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On the Structure of Quasi-Ordering Lattices

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We investigate the structure of the lattice $\Omega(I)$ of all quasi-orderings on a set I. We describe a natural set of the so called fundamental inequalities defining all minimal inequalities in $\Omega(I)$ and develop an axiomatic characterization of $\Omega(I)$. We further describe the automorphism group of $\Omega(I)$.

Introduction

It was proved in [2] that every finite quasi-ordering lattice is isomorphic to an interval in the subgroup lattice of a finite group. To understand fully the structure of intervals in quasi-ordering lattices remains an interesting unsolved problem related to this work. Recently, Wehrung in a long and important paper [3] studied the so called D-valued posets. These posets are equivalent to $\{0, \vee\}$-preserving mappings from the semilattice of finitely generated quasi-orders on an infinite set Ω into a distributive lattice D.

In this paper we prove several basic facts about the structure of quasi-ordering lattices on arbitrary sets. We describe a set of minimal inequalities generating all inequalities among atoms in these lattices. This set of minimal inequalities is then applied to prove that quasi-ordering lattices are simple. More significantly, we use this set to prove a structure theorem for quasi-ordering lattices similar to that one of Ore [1] about partition lattices. Finally, we describe the automorphism groups of quasi-ordering lattices.

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A *quasi-ordering* on a non-empty set I is a reflexive and transitive relation on I. The set of all quasi-orderings on I ordered by inclusion is a complete lattice that will be denoted by $\Omega(I)$ and called the *quasi-ordering lattice* on I. The least element of $\Omega(I)$ will be denoted by 0_I and the greatest element by 1_I. The quasi-ordering lattice $\Omega(I)$ is an atomistic lattice, the atoms are the quasi-orders containing exactly one non-trivial pair (i, j), $i \neq j$. In this way, the atoms in $\Omega(I)$ are in one-to-one correspondence with the edges of the complete oriented loopless graph on I. This observation will play an important role in the paper.

If $a, b_k, k \in K$ are atoms in a complete lattice L, and $a \leq \bigvee_{k \in K} b_k$, then this inequality is called *minimal* if $a \nleq \bigvee_{j \in J} b_j$ for any proper subset J of K. A complete atomistic lattice L is *algebraic* if and only if all minimal inequalities in L have finite right-hand sides.

From the definition of join in $\Omega(I)$ it is obvious that all minimal inequalities have a finite right-hand side, hence $\Omega(I)$ is an algebraic lattice. The most important minimal inequalities in $\Omega(I)$ are those of the form $p \leq q \vee r$, since every other inequality involving atoms can be deduced from them. Inequalities of the form $p \leq q \vee r$ (p, q, r atoms in $\Omega(I)$) will be called *fundamental inequalities*. We may proceed further with our identification of atoms in $\Omega(I)$ with edges of the complete oriented loopless graph on I and identify fundamental inequalities with ordered triples (i, j, k) of different elements of I. An ordered triple (i, j, k) corresponds to the fundamental inequality $(i, k) \leq (i, j) \vee (j, k)$. The three atoms $(i, k), (i, j)$ and (j, k) generate a 7-element sublattice D_1 of $\Omega(I)$ in which they are atoms and the remaining elements are the least and the greatest elements, $(i, j) \vee (i, k)$ and $(i, k) \vee (j, k)$. Note that $(i, j) \vee (j, k)$ is the largest element of D_1. In fact, this sublattice is a lower interval in $\Omega(I)$. The fact that D_1 is a simple lattice leads to a proof of simplicity of $\Omega(I)$.

Theorem 1. *The quasiordering lattice $\Omega(I)$ is simple if and only if $|I| > 2$.*

Proof. If $|I| = 2$, then $\Omega(I)$ is isomorphic to the product of two 2-element chains. So assume $|I| \geq 3$, and let π be a congruence on $\Omega(I)$ identifying two different quasi-orderings α, β. We may assume $\alpha > \beta$. Take a pair $(i, j) \in \alpha \setminus \beta$. Then the atom (i, j) is congruent to 0_I in π, since $(i, j) = (i, j) \wedge \alpha$, $0_L = (i, j) \wedge \beta$ and $(\alpha, \beta) \in \pi$. By the remark preceding Theorem 1, we get that both (i, k), (j, k) and (k, i), (k, j) are congruent to 0_I in π, for every $k \neq i, j$. By the same argument, replacing (i, j) by (i, k) and k by an element $l \neq i, k$, we get that $(i, l), (k, l), (l, i)$ and (l, k) are congruent to 0_I. Hence all atoms are congruent to 0_I in π.

