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Exchangeable Partial Groupoids II

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In the paper, the problem of distances between finite latin squares and groups is studied.

V článku se studuje problém vzdálenosti konečných latinských čteverců od grup.

В статье изучается проблема расстояния между конечными латинскими квадратами и группами.

The present note is an immediate continuation of [1] and the reader is referred to [1] as for notation, terminology, etc.

13. An example

Let k_1, k_2, k_3 be non-negative integers, $k = k_1 + k_2 + k_3$, $k_0 = 0$ and $k_4 = k$. Further, put $T = \{1, 2, 3\}$ and define a permutation t of T by $t(1) = 2$, $t(2) = 3$ and $t(3) = 1$. For $1 \leqq i \leqq k$ let $s(i) = j \in T$ iff $\sum_{u=0}^{j-1} k_u < i \leqq \sum_{u=0}^{j} k_u$. For $j \in T$ let $r(j) = \sum_{u=1}^{j} k_u$ if $k_j \neq 0$ and $r(j) = k + j$ in the opposite case. Finally, let $v(j) = 1 + \sum_{u=0}^{j-1} k_j$ and $V = \{v(j); j \in T\}$.

Now, we shall define a balanced partial groupoid $Y = Y(\circ) = Y(k_1, k_2, k_3, \circ)$ as follows: $B(Y) = \{b_1, b_2, b_3\}$, $C(Y) = \{c_1, c_2, \ldots, c_{k+2}\}$, $D(Y) = \{1, 2, \ldots, k + 3\}$ and $M(Y) = \{(b_{s(i)}, c_i); 1 \leqq i \leqq k\} \cup \{(b_{ts(i)}, c_i); 1 \leqq i \leqq k\} \cup B(Y) \times \{c_{k+1}, c_{k+2}\}$. We put $b_{s(i)} \circ c_i = i$ for any $1 \leqq i \leqq k$, $b_{ts(i)} \circ c_i = i - 1$ for every $1 \leqq i \leqq k$ such that $i \notin V$, $b_j \circ c_{k+2} = k + j$ and $b_{t(j)} \circ c_{k+1} = r(j)$ for every $j \in T$ and $b_{t(j)} \circ v_{v(j)} = k + j$ for every $j \in T$ with $k_j \neq 0$. This definition is correct since for $j \in T$ we have $sv(j) = j$ if $k_j \neq 0$ and $v(j) \in \{v(j + 1), k + 1\}$ otherwise.

13.1. Lemma. (i) Y is a balanced cancellative reduced partial groupoid.

(ii) $\operatorname{card}(Y) = 2k + 8$, $m = 2k + 6$, $p = 3$, $q = k + 2$, $o = k + 3$, $\delta(p) = 2k$, $\delta(q) = 2$, $\delta(o) = 0$ and $\delta = 2k + 2$.

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(iii) $p(b_j) = k + 2 - k_{t(j)}$ for every $j \in T$.
(iv) $q(c_i) = 2$ for every $1 \leqq i \leqq k$.
(v) $q(c_i) = 3$ for $i = k + 1, k + 2$.
(vi) $o(i) = 2$ for every $1 \leqq i \leqq k + 3$.

Proof. Let $j \in T$ and let S be the set of all i such that $b_j \circ c = i$ for some $c \in C(Y)$. Then it is easy to see that $S = \{i;\ 1 \leqq i \leqq k + 3\} - (\{i;\ \sum_{u=0}^{t-1} k_u < i \leqq \sum_{u=0}^{t} k_u\} \cup \cup \{k + t\})$, $t = t(j)$.

13.2. Lemma. Suppose that $k_1 \geqq k_2 \geqq k_3$.
(i) If $k_3 \geqq 1$ then $m \geqq 12$.
(ii) If $k_3 = 0$ then Y is regular, $H(Y)$ is a cyclic group of order $k_1 k_2 + 2k_1 + 2k_2 + 3$ and no non-trivial subgroup of $H(Y)$ is regular.

