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On a Class of Subdirectly Irreducible Groupoids

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Let G be a groupoid. Define a relation t by $(a, b) \in t$ iff $a, b \in G$ and $ax = bx$, $xa = xb$ for every $x \in G$. Then t is a congruence of G. In the paper, there are found some necessary and sufficient conditions for a groupoid G to be isomorphic to H/t for a subdirectly irreducible groupoid H with $t \neq id_H$.

В статье найдены некоторые достаточные условия для того, чтобы группоид G был изоморфен группоиду H/t_H для некоторого подпрямо нерозложимого группоида H.

V článku se vyšetřují některé nutné a postačující podmínky pro to, aby groupoid G byl isomorfní faktoru subdirektně nerozložitelného grupoidu podle jeho nejmenší kongruence.

1. Introduction

This paper is a continuation of [1] and the reader is referred to [1] for definitions, terminology, notation, references, etc.

2. Pseudocongruences

Let G be a groupoid and $a \in G$. A relation r defined on G is said to be a pseudocongruence with companion a of G if the following two conditions are satisfied:

(i) r is compatible, reflexive and symmetric.

(ii) If $x, y, z \in G$, $y \neq a$, $(x, y), (y, z) \in r$ then $(x, z) \in r$.

Let r with a companion a be a pseudocongruence of a groupoid G. Put $M = \{x \mid (x, a) \in r\} \setminus \{a\}$. It is clear that $s = r \mid M$ is an equivalence on M. We shall say that r is of class (at most, at least) n, where $0 \leqq n$ is an integer, if s has (at most, at least) n blocks.

2.1 Lemma. The following conditions are equivalent for a pseudocongruence r:

(i) r is a congruence.

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(ii) r is transitive.

(iii) r is of class at most 1.

Proof. Obvious.

2.2 Lemma. Let r with a companion a be a pseudocongruence of G. Put $K = G \setminus \{a\}$, $M = \{x \in K \mid (x, a) \in r\}$ and $N = \{x \in K \mid (x, a) \notin r\}$. Then:

(i) $K = M \cup N$, $M \cap N = \emptyset$.

(ii) $r \mid K$, $r \mid M$ and $r \mid N$ are equivalences.

(iii) The relation $s = r \cup (M \times M)$ is a congruence of G. Moreover, it is the least congruence containing r and $s \mid N = r \mid N$.

Proof. Easy.

In the following lemmas, let r with a companion a be a pseudocongruence of a groupoid G.

2.3 Lemma. Suppose that G contains an element b such that $ba \neq a$ $(ab \neq a)$ and L_b (R_b) is a permutation of a finite order. Then r is a congruence.

Proof. It remains to show that r is transitive. Assume that $ba \neq a$ and $L_b^n = \mathrm{id}_G$. Let $x, y \in G$, $(x, a), (a, y) \in r$. We have $(bx, ba), (ba, by) \in r$, $ba \neq a$ and consequently $(bx, by) \in r$. Since r is compatible, $(x, y) \in r$.

2.4 Lemma. Suppose that G is a division groupoid and s is left (right) cancellative, where s is the least congruence containing r. Then $r = s$.

Proof. First, assume that $(a, b) \notin r$ for some $b \in G$. Let $x, y \in G$, $(x, a), (a, y) \in r$. There are $u, v \in G$ with $x = ub$ and $y = uv$. We have $(ub, uv), (b, v) \in s$, since s is left cancellative. From this, $(b, v) \in r$ and $(x, y) \in r$. Now, let $s = G \times G$. Suppose that $r \neq s$. Then r is of class at least 2 and the equivalence $r \mid A$, $A = G \setminus \{a\}$, has at least two blocks, say $N, K, \dots$. Farther, there exists $c \in G$ such that $a \neq ac \in N$. For every $x \in G$, $(a, c), (x, a), (xa, ac) \in r$, $xa \in N \cup \{a\}$. Thus $G \subseteq N \cup \{a\}$, a contradiction.

2.5 Lemma. Suppose that G is simple and $a \neq aa$. Then r is a congruence.

Proof. Let $r \neq \mathrm{id}_G$. It is an easy consequence of 2.2(iii) that $(x, a) \in r$ for every $x \in G$. Put $M = G \setminus \{a\}$ and denote by A the block of $r \mid M$ containing aa. If $x, y \in G$, then $(x, a), (y, a) \in r$, $(xy, aa) \in r$ and $xy \in B = A \cup \{a\}$. Hence $GG \subseteq B$, B is an ideal of G and $B = G$, since G is simple. From this, $r = G \times G$.