Now if $\gamma \in \Omega(I)$ is congruent to 0_I in π and δ covers γ in $\Omega(I)$, then for every pair $(k, l) \in \delta \setminus \gamma$ we get $(k, l) \vee \gamma = \delta$. Since (k, l) is congruent to 0_L in π, we get $(\gamma, \delta) \in \pi$, hence also $(0_I, \delta) \in \pi$. Thus the greatest element of $\Omega(I)$ congruent to 0_I in π must be 1_I, since any other element of $\Omega(I)$ is covered by another element of $\Omega(I)$. $\square$

The structure theorem

We have already noted that fundamental inequalities $p \leq q \vee r$ generate all minimal inequalities among atoms in $\Omega(I)$. This concept is made precise by the following definition.

Definition. Let L be an atomistic lattice and Φ a set of minimal inequalities of the form $a \leq \bigvee\{b_j : j \in J\}$. Then Φ is a *generating set* for inequalities in L whenever $a \leq \bigvee\{b_k : k \in K\}$ is a minimal inequality, then either it belongs to Φ or there is a subset $J \subset K$ such that $c \leq \bigvee\{b_j : j \in J\}$ belongs to Φ and

$$a \leq c \vee \bigvee\{b_k : k \in K \backslash J\}$$

is a minimal inequality. The elements of Φ will be called *fundamental inequalities* in L.

Stating otherwise, Φ is a generating set if every minimal inequality in L can be obtained from the set Φ using only substitutions. We shall investigate those atomistic algebraic lattices, in which the set Φ of minimal inequalities of the form $a \leq b \vee c$ is a generating set. Such lattices will be called *2-lattices*. The quasi-ordering lattice $\Omega(I)$ is obviously a 2-lattice. The following lemma states that the set of fundamental inequalities fully characterizes quasi-ordering lattices. By $At(L)$ we denote the set of atoms of L.

Lemma 1. *Let L be a 2-lattice. Then L is isomorphic to $\Omega(I)$ if and only if there exists a bijection $Z : At(L) \to At(\Omega(I))$ such that $A \leq b \vee c$ is a minimal inequality in L if and only if $Z(a) \leq Z(b) \vee Z(c)$ is a minimal inequality in $\Omega(I)$.*

Proof. Let a bijection $Z : At(L) \to At(\Omega(I))$ exist. We extend Z to a mapping from L to $\Omega(I)$ by

$$Z(x) = \bigvee\{Z(a) : a \leq x\}.$$

The mapping Z is obviously order-preserving and we shall prove that it is also a bijection.

First of all we prove that if $p \in At(\Omega(I))$ and $p \leq Z(x)$, then $Z^{-1}(p) \leq x$. If $p \leq Z(x) = \bigvee\{Z(a) : a \leq x\}$, then there is a finite set $\{a_1, a_2 \ldots, a_k\}$ of atoms under x in L such that $p \leq Z(a_1) \vee \ldots Z(a_k)$. We may suppose that the last inequality is minimal. If it is a fundamental inequality (i.e. $k = 2$), then $Z^{-1} \leq a_1 \vee a_2$. Since both $a_1 \leq x$ and $a_2 \leq x$, we get also $Z^{-1} \leq x$.

If $k \geq 3$, then we may assume that there is $r \in At(\Omega(I))$ such that $r \leq Z(a_1) \vee Z(a_2)$ and $p \leq r \vee Z(a_3) \vee \ldots Z(a_k)$ are minimal inequalities. From the previous part of the proof we may conclude $Z^{-1}(r) \leq x$. Now a simple induction on k proves that $Z^{-1}(p) \leq x$.

Now we can prove that Z is injective. Take $x < y$ in L and find an atom $a \in At(L)$ such that $a \leq y$ and $a \nleq x$. Suppose $Z(x) = Z(y)$. Then $Z(a) \leq Z(x)$, hence by the previous part of the proof $a \leq x$, contrary to our assumption $A \nleq x$. Hence Z is injective.

Further, take a quasi-ordering $\alpha \in \Omega(I)$ and consider all $a \in At(L)$ such that $Z(a) \leq \alpha$. We shall prove that $\alpha = Z(x)$, where $x = \bigvee\{a \in At(L) : Z(a) \leq \alpha\}$. The inequality $\alpha \leq Z(x)$ is obvious. On the other hand, if $b \in At(L)$ is such that $b \leq x$, then there are $a_1, \ldots, a_m \in At(L)$ such that $Z(a_j) \leq \alpha$ and $b \leq a_1 \vee \ldots \vee a_m$. Similarly as in the first part of the proof we can prove that $Z(b) \leq \alpha$, hence $Z(x) \leq \alpha$. It completes the proof that Z is onto, therefore Z is an isomorphism. $\square$

Now we list several properties of the set of fundamental inequalities $a \leq b \vee c$ in 2-lattices and prove that these properties characterize quasi-ordering lattices. All these properties are easily verified in quasi-ordering lattices. We leave it as an exercise, although occasionally we give a hint why a certain property holds.