Proof. Suppose that $k_3 = 0$, $k_1 \neq 0$ and consider the group $H \doteq G/P(Y, (b_1, c_1))$ (see [2]). Then $b_1 = c_1 = 1$ in H and $b_2 = c_{k+2} = x^{-1}$, $x = c_2$. Further, $c_i = x^{i-1}$ for $1 \leqq i \leqq k_1$, $c_{k+1} = x^{k_1}$ and from $k + 3 = b_3 \circ c_{k+2} = b_1 \circ c_{k+1}$ it follows that $b_3 = x^{k_1+1}$. If $k_2 = 0$ then from $b_3 \circ c_{k+1} = b_2 \circ c_{k+2}$ it follows that $x^{2k+3} = 1$. If $k_2 \neq 0$ then $c_{k_1+i} = x^{-ik_1-2i-1}$ for $1 \leqq i \leqq k_2$ and hence from $b_2 \circ c_k = b_3 \circ c_{k+1}$ it follows that $x^l = 1$, $l = k_1 k_2 + 2k_1 + 2k_2 + 3$.

13.3. Lemma. Y is primary and strictly exchangeable.

Proof. The result is an easy consequence of 8.1, 8.2 and 8.3.

By 13.3 there exists a unique partial groupoid $Y(*) = Y(k_1, k_2, k_3, *)$ such that $I = I(k_1, k_2, k_3) = (Y(\circ), Y(*))$ is a couple of companions.

13.4. Lemma. Let d be a permutation of T. Then the partial groupoids $Y(k_1, k_2, k_3)$ and $Y(k_{d(1)}, k_{d(2)}, k_{d(3)})$ are isomorphic.

Proof. We shall construct a mapping f of $Y(k_{d(1)}, k_{d(2)}, k_{d(3)})$ onto $Y(k_1, k_2, k_3)$ as follows: Let $j \in T$. Put $f(b_j) = b_{d(j)}$, $f(c_{k+1}) = c_{k+1}$, $f(c_{k+2}) = c_{k+2}$, $f(k + j) = k + d(j)$ and $f(y) = i + \sum_{u=0}^{j-1} k_u$, $f(c_y) = c_{f(y)}$ for every $1 \leqq i \leqq k_{d(j)}$ and $y = i + \sum_{u=0}^{d(j)-1} k_u$.

13.5. Lemma. Suppose that $k_1 \geqq k_2 \geqq k_3$ and let $k'_1 \geqq k'_2 \geqq k'_3$ be non-negative integers. Then the partial groupoids $Y(k_1, k_2, k_3)$ and $Y(k'_1, k'_2, k'_3)$ are isomorphic iff $(k_1, k_2, k_3) = (k'_1, k'_2, k'_3)$.

Proof. Easy.

13.6. Lemma. (i) If $k \neq 1$ then the couples I and $\bar{I}$ are not isomorphic.

(ii) The couples I and I^{-1} are not isomorphic.

(iii) The couples $\bar{I}$ and I^{-1} are not isomorphic.

(iv) If $k \neq 0$ then the couples I and ${}^{-1}I$ are not isomorphic.

(v) The couples $\bar{I}, I^{-1}$ and ${}^{-1}I$ are pair-wise non-isomorphic.

Proof. Easy.

14. Auxiliary results

In this section, some results of auxiliary character are formulated. In fact, they are slight modifications of the material contained in the eighth section, and so the proves are omitted.

Let $I = (K(\circ), K(*))$ be a couple of companions and suppose that $(a, c), (b, c) \in \in M(I)$, $a \neq b$, $o(a \circ c) = 2$. A finite sequence $(c_1, \ldots, c_r)$ is called pseudoadmissible if $r \geqq 2$, $c_1, \ldots, c_r$ are different elements of $C(I)$, $c_1 = c$, $a \circ c_i = b \circ c_{i+1}$ and $o(a \circ c_i) = 2$ for every $1 \leqq i < r$ and $q(c_j) = 2$ for every $1 < i < r$. We shall say that the sequence is maximal if it has no pseudoadmissible prolongation.