2.6 Lemma. Suppose that G is simple idempotent and a is neither a left nor a right zero of G. Then r is a congruence.

Proof. Let $r \neq \mathrm{id}_G$ and $M = G \setminus \{a\}$. There are $b, c \in G$ with $ca \neq a \neq ab$. Denote by A the block of $r \mid M$ containing ab. We have $(b, v), (b, a), (b, ab) \in r$ and $b \in A$. Now, for every $x \in G$ and every $y \in A$, $(x, a), (a, b), (y, b), (xa, ab), (xy, ab) \in r$ and we see that $Ga \subseteq B$ and $GA \subseteq B$, $B = A \cup \{a\}$. Consequently,

$GB \subseteq B$ and $ca \in A$, since $ca \neq a$. Proceeding similarly, we can show that $BG \subseteq B$. Thus B is an ideal, $B = G$ and $r = G \times G$.

2.7 Proposition. Let r with a companion a be a pseudocongruence of a groupoid G. Then r is a congruence, provided at least one of the following conditions holds:

(i) There exists $b \in G$ such that $ba \neq a$ and L_b is a permutation of finite order.

(ii) There exists $b \in G$ such that $ab \neq a$ and R_b is a permutation of finite order.

(iii) G is a division groupoid and the least congruence containing r is either left or right cancellative.

(iv) G is simple and $a \neq aa$.

(v) G is simple idempotent and a is neither a left nor a right zero.

Proof. Apply 2.3, 2.4, 2.5 and 2.6.

2.8 Corollary. Every pseudocongruence of a groupiod G is a congruence, provided at least one of the following conditions holds:

(i) G is a division groupoid and every non-trivial congruence of G is either left or right cancellative.

(ii) G is a finite quasigroup.

(iii) G is a simple division groupoid.

(iv) G is a simple groupoid without idempotents.

(v) G is a simple idempotent groupoid containing no left and no right zeros.

3. Congruences of Primitive Groupoids

Throughout this section, let G be a primitive groupoid, $a, b \in G$, $a \neq b$, $(a, b) \in t$, $t = t_G$, $H = G/t$. Farther, let k denote the natural homomorphism of G onto H and $c = k(a)$. Finally, let r be a congruence of G and $s = k(r)$.

3.1 Lemma. s is a pseudocongruence with companion c of the groupoid H. Moreover, s is of class at most 2.

Proof. Easy.

We shall assume in the remaining part of this section that $(a, b) \notin r$. Denote by A and B the blocks of r containing a and b, resp. Obviously, $r \cap t = \mathrm{id}_G$.

3.2 Lemma. Let w be the least congruence of H containing s. Then $w \cap t_H = \mathrm{id}_H$.

Proof. First, let $x, y \in G$, $(k(x), k(y)) \in s \cap t_H$. There are $u, v \in G$ with (x, u), $(y, v) \in t$, $(u, v) \in r$. Moreover, $(xz, yz) \in t$ and $(zx, zy) \in t$ for every $z \in G$. But $xz = uz$, $zx = zu$, $yz = vz$ and $zy = zv$. Thus $(uz, vz) \in t \cap r$, $uz = vz$, $zu = zv$, $(u, v) \in t$, $(u, v) \in t \cap r$ and $u = v$, $k(x) = k(y)$. We have proved that $s \cap t_H = \mathrm{id}_H$. Now, let $x, y \in G$, $(k(x), k(y)) \in w \cap t_H$. We can assume that $(k(x), k(y)) \notin s$. Then $(k(x), c), (k(y), c) \in s$ and there are $u, v \in G$ with $(u, x), (v, y) \in t$, $(u, a), (v, b) \in r$.

Since $(k(x), k(y)) \in t_H$, $(uz, vz), (zu, zv) \in t$ for every $z \in G$. On the other hand, $(uz, az), (bz, vz) \in r$, $az = bz$, $(uz, vz) \in t \cap r$, $uz = vz$. Similarly, $zu = zv$, $(u, v) \in \in t$, $(x, y) \in t$ and $k(x) = k(y)$.

3.3 Lemma. Let H be regular and $x, y, z \in G$.

(i) If $(k(x), k(y)) \in s$ and $k(x)\,k(z) \neq k(y)\,k(z)$ then either $a \notin xG$ or $b \notin xG$.

