the diagram

$$\begin{array}{ccccccccc} O & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & O \\ & & \downarrow & & f\downarrow & & \downarrow & & \\ O & \rightarrow & E(A) & \rightarrow & E(A) \oplus E(C) & \rightarrow & E(C) & \rightarrow & O \end{array}$$

is commutative. Hence $B \in \mathcal{F}_r$ by 2.4 and [5], Prop. 2.1. Further, let $M \in R\text{-}Mod$ be arbitrary. The exact sequence $O \to r(M) \to T \to T/r(M) \to O$, where $T/r(M) = = r(M/r(M))$, yields $T \in \mathcal{T}_r$. Thus $T \subseteq r(M)$ and $r(M/r(M)) = O$.

2.6. Theorem: Let r be a preradical. Then

(i) r is stable iff $h(r)$ is so,

(ii) if r is stable then $\bar{r}, \tilde{r}, h(r)$ are so and $h(r) = h(\bar{r}) = h(\tilde{r}) = \widetilde{h(r)}$.

Proof: (i) If Q is injective then $h(r)(Q) = Q \cap r(Q) = r(Q)$ be the definition of $h(r)$, and the assertion easily follows.

(ii) For every injective module Q, $Q = r(Q) \oplus F$, and hence $r(Q) = r(r(Q)) \oplus r(F)$. Thus $r(Q) = r(r(Q))$, $r(Q/r(Q)) \cong r(F) = O$, so that $\bar{r}(Q) = \tilde{r}(Q) = r(Q)$. Consequently, $\bar{r}$ and $\tilde{r}$ are stable by the definition of stability. Further, by [5], Prop. 3.2, $h(r) = h(\tilde{r}) = \widetilde{h(r)}$ and $h(r)(M) = M \cap r(E(M)) = M \cap \bar{r}(E(M)) = h(\bar{r})(M)$, for all $M \in R\text{-}Mod$, finishes the proof.

2.7. Theorem: A preradical r for $R\text{-}Mod$ is stable iff the injective hulls of cyclic modules split in r.

Proof: With respect to 2.6, it suffices to prove that $h(r)$ is stable. Let $T \in \mathcal{T}_{h(r)}$ and $x \in E(T)$ be arbitrary. Then $T \subseteq h(r)(E(T))$ and $E(Rx) = r(E(Rx)) \oplus F = = h(r)(E(Rx)) \oplus F$. As $Rx, h(r)(E(T))$ are essential in $E(Rx), E(T)$ respectively, for every nonzero $y \in E(Rx)$ there are $s, t \in R$ such that $O \neq sy \in Rx$ and $O \neq tsy \in \in Rx \cap h(r)(E(T)) = h(r)(Rx)$, and therefore $h(r)(Rx)$ is essential in $E(Rx)$. Since $h(r)(Rx) \cap F \subseteq h(r)(E(Rx)) \cap F = O$, $E(Rx) = h(r)(E(Rx))$. Thus $x \in h(r)(E(T))$. According to 2.4, $h(r)$ is stable, and we are through.

2.8. Theorem: Let r be a hereditary preradical for $R\text{-}Mod$. Then the following are equivalent:

(i) every module $M \notin \mathcal{T}_r$ contains a nonzero submodule $N \in \mathcal{F}_r$,

(ii) the class $\mathcal{T}_r$ is closed under envelopes,

(iii) if $A \subseteq B \subseteq C$ are modules and $B/A \in \mathcal{T}_r$ then there is a submodule D of C such that $D \cap B = A$ and $C/D \in \mathcal{T}_r$,

(iv) if $I \supseteq K$ are left ideals of R and $K/I = r(R/I)$ then there is a left ideal L such that $L \cap K = I$ and $R/L \in \mathcal{T}_r$,

(v) if $I \subseteq K \subsetneqq R$ are left ideals of R and $K/I = r(R/I)$ then there is a left ideal L such that $L \neq I$ and $L \cap K = I$,

(vi) r is stable.

Proof: (i) implies (ii). Let $N \in \mathcal{T}_r$ be essential in the module M. If $M \notin \mathcal{T}_r$ then there is a nonzero submodule $K \subseteq M$ with $r(K) = O$. Hence $K \cap N \in \mathcal{T}_r \cap \mathcal{F}_r = O$, r being hereditary, and consequently $K = O$, a contradiction.