14.1. Lemma. Let $(c_1, \ldots, c_r)$ be a pseudoadmissible sequence. Then $b * c_i = = a * c_{i+1} = a \circ c_i$ for every $1 \leqq i < r$.

14.2. Lemma. Let $(c_1, \ldots, c_r)$ be a maximal pseudoadmissible sequence. Then at least one of the following three conditions is satisfied:

(1) $q(c_r) = 2$ and $a \circ c_r = b \circ c_1 = a * c_1 = b * c_r$.

(2) $q(c_r) \geqq 3$.

(3) $o(a \circ c_r) \geqq 3$.

14.3. Lemma. Suppose that I is simple and $(c_1, \ldots, c_r)$ is a maximal pseudoadmissible sequence satisfying (1). Then $M(I) = \{a, b\} \times \{c_1, \ldots, c_r\}$.

14.4. Lemma. Suppose that I is finite and there exists at least one pseudoadmissible sequence. Then there exists at least one maximal pseudoadmissible sequence.

15. Couples with $\delta(q) = 2$ and $\delta(o) = 0$

15.1. Lemma. Let $I = (K(\circ), K(*))$ be a simple couple of finite companions with $\delta(q) = 2$ and $\delta(o) = 0$. Then there exist $u, v \in C(I)$ such that $u \neq v$, $q(u) = = q(v) = 3$ and $q(w) = 2$ for any $w \in C(I)$, $u \neq w \neq v$.

Proof. Put $K = K(\circ)$ and take an element $u \in C(K)$ with $q(u) \geqq 3$. We have $2 = \delta(q) = \sum(q(c) - 2)$, and therefore such an element exists. Further, let $a \in B(u)$ (see the eighth section). Since $\delta(o) = 0$, $o(x) = 2$ for every $x \in D(K)$. In particular, $o(a \circ u) = 2$ and $a \circ u = b \circ c$ for some $(b, c) \in M(K)$, $a \neq b$, $u \neq c$. By 8.1, $a \circ u =$

$= a * c = b * u$. If $q(c) \geqq 3$, then $q(u) = 3 = q(c)$ and we can put $v = c$. Suppose that $q(c) = 2$. By 14.4, let $(c_1, \ldots, c_r)$ be a maximal pseudoadmissible sequence, $c_1 = u$, $c_2 = c$. Now, with respect to 14.3 and 14.2, $q(c_r) \geqq 3$ and we can put $v = c_r$.

15.2. Proposition. Let $I = (K(\circ), K(*))$ be a simple couple of finite balanced companions such that $\delta(q) = 2$ and $\delta(o) = 0$. Then there exist uniquely determined non-negative integers k_1, k_2, k_3 such that $k_1 \geqq k_2 \geqq k_3$ and the couple I is isomorphic to $I(k_1, k_2, k_3)$.

Proof. Put $K = K(\circ)$. By 15.1, there are $u, v \in C(K)$ such that $q(u) = q(v) = 3$ and $u \neq v$. Let $B = B(u) = \{a, b, c\}$. There is a permutation s of B and mapping t of B into $C(K)$ such that $s(x) \neq x$ and $x \circ u = s(x) \circ t(x)$ for each $x \in B$. It is easy to see that $t(x) = v$ whenever $x, y \in B$, $x \neq y$ and $t(x) = t(y)$. Further, we can assume that $s(a) = b$, $s(b) = c$ and $s(c) = a$. The rest of the proof is divided into four parts.

(i) Let card $(t(B)) = 1$. Then clearly I is isomorphic to $I(0, 0, 0)$.

(ii) card $(t(B)) = 2$. Then t is not injective and we can assume that $t(a) = c_2 \neq v$, $u = c_1$ and $t(b) = t(c) = v$. Let $(c_1, c_2, \ldots, c_r)$, $r \geqq 2$, be a maximal pseudoadmissible sequence (by 14.4). We have $c_r = v$ and, since I is simple, I is isomorphic to $I(r - 1, 0, 0)$.