(ii) If $(k(x), k(y)) \in s$ and $k(z)\,k(x) \neq k(z)\,k(y)$ then either $a \notin Gx$ or $b \notin Gx$.

Proof. There are $x', y' \in G$ with $(x', y') \in r$, $(x, x'), (y, y') \in t$. Then $xz = x'z$, $z = y'z$ and $(x'z, y'z) \in t$. Moreover, $xG = x'G$ and we can assume that $x = x'$ and $y = y'$. Now, suppose that $a = xu$ and $b = xv$ for some $u, v \in G$. We have $k(x)\,k(u) = k(x)\,k(v)$, $(k(u), k(v)) \in q_H$ and $(wu, wv) \in t$ for every $w \in G$. Thus $yu \neq y$, $yu, yv \in \{a, b\}$, $yu = a = xu$, $yv = b = xv$ (if $yu = yv$ then $(a, b) \in r$, a contradiction). From this, $k(x)\,k(u) = k(y)\,k(u)$ and $k(x)\,k(z) = k(y)\,k(z)$, a contradiction.

3.4 Lemma. Let H be regular and let $x \in G$ be such that $(x, y) \in r$ for some $x \neq y$. Then either $a \notin xG \cap Gx$ or $b \notin xG \cap Gx$.

Proof. Since $(a, b) \notin r$ and $x \neq y$, $k(x) \neq k(y)$. But $(k(x), k(y)) \in s$. By 3.2, $(k(x), k(y)) \notin t_H$. The rest follows from 3.3.

3.5 Lemma. (i) If s is transitive then either card $A = 1$ or card $B = 1$.

(ii) If G is strongly primitive then $3 \leqq$ card G/r.

Proof. (i) Let $x \in A$, $y \in B$, $x \neq a$, $y \neq b$. We have $k(a) = c = k(b)$, $(k(x), c)$, $(c, k(y))$, $(k(x), k(y)) \in s$, $(x, y) \in r$, a contradiction.

(ii) Let card $G/r \leqq 2$. Then, taking into account the equalities $aa = ba = ab = bb$, we see that G/r is a Z-groupoid. Consequently, either $GG \subseteq A$ or $GG \subseteq B$, a contradiction, since G is strongly primitive.

3.6 Lemma. Suppose that H is simple and $r \neq \mathrm{id}_G$. Then G is not strongly primitive.

Proof. Denote by w the least congruence of H containing s. Since $r \neq \mathrm{id}_G$ and $(a, b) \notin r$, $w \neq \mathrm{id}_H$ and $w = H \times H$. Hence card $G/r \leqq 2$ (use 3.1 and 2.2(iii)) and we can apply 3.5.

3.7 Lemma. Suppose that $r \neq \mathrm{id}_G$, H is a division groupoid and either G or H is regular.

(i) Either $2 \leqq$ card A or $2 \leqq$ card B.

(ii) If C is a block of r and $A \neq C \neq B$ then $2 \leqq$ card C.

(iii) Let card $A = 1$. Then $a \neq xx$ for every $x \in G$. Moreover, if $a = yz$ and Y, Z are the blocks of r containing y, z, resp., then Y is contained in a block of p_G and Z in a block of q_G.

Proof. (i) There is a block C of r with $2 \leqq$ card C. Let $x, y \in C$, $x \neq y$. We have $(x, y) \notin t$, and so either $(x, y) \notin p$ or $(x, y) \notin q$. Assume $(x, y) \notin p$. Farther,

$k(xz) = c$ for some $z \in G$. We have $xz \in \{a, b\}$. If $xz = a$ then $Cz \subseteq A$ and $2 \leqq$ $\leqq$ card A by [1, Lemma 3.6]. Similarly, if $xz = b$.

(ii) Let $C \neq A, B$ be a block of r and let $x \in C$. By (i), either $2 \leqq$ card A or $2 \leqq$ $\leqq$ card B. Suppose that $2 \leqq$ card A and $(y, a) \notin p$ for some $y \in A$. We have $k(x) = k(yz)$ for some $z \in G$, and so $x = yz \neq az$, $yz, az \in C$.

(iii) Suppose that Y is not contained in a block of p. Then $2 \leqq$ card Y and there is $u \in Y$ with $(y, u) \notin p$. Hence $a = yz \neq uz$, $uz \in A$ and $2 \leqq$ card A, a contradiction. Similarly for Z. Finally, let $a = xx$ and let X be the block of r with $x \in X$. As we have proved, X is contained in a block of t, card $X = 1$ and $X = A$ by (i) and (ii). Thus $x = a$, $a = aa = bb$ and card $B = 1$, a contradiction.