(ii) implies (iii). Let D be a submodule of C such that D/A is maximal with respect to

$D/A \cap B/A = O$. Then $D \cap B = A$ and $(B + D)/D \cong B/(D \cap B) = B/A \in \mathcal{T}_r$. According to (ii), it remains to show that $(B + D)/D$ is essential in C/D. If $(B + D)/ /D \cap G/D = O$ then $(B + D) \cap G = D$, hence $B \cap (G + D) = A$, and therefore $B/A \cap (G + D)/A = O$. Thus $G + D = D$ by the maximality of D/A, and consequently $G = D$, as desired.

(iii) implies (iv). Obvious.

(iv) implies (v). Obvious.

(v) implies (vi). With respect to 2.4, it suffices to show that $\mathcal{T}_r$ is stable. Let $T \in \mathcal{T}_r$ and $x \in E(T)$ be arbitrary. Then $r(Rx)$ is essential in Rx (as in the proof of 2.7) and there are left ideals I, K such that $Rx \cong R/I$ and $K/I = r(R/I)$. If $x \notin r(E(T))$ then $r(Rx) = = Rx \cap r(E(T)) \neq Rx$, and hence there is a left ideal L such that $L \neq I$ and $L \cap K = = I$. Then $L/I \cap K/I = O$, a contradiction.

(vi) implies (i). If $M \notin \mathcal{T}_r$ then $r(M)$ is not essential in M, for otherwise $E(M) \in \mathcal{T}_r$ by 2.4 and $M \in \mathcal{T}_r$. Thus there is a nonzero submodule $N \subseteq M$ with $r(N) = r(M) \cap N = O$.

2.9. Corollary: Every hereditary preradical with CSP is stable.

2.10. Theorem: Let r be a preradical such that $\mathcal{T}_r$ is closed under direct products and every cocyclic module splits in r. Then r is stable.

Proof: First, we shall show that every module M can be imbedded into a direct product of cocyclic modules. For every nonzero $x \in M$, let M_x be a submodule of M maximal with respect to $x \notin M_x$, and $C_x = M/M_x$. One can easily show that all C_x are cocyclic and there is a monomorphism $f: M \to \prod_{i \in I} C_x$. Now, let Q be an injective module. Then $Q \subseteq \prod_{i \in I} C_i$, where C_i are cocyclic, so that $\prod_{i \in I} C_i = Q \oplus X$. Since every cocyclic module is directly indecomposable, it belongs either to $\mathcal{T}_r$ or to $\mathcal{F}_r$, and consequently $\prod_{i \in I} C_i = T \oplus F$, where $T \in \mathcal{T}_r$ and $F \in \mathcal{F}_r$. Further, $r(\prod_{i \in I} C_i) = T = = r(Q) \oplus r(X)$, hence $\prod_{i \in I} C_i = r(Q) \oplus r(X) \oplus F$, which yields $Q = r(Q) \oplus \oplus (Q \cap (r(X) \oplus F))$.

2.11. Theorem: Let r be a stable preradical. If $\bar{r}$ is hereditary then $\tilde{r}$ is hereditary.

Proof: Obviously, $\tilde{r} \subseteq h(\tilde{r})$. On the other hand, 2.6 yields $h(\tilde{r}) = h(\bar{r}) = \bar{r} \subseteq \tilde{r}$.

2.12. Corollary: Let r be a stable preradical. Then the following are equivalent:

(i) r is a radical and $\mathcal{T}_r$ is hereditary,

(ii) r is idempotent and $\mathcal{T}_r$ is hereditary,

(iii) r is hereditary,

(iv) r is a hereditary radical,

(v) r is a radical and $\bar{r}$ is hereditary.

3. Costable Preradicals

3.1. Definition: A class $\mathfrak{M}$ of modules is said to be costable if every module $M \in \mathfrak{M}$ has a projective presentation $O \to K \to P \to M \to O$ such that $P \in \mathfrak{M}$.

3.2. Proposition: Let $\mathfrak{M}$ be an abstract class of modules such that $\mathfrak{M}$ is closed under direct summands and every module from $\mathfrak{M}$ has a projective cover. Then $\mathfrak{M}$ is costable iff it is closed under projective covers.