From now we shall assume that $|I| \geq 3$, i.e. $\Omega(I)$ has more than two atoms. Let L be a 2-lattice with more than two atoms. The atoms $a, c \in At(L)$ are called *collinear* in L if there is a fundamental inequality containing both of them.

Two collinear atoms a, c are *related* in L if they appear in different sides of a fundamental inequality (i.e. if there is $b \in At(L)$ such that $a \leq b \vee c$ is a fundamental inequality), and they are *chained* in L, if they appear on the right-hand side of a fundamental inequality (i.e. if $b \leq a \vee c$ for some $b \neq a, c$).

Two atoms in $\Omega(I)$ are collinear if the corresponding edges have exactly one common vertex, and they are related if the edges have either the same initial vertex or the same terminal vertex. The following two conditions are then obviously satisfied in $\Omega(I)$. The first one is the only global condition we shall be using.

(A0) For any two different atoms $a, b \in L$ there exists a sequence of atoms $a = a_0, a_1, \ldots, a_k = b$ such that every two subsequent atoms are related.
(A1) If the inequality $a \leq b \vee c$ belongs to Φ and $d \in At(L)$, then d is collinear either with exactly two elements of $\{a, b, c\}$ or with none of them.

To verify the following condition in $\Omega(I)$ it is sufficient to consider only edges on a 3-element subset of I.

(A2) If $a \leq b \vee c$ is in Φ, then there is a unique $c' \in At(L)$, $c' \neq c$, such that $b \leq a \vee c'$ is a fundamental inequality in L.

The element c' will be called *opposite* to c. The definition depends on a fundamental inequality containing c on the right-hand side. However, the next condition states that, in fact the element c' is independent of the inequality involved. Note moreover that it follows from (A2) that $c'' = c$ (there is a unique c'' such that $a \leq b \vee c''$).

(A3) If $a \leq b \vee c$, $b \leq a \vee c'$, and $e \leq f \vee c$, then $f \leq e \vee c'$.

The following condition is a refinement of (A1). To verify this condition in $\Omega(I)$ we have to consider a subset of I with four elements. The same is true about (A3) and (A5).

(A4) If $a \leq b \vee c$ and $a \leq d \vee e$ are different fundamental inequalities, then b is related to d or to c (but not to both, by (A1)).
(A5) If $a \leq b \vee c$, $a \leq d \vee e$ and $b \leq f \vee d$, then $e \leq f \vee c$.

To verify the last condition in $\Omega(I)$ requires to consider a 5-element subset of I.

(A6) If a, b, c, d are different atoms in L such that five of the possible six pairs are pairs of related atoms, then all the four atoms are mutually related.

Theorem 2. *A 2-lattice L is isomorphic to a quasi-order lattice $\Omega(I)$, $|I| \geq 3$, if and only if it satisfies the conditions (A0)–(A6).*

We start with a series of claims. We assume that L is a 2-lattice satisfying the conditions (A0)–(A6).

Claim 1. *(a) If $a \leq b \vee c$ and $a \leq b \vee d$, then $c = d$, and*
(b) If $a \leq b \vee c$ and $d \leq b \vee c$, then $a = d$.

Proof. (a) Let c', d'' be opposite to c, d, resp. Then $b \leq a \vee c'$ and $b \leq a \vee d'$. Hence both c', d' are opposite to c (and d as well). By (A2), $c' = d''$. Thus both c and d are opposite to $c' = d'$. It proves $c = d$, by another application of (A2).
(b) We get $b \leq a \vee c'$ and $b \leq d \vee c'$. By (a) we obtain $a = d$. □

Claim 2. *Two collinear atoms are either related or chained, but not both.*

Proof. Suppose that $a, b \in At(L)$ are both related and chained. Then there are $c, d \in At(L)$ such that $a \leq b \vee c$ and $d \leq a \vee b$. Using (A2) we obtain also $a \leq d \vee b'$. The two fundamental inequalities $a \leq b \vee c$ and $a \leq d \vee b'$ are different (since $b \neq b'$). So we get by (A4) that b is related either to b' or to d.