3.8 Lemma. Suppose that G is superprimitive and either G or H is regular. Then $2 \leqq$ card A, card B, provided $r \neq \mathrm{id}_G$.

Proof. Since $r \nsubseteq t$, either $r \nsubseteq p$ or $r \nsubseteq q$. Let $r \nsubseteq p$. Then $(x, y) \notin p$ for some $x, y \in G$, $(x, y) \in r$. There is $z \in G$ with $xz = a$. Now, $xz \neq yz$, $xz, yz \in A$ and $2 \leqq$ $\leqq$ card A. Similarly for B.

3.9 Lemma. Suppose that $r \neq \mathrm{id}_G$, H is a division groupoid and either G or H is regular. Then $2 \leqq$ card X for every block X of r, provided at least one of the following conditions is satisfied:

(i) H is commutative and $a, b \in GG$.

(ii) H is left (right) faithful and $a, b \in GG$.

(iii) $a = xx$ and $b = yy$ for some $x, y \in G$.

(iv) G is superprimitive.

Proof. Apply 3.7 and 3.8.

3.10 Lemma. Suppose that G is regular superprimitive, p_G is a congruence of G and $\bar{p}_G = G \times G = q_{G,o}$, where o is the first infinite ordinal. Then $r = \mathrm{id}_G$.

Proof. Let $r \neq \mathrm{id}_G$. Then $w = r \cap p \neq \mathrm{id}_G$ and $(a, c) \in w$ for some $c \neq a$ (use 3.8). Since $q_{G,o} = G \times G$, there is a natural number $1 \leqq n$ with $x_1(\ldots(x_n a)) = x_1(\ldots(x_n c))$ for all $x_1, \ldots, x_n \in G$. Farther, $a = da$ for a $d \in G$. Then $a = L_d^n(c)$. Let $1 \leqq m$ be the least natural number with $a = L_d^m(c)$. Put $e = L_d^{m-1}(c)$. Then $a \neq e$, $(a, e) \in p$ and $da = a = de$. Since G is regular, $(a, e) \in q$, $(a, e) \in t$, $e = b$. However, $(a, e) \in r$, a contradiction.

4. Main Results

4.1 Theorem. Let G be a strongly primitive groupoid and $H = G/t$. Then G is subdirectly irreducible, provided at least one of the following conditions is satisfied:

(i) G is torsion.

(ii) H is torsion.

(iii) p_G, q_G are congruences of G and $\bar{p}_G = G \times G = \bar{q}_G$.

(iv) If $r \neq \mathrm{id}_H$ is a congruence of H then $r \cap t_H \neq \mathrm{id}_H$.
(v) H is subdirectly irreducible and primitive.
(vi) Every proper factorgroupoid of G is semifaithful.
(vii) H is simple.

Proof. Apply [1, Lemma 2.16], 3.2 and 3.6.

4.2 Theorem. Let G be a superprimitive groupoid such that G/t is regular. Then G is subdirectly irreducible.

Proof. Apply 3.4.

4.3 Proposition. Let G be a primitive groupoid, $H = G/t$, $a, b \in G$, $a \neq b$, $(a, b) \in t_G$. Denote by k the natural homomorphism of G onto H and put $c = k(a)$. Suppose that every pseudocongruence of H with companion c and of class at most 2 is a congruence. Then G is subdirectly irreducible, provided either G or H is regular and at least one of the following conditions is satisfied:
(i) G is superprimitive.
(ii) H is commutative, H is a division groupoid and $a, b \in GG$.
(iii) H is a left (right) faithful division groupoid and $a, b \in GG$.
(iv) H is a division groupoid and $a = xx$, $b = yy$ for some $x, y \in G$.

Proof. Appy 3.1, 3.5(i), 3.8 and 3.9.

4.4 Theorem. Let G be a regular superprimitive groupoid and $H = G/t$. Then G is subdirectly irreducible, provided at least one of the following conditions is satisfied:
(i) H is a division groupoid and every non-trivial congruence of H is either left or right cancellative.
(ii) $\bar{p}_G = G \times G = q_{G,o}$ and p_G is a congruence of G.
(iii) $\bar{q}_G = G \times G = p_{G,o}$ and q_G is a congruence of G.

Proof. Apply 4.3, 2.8 and 3.10.