Proof: Let $M \in \mathfrak{M}$, $O \to K \to F \xrightarrow{f} M \to O$ be a projective presentation of M with $F \in \mathfrak{M}$ and $O \to L \to P \xrightarrow{g} M \to O$ be a projective cover of M. There is $h \in Hom(F, P)$ such that $hg = f$, and h is an epimorphism, since L is superfluous in P. Thus P is a direct summand in F. The converse is obvious.

3.3. Definition: A preradical r for R-Mod is said to be costable if every projective module splits in r.

3.4. Proposition: Let r be preradical for R-Mod. Then

(i) if r is costable then $\mathscr{F}_r$ is costable,

(ii) if r is a radical and $\mathscr{F}_r$ is costable then r is costable.

Proof: (i) Let $F \in \mathscr{F}_r$ be arbitrary and $O \to L \to P \to F \to O$ be a projective presentation of F. Then $P = r(P) \oplus Q$. Since $r(F) = O$, $r(P) \subseteq L$ and $O \to L/r(P) \to$ $\to Q \to F \to O$ is a projective presentation of F with $Q \in \mathscr{F}_r$.

(ii) Let P be a projective module. According to the hypothesis, there is an exact sequence $O \to L \to Q \xrightarrow{f} P/r(P) \to O$ with $Q \in \mathscr{F}_r$ projective. Further, there is a homomorphism h such that the diagram

$$\begin{array}{ccccccc} & & & & P & & \\ & & & \swarrow^{h} & \downarrow g & & \\ O \to & L \to & Q & \xrightarrow{f} & P/r(P) & \to O \end{array}$$

is commutative. Since $r(Q) = O$, h induces a homomorphism $k: P/r(P) \to Q$ such that $gk = h$. Therefore $P/r(P)$ is projective and P splits.

3.5. Proposition: Every costable cohereditary radical r is idempotent.

Proof: First, we shall prove that $\mathscr{F}_r$ is closed under extensions. For, let $O \to A \to$ $\to B \to C \to O$ be an exact sequence with $A, C \in \mathscr{F}_r$. There is a homomorphism f such that the diagram

$$\begin{array}{ccccccccc} O \to & P_1 & \to & P_1 \oplus P_2 & \to & P_2 & \to O \\ & \downarrow & & f\downarrow & & \downarrow & \\ O \to & A & \longrightarrow & B & \longrightarrow & C & \to O \\ & \downarrow & & & & \downarrow & \\ & O & & & & O & \end{array}$$

where P_1, P_2 are projective modules from $\mathscr{F}_r$, is commutative. Obviously f is an epimorphism and [5], Prop. 4.1 gives $B \in \mathscr{F}_r$. Further, let $M \in R$-Mod be arbitrary. Consider the exact sequence $O \to r(M)/r(r(M)) \to M/r(r(M)) \to M/r(M) \to O$. The first part of the proof yields $M/r(r(M)) \in \mathscr{F}_r$, which implies that $r(M) = r(r(M))$.

3.6. Proposition: The following conditions for a preradical r for R-Mod are equivalent:

(i) R splits in r,

(ii) every free module splits in r,

(iii) r is costable.

Proof: (i) implies (ii). It follows from the fact that $r(\coprod_{i\in I} M_i) = \coprod_{i\in I} r(M_i)$ for every family of modules $M_i, i \in I$.
(ii) implies (iii). If P is projective then there is a free module F such that $F = P \oplus Q$. Hence $F = r(F) \oplus H = r(P) \oplus r(Q) \oplus H$ and $P = r(P) \oplus (P \cap (r(Q) \oplus H))$.
(iii) implies (i). Obvious.

3.7. Corollary: Every preradical r with CSP is costable.

3.8. Theorem: Let r be a preradical for R-*Mod*. Then
(i) r is costable iff $ch(r)$ is so,
(ii) if r is costable then $\bar{r}$, $\tilde{r}$, $ch(r)$ are so and $ch(r) = ch(\tilde{r}) = ch(\bar{r}) = \overline{ch(r)}$.

Proof: (i) By 3.6, since $ch(r)(R) = r(R)$.
(ii) If r is costable then $R = r(R) \oplus F$, hence $r(R) = r(r(R)) \oplus r(F)$, which implies that $r(R) = r(r(R)), r(R/r(R)) \cong r(F) = O$. Thus $\bar{r}(R) = \tilde{r}(R) = r(R)$, and hence $\bar{r}$, $\tilde{r}$ are costable. Further, $ch(r) = ch(\bar{r}) = \overline{ch(r)}$ by [5], Prop. 5.3. Finally, $ch(r)(M) = r(R)M = \tilde{r}(R)M = ch(\tilde{r})(M)$ for every module M.