(iii) card $(t(B)) = 3$ and $v \in t(B)$. We can assume that $t(c) = v$, $t(a) = c_2$, $t(b) = c'_2$. We have $a \circ u = b \circ c_2$, $b \circ u = c \circ c'_2$, $c \circ u = a \circ v$. Let $(c_1, \ldots, c_r)$ and $(c'_1, \ldots, c'_d)$ be maximal pseudoadmissible sequences such that $c_1 = u = c'_1$, $a \circ c_2 = b \circ c_3, \ldots, b \circ c'_2 = c \circ c'_3, \ldots$. Obviously $c_r = v = c'_d$ and I is isomorphic to $I(r - 1, d - 1, 0)$.

(iv) card $(t(B)) = 3$ and $v \notin t(B)$. Let $t(a) = c_2$, $t(b) = c'_2$, $t(c) = c''_2$ and let $(c_1, \ldots, c_r)$, $(c'_1, \ldots, c'_d)$, $(c''_1, \ldots, c''_e)$ be maximal pseudoadmissible sequences such that $u = c_1 = c'_1 = c''_1$, $a \circ c_2 = b \circ c_3$, $b \circ c'_2 = c \circ c'_3$, $c \circ c''_2 = a \circ c''_3$. Then $c_r = c'_d = c''_e = v$ and I is isomorphic to $I(r - 1, d - 1, e - 1)$.

15.3. Corollary. Let k_1, k_2, k_3 be non-negative integers. Then the partial groupoids $Y(k_1, k_2, k_3, \circ)$ and $Y(k_1, k_2, k_3, *)$ are isomorphic.

16. An example

Let k_1, k_2, k_3 be non-negative integers, $k = k_1 + k_2 + k_3$, $k_0 = 3$, $n_i = k_i + 1$, $n = k + 3$, $n_0 = 0$. Put $T = \{1, 2, 3\}$, $t(1) = 2$, $t(2) = 3$, $t(3) = 1$.

Now, we shall define a balanced partial groupoid $X = X(\circ) = X(k_1, k_2, k_3, \circ)$ as follows: $B(X) = \{b_1, b_2, b_3\}$, $C(X) = \{c_1, c_2, \ldots, c_{k+4}\}$, $D(X) = \{1, 2, \ldots, k + 4\}$, $b_j \circ c_j = n + 1$, $b_{t(j)} \circ c_j = 1 + \sum_{u=0}^{j-1} n_u$, $b_j \circ c_{k+4} = \sum_{u=1}^{j} n_u$, $b_j \circ c_i = i - 4 + j$, $b_{t(j)} \circ c_i = i - 3 + j$ for all $j \in T$ and $\sum_{u=0}^{j-1} k_u < i \leqq \sum_{u=0}^{j} k_u$.

16.1. Lemma. (i) X is a balanced cancellative reduced partial groupoid.

(ii) $\operatorname{card}(X) = 2k + 11$, $m = 2k + 9$, $p = 3$, $q = k + 4$, $o = k + 4$, $\delta(p) = 2k + 3$, $\delta(q) = 1$, $\delta(o) = 1$ and $\delta = 2k + 5$.

(iii) $p(b_j) = k + 4 - k_{t(j)}$ for every $j \in T$.

(iv) $q(c_i) = 2$ for every $1 \leqq i \leqq k + 3$ and $q(c_{k+4}) = 3$.

(v) $o(i) = 2$ for every $1 \leqq i \leqq k + 3$ and $o(k + 4) = 3$.

Proof. Easy.

16.2. Lemma. Suppose that $k_1 \geqq k_2 \geqq k_3$.

(i) If $k_3 \geqq 1$ then $m \geqq 15$.

(ii) If $k_3 = 0$ then X is regular, $H(X)$ is a cyclic group of order $k_1k_2 + 3k_1 + 3k_2 + 7$ and no non-trivial subgroup of $H(X)$ is regular.

Proof. Easy.