4.5 Theorem. Let G be a primitive groupoid such that H is a division groupoid and either G or H is regular, where $H = G/t$. Suppose that every non-trivial congruence of H is either left or right cancellative. Then G is subdirectly irreducible, provided at least one of the following conditions is satisfied:
(i) H is commutative and $a, b \in GG$, where $a, b \in G$, $a \neq b$ and $(a, b) \in t_G$.
(ii) H is left (right) faithful and $a, b \in GG$.
(iii) $a = xx$ and $b = yy$ for some $x, y \in G$.

Proof. Apply 4.3, 2.8.

4.6 Corollary. Let G be a strongly primitive groupoid such that G/t is a quasigroup having only cancellative congruences. Then G is subdirectly irreducible.

4.7 Theorem. Let G be a primitive division groupoid and $H = G/t$. Then G is subdirectly irreducible, provided at lest one of the following conditions is satisfied:

(i) H is regular.

(ii) G is regular and every non-trivial congruence of H is either left or right cancellative.

Proof. Apply 4.2 and 4.4.

4.8 Corollary. Every primitive medial division groupoid is subdirectly irreducible.

4.9 Example. Consider the following groupoid $G : G = \{a, b, c, d\}$, $aa = ab = ac = ad = ba = bb = bd = a$, $ca = cb = cc = cd = b$, $da = db = dc = dd = d$. One may verify easily that G is strongly primitive, G satisfies (C5) and G is not subdirectly irreducible.

4.10 Example. Let G be a commutative loop possessing a congruence r such that G/r and blocks of r are infinite countable sets. Let $A_1, A_2, \ldots$ be the blocks of r and suppose that $1 \in A_1$, $A_1 = B \cup C$, $B \cap C = \emptyset$, $1 \in B$ and $\operatorname{card} B = \operatorname{card} C$. Let f be transformation of G such that $f \mid A_i$ is a biunique mapping of A_i onto A_{i-1} for every $3 \leqq i$, $f \mid A_2$ is a biunique mapping of A_2 onto B, $f(1) = 1$ and $f \mid A_1$ is a biunique mapping of A_1 onto $C \cup \{1\}$. There is just one element $a \in G$ with $a \neq 1$ and $f(a) = 1$ and $\ker f = \{(a, 1), (1, a)\} \cup \mathrm{id}_G$. Now, put $x \circ y = f(x) f(y)$ for all $x, y \in G$. Then $G(\circ)$ is a primitive regular commutative division groupoid. It is easy to check that r is a congruence of $G(\circ)$. But $a \in A_2$ and $(1, a) \notin r$. Hence $G(\circ)$ is not subdirectly irreducible.

4.11 Theorem. Let H be a groupoid. Then H is isomorphic to G/t for a subdirectly irreducible primitive groupoid G if at least one of the following conditions is satisfied:

(i) H is regular and satisfies (C7).

(ii) H is a regular division groupoid and $2 \leqq \operatorname{card} A = \operatorname{card} B$ for any two blocks A, B of t_H.

(iii) H is a Z-groupoid.

(iv) H is a semifaithful regular division groupoid and $2 \leqq \operatorname{card} A$, $2 \leqq \operatorname{card} B$ for every block A of p_H and B of q_H.

(v) H is a semifaithful regular division groupoid, every non-trivial congruence of H is either left or right cancellative and $xx = yy$ for some $x, y \in G$, $x \neq y$.

(vi) H is a left faithful regular division groupoid and every congruence of H is either left or right cancellative.

(vii) H is a quasigroup and every congruence of H is either left or right cancellative.

(viii) H is a finite quasigroup.

(ix) H is simple and not injective.

(x) H is subdirectly irreducible and primitive.

(xi) H is a torsion division groupoid and card $B \leqq 2^{\text{card}A}$, whenever A is a block of p_H and B of t_H.

Proof. (i) follows from [1, 10.1(iii)] and 4.2, (ii) follows from (i) and [1, 10.2(iib)], (iii) follows from (i) and [1, 10.2 (iia)], (iv) follows from (i) and [1, 10.2(iic)], (v) follows from [1, 10.1(ii), 10.2(ia)] and 4.5(iii), (vi) follows from [1, 10.1 (ii), 10.2(ia)] and 4.5(ii), (viii) and (viii) follow from (vi), (ix) follows from [1, 10.1(ii), 10.2(ia)] and 4.1(vii), (x) follows from [1, 10.1(ii), 10.2(ie)] and 4.1(v) and (xi) follows from [1, 10.1(ii), 10.2(ib)] and 4.1(ii).

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