3.9. Theorem Let R be a left perfect ring and r be a preradical for R-*Mod*. The following are equivalent:
(i) r is costable,
(ii) if C is a cocyclic module and $O \to K \to P \to C \to O$ is a projective cover of C then P splits in r.

Proof: (i) implies (ii). Obvious.
(ii) implies (i). First, we shall prove that $\mathscr{F}_{ch(r)}$ is costable. Let $F \in \mathscr{F}_{ch(r)}$ be arbitrary and $O \to K \to P \to F \to O$ be a projective cover of F. Suppose that $ch(r)(P) \neq O$ and $x \in ch(r)(P)$, $x \neq O$. Further, let Q be a submodule of P maximal with respect to $x \notin Q$, and $O \to L \to S \to P/Q \to O$ be a projective cover of P/Q. Obviously, P/Q is cocyclic. As $ch(r)(F) = O$, it must be $r(P) = ch(r)(P) \subseteq K$. According to the hypothesis, $S = r(S) \oplus H$. Since L is superfluous in S, there is an epimorphism of P onto S, and hence $r(S)$ is isomorphic to a direct summand in P, S being projective. Thus $ch(r)(S) = r(S) = O$, since $r(P)$ is superfluous in P. However $ch(r)$ is cohereditary, hence $ch(r)(P/Q) = O$. On the other hand, $x + Q$ is a nonzero element of $ch(r)(P/Q)$, which yields a contradiction. Now, an application of 3.4 and 3.6 finishes the proof.

3.10. Theorem: Let r be a cohereditary radical for R-*Mod*. Consider the following conditions:
(i) r is costable,
(ii) every module $M \notin \mathscr{F}_r$ has a nonzero factormodule which belongs to $\mathscr{T}_r$,
(iii) $\mathscr{F}_r$ is closed under covers.
Then (i) implies (ii) and (ii) implies (iii). Moreover, if R is left perfect then (iii) implies (i).

Proof: (i) implies (ii). Let $M \notin \mathscr{F}_r$ be arbitrary. With respect to 3.4, we get the commutative diagram

$$\begin{array}{ccccccccc} O & \to & K & \to & P & \to & M/r(M) & \to & O \\ & & & & f\downarrow & & \| & & \\ O & \to & r(M) & \to & M & \to & M/r(M) & \to & O \end{array}$$

with $P \in \mathscr{F}_r$ projective. If $r(M)$ is superfluous in M then f is an epimorphism, and hence $M \in \mathscr{F}_r$, r being cohereditary. Thus there is $K \subsetneqq M$ with $r(M) + K = M$. Then $O \neq M/K \cong r(M)/(r(M) \cap K) \in \mathscr{F}_r$, since r is idempotent by 3.5.

(ii) implies (iii). Let $F \in \mathscr{F}_r$ and $O \to K \xrightarrow{f} P \to F \to O$ be a cover of F. If $P \notin \mathscr{F}_r$ then there is a nonzero factormodule $P/L \in \mathscr{T}_r$. However r is cohereditary, and so $P/(f(K) + L) \in \mathscr{T}_r \cap \mathscr{F}_r$. Thus $P = L$, since K is supperfluous in P, a contradiction. The rest is obvious.

3.11. Theorem: Let r be a costable preradical for *R-Mod*. If $\tilde{r}$ is cohereditary then $\bar{r}$ is cohereditary.

Proof: Obviously $ch(\bar{r}) \subseteq \bar{r}$. On the other hand, 3.8 yields $\bar{r} \subseteq \tilde{r} = ch(\tilde{r}) = ch(\bar{r})$.

3.12. Corollary: Let r be a costable preradical for *R-Mod*. The following are equivalent:

(i) r is idempotent and $\mathscr{F}_r$ is cohereditary,
(ii) r is a radical and $\mathscr{F}_r$ is cohereditary,
(iii) r is cohereditary,
(iv) r is an idempotent cohereditary radical,
(v) r is idempotent and $\tilde{r}$ is cohereditary.

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