16.3. Lemma. X is primary and strictly exchangeable.

Proof. Let $(b_j, c_i) \in M(X)$. If $i < k + 4$ then $b_j * c_i$ is determined uniquely, since $q(c_i) = 2$. If $i = k + 4$ then $b_j * c_i$ is uniquely determined as well, since $o(b_j \circ c_i) = 2$ (use 8.1).

By 16.3 there exists a unique partial groupoid $X(*) = X(k_1, k_2, k_3, *)$ such that $J = J(k_1, k_2, k_3) = (X(\circ), X(*))$ is a couple of companions.

16.4. Lemma. Let k'_1, k'_2, k'_3 be non-negative integers. Then the groupoids $X(k_1, k_2, k_3, \circ)$ and $X(k'_1, k'_2, k'_3, \circ)$ are isomorphic iff $k_i = k_{s(i)}$ for a permutation s of T.

Proof. Easy.

17. An example

Let k_1, k_2 be non-negative integers, $k = k_1 + k_2$, $k_0 = 3$, $n_i = k_i + 1$, $n = k + 2$. Put $T = \{1, 2, 3\}$, $t(1) = 2$, $t(2) = 3$, $t(3) = 1$.

We shall define a balanced partial groupoid $X = X(\circ) = X(k_1, k_2, \circ)$ as follows: $B(X) = \{b_1, b_2, b_3\}$, $C(X) = \{c_1, \ldots, c_{k+3}\}$, $D(X) = \{1, \ldots, k + 3\}$, $b_j \circ c_j = k + 3$ for every $j \in T$, $b_{t(j)} \circ c_i = i - 4 + j$, $b_{tt(j)} \circ c_i = i - 3 + j$, $b_{t(j)} \circ c_1 = \sum_{u=1}^{j} n_u$ for every $1 \leqq j \leqq 2$ and $\sum_{u=0}^{j-1} k_u < i \leqq \sum_{u=0}^{j} k_u$ and $b_1 \circ c_3 = n_1 + 1$, $b_3 \circ c_2 = 1$.

17.1. Lemma. (i) X is a balanced cancellative reduced partial groupoid.

(ii) $\operatorname{card}(X) = 2k + 9$, $m = 2k + 7$, $p = 3$, $q = k + 3$, $o = k + 3$, $\delta(p) = 2k + 1$, $\delta(q) = 1$, $\delta(o) = 1$, $\delta = 2k + 3$.

(iii) $p(b_1) = k_2 + 2$, $p(b_2) = k_1 + 2$, $p(b_3) = k + 3$.
(iv) $q(c_i) = 2$ for every $2 \leqq i \leqq k + 3$ and $q(c_1) = 3$.
(v) $o(i) = 2$ for every $1 \leqq i \leqq k + 2$ and $o(k + 3) = 3$.

Proof. Easy.

17.2. Lemma. X is regular, $H(X)$ is a cyclic group of order $k_1k_2 + 2k_1 + 2k_2 + + 2$ and $H(X)$ has no non-trivial regular subgroups.

Proof. Easy.

17.3. Lemma. X is strictly exchangeable but not primary.

Proof. For $1 < i \leqq k + 3$, the element $b_j * c_i$ is determined uniquely, since $q(c_i) = 2$. Now, it is easy to see that X is strictly exchangeable. On the other hand, the subset $\{b_1, b_2\} \times (\{c_1, c_3\} \cup \{c_i; k_1 + 3 < i \leqq k + 3\})$ of $M(X)$ is admissible. Hence X is not primary.

By 17.3, there exists a unique partial groupoid $X(*) = X(k_1, k_2, *)$ such that $J = J(k_1, k_2) = (X(\circ), X(*))$ is a couple of companions.

17.4. Lemma. J is a simple couple.

Proof. Easy.

17.5. Lemma. Let k'_1, k'_2 be non-negative integers. Then the partial groupoids $X(k_1, k_2, \circ)$ and $X(k'_1, k'_2, \circ)$ are isomorphic iff $k_1 = k'_1$ and $k_2 = k'_2$.

Proof. The partial groupoid $X(k_1, k_2, \circ)$ contains a partial subgroupoid isomorphic to $Z(k_2 + 2, \circ)$. On the other hand, if Z is a partial subgroupoid of $X(k_1, k_2, \circ)$ and Z is isomorphic to $Z(i, \circ)$ for some i then $i = k_2 + 2$. The rest is clear.

17.6. Lemma. $X(k_1, k_2, *)$ is isomorphic to $X(k_2, k_1, \circ)$.

Proof. Easy.

18. Couples with $\delta(q) = 1 = \delta(o)$

18.1. Proposition. Let $I = (K(\circ), K(*))$ be a simple couple of finite balanced companions such that $\delta(q) = 1 = \delta(o)$. Then there exist non-negative integers k_1, k_2, k_3 such that I is isomorphic either to $J(k_1, k_2)$ or to $J(k_1, k_2, k_3)$.

Proof. Let $T = \{1, 2, 3\}$, $t(1) = 2$, $t(2) = 3$ and $t(3) = 1$. Since $\delta(o) = 1$, there is just one element $a \in D(K)$ with $o(a) = 3$ and consequently there exist elements $b_j \in B(K)$, $c_j \in C(K)$, $j \in T$, with $b_j \circ c_j = a = b_{t(j)} * c_j$. Without loss of generality, we can assume that $q(c_1) \geqq q(c_2) \geqq q(c_3) = 2$. Let us distinguish the following two cases:

(i) $q(c_1) = 3$. Put $e_1 = c_2$, $e'_1 = c_3$. We have $q(e_1) = q(e'_1) = o(b_3 \circ e_1) = o(b_1 \circ e'_1) = 2$. Now, we can consider maximal admissible sequences $(e_1, \ldots, e_s)$, $(e'_1, \ldots, e'_d)$ such that $b_3 \circ e_i = b_2 \circ e_{i+1}$ for every $1 \leqq i < r$ and $b_1 \circ e'_i = b_3 \circ e'_{i+1}$ for every $1 \leqq i < d$. Since $(b_2, c_3) \notin M(K)$, we have $c_3 \neq e_i$ for every $1 \leqq i \leqq r$. Hence $o(b_3 \circ e_r) = 2$, and so by 8.8 $q(e_r) \geqq 3$ and consequently $e_r = c_1$. Similarly, $e'_d = c_1$. Now it is easy to see that I is isomorphic to $J(r-1, d-1)$ (use 8.4).

(ii) $q(c_1) = q(c_2) = q(c_3) = 2$. If $b_{t(j)} \circ c_j = b_{tt(j)} \circ c_{t(j)}$ then, by 8.1, $(b_{tt(j)}, c_j) \in M(K)$, a contradiction with $q(c_j) = 2$. Hence the elements $b_{t(j)} \circ c_j$ are pair-wise different. Further, consider three maximal admissible sequences $(e_1, \ldots, e_r)$, $(e'_1, \ldots, e'_d)$ $(e''_1, \ldots, e''_f)$ such that $e_1 = c_1$, $e'_1 = c_2$, $e''_1 = c_3$, $e_i \circ b_2 = e_{i+1} \circ b_1$ for every $1 \leqq i \leqq r-1$, $e'_i \circ b_3 = e'_{i+1} \circ b_2$ for every $1 \leqq i \leqq d-1$ and $e''_i \circ b_1 = e''_{i+1} \circ b_3$ for every $1 \leqq i \leqq f-1$. Further, since $q(c_j) = 2$, we have $e_r, e'_d, e''_f \notin \{c_1, c_2, c_3\}$, $o(e_r) = o(e'_d) = o(e''_f) = 3$ and $e_r = e'_d = e''_f$. In the rest we can proceed similarly as in (i).

18.2. Corollary. Let k_1, k_2, k_3 be non-negative integers. Then the partial groupoids $X(k_1, k_2, k_3, \circ)$ and $X(k_1, k_2, k_3, *)$ are isomorphic.

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