If b is related to b', then there exists f such that $b \leq b' \vee f$. By another application of (A2) we get $f \leq b \vee b$. But this is not a fundamental inequality. This contradiction proves that b is related to d. Hence there is f such that $b \leq f \vee d$. By (A5) we get also $c \leq f \vee b'$. By (A2) we get $f \leq c \vee b$. Since $a \leq b \vee c$, we get $a = f$ by Claim 1(b). Applying (A2) to $b \leq f \vee d$, we obtain $d \leq b \vee f'$. Since $d \leq a \vee b$, we find $a = f'$, by Claim 1(a). Hence $a = f = f'$, contrary to (A2). This contradiction proves that the pair a, b cannot be simultaneously related and chained. □

Claim 3. *If $a \leq b \vee c$ and $d \leq e \vee f$ are different fundamental inequalities, then $\{a, b, c\} \neq \{d, e, f\}$.*

Proof. If, for example, $a \leq b \vee c$ and $b \leq a \vee c$, then $c = c'$, which contradicts (A2). □

Claim 4. *If $a \leq b \vee c$, then $a' \leq b' \vee c'$.*

Proof. We apply three times. From $a \leq b \vee c$ we obtain subsequently $b \leq a \vee c'$, $c' \leq b \vee a'$, and $a' \leq b' \vee c'$. □

We define a *net* in a 2-lattice L as a set of mutually related atoms. A *maximal net* in L is a net which is not properly contained in any other net in L. Since we assume that L satisfies the condition (A0), and has at least two different atoms, every atom appears in a fundamental inequality. This proves that maximal nets in L have at least two elements. We do not have to use Zorn Lemma to prove the existence of maximal nets.

Claim 5. *Let a, b be two related atoms. Define*

$$N(a, b) = \{c \in At(L) : c \text{ is related to both } a \text{ and } b\} \cup \{a, b\}.$$

Then $N(a, b)$ is a maximal net in L.

Proof. If $c, d \in N(a, b)$, $c, d \neq a, b$, then c is related to both a and b, d is related to both a and b, and a, b are related. By (A6), c is related to d. It proves that every two elements of $N(a, b)$ are related. If N is a net containing both a and b, then $N \subseteq N(a, b)$ by the very definition of a net. This proves maximality of $N(a, b)$. $\square$

The following chlaim is a direct consequence of Claim 5.

Claim 6. *Every two maximal nets in L have at most one-element in common.* $\square$

Let M be a net in L. Define $M' = \{a' : a \in M\}$. Then M' is also a net, by Claim 4. Obviously, $M'' = M$, and M is maximal if and only if M' is maximal. Then net M' is called *opposite* to M.

Claim 7. *Let $M, N \neq M'$ be two different maximal nets in L. Then either $M \cap N = \emptyset$ or $M \cap N' = \emptyset$.*

Proof. Suppose that $a \in M \cap N$ and $b \in M \cap N'$. Then a, b are related, since both belong to M. The atoms a', b are also related, since they belong to N'. Hence there is some $c \in At(L)$ such that $b \leq a' \vee c$. By (A2), $c \leq b \vee a$. By (A2), $c \leq b \vee a$. It proves that a, b are not only related, but also chained, contrary to Claim 2. $\square$

Claim 8. *Let M be a maximal net in L and $d \in At(L)$. Then one of the following three possibilities holds:*

(i) $d \in M$,

(ii) $d \in M'$,

(iii) there are $a, b \in M$ such that $b \leq a \vee d$.

Proof. Suppose that $d \notin M$ and $d \notin M'$. By (A0), there exists for every $a \in M$ a sequence of atoms $a = a_0, a_1, \ldots, a_k = d$ such that every two subsequent atoms are related. Choose the atom $a \in M$ and the sequence such that k is the smallest possible. We shall prove that $k = 1$. For then there is $b \in At(L)$ such that $a \leq b \vee d$. If $b \in M$, then we are done. If $b \notin M$, then take $c \in M$, $c \neq a$. It exists since maximal nets have at least two elements. Then by (A4), c is related either to

b or to d. It cannot be related to d, since then it would be $d \in N(a, c) = N$. So c is related to b, hence $b \in N(a, c) = N$.

So it remains to prove $k = 1$. Suppose $k \geq 2$. We have $a_1 \notin M$ (otherwise we could replace a by a_1, thus contradicting our choice of a), and there is $b_1 \in At(L)$ satisfying $a \leq b_1 \vee a_1$. By the first paragraph of the proof (replacing d by a_1), we get $b_1 \in M$. By (A2) we get $a_1 \leq a \vee b'$. Since a_1 is related to a_2, there exists $c \in At(L)$ such that $a_1 \leq c \vee a_2$. By (A4), a_2 is related either to a or to b'. To be related to a would contradict our choice of the sequence $a_0, a_1, \ldots, a_k$, since then a_1 could be omitted. So a_2 is related to b'_1. Hence there exists $b_2 \in At(L)$ such that $a_2 \leq b'_1 \vee b_2$. By (A2), we get $b_2 \leq a_2 \vee b_1$. By (A5), we get $a \leq c \vee b_2$, hence a is related to b_2. So b_2 is related to both a and b_1, hence $b_2 \in M$. So a_2 is related to $b_2 \in M$. By replacing a by b_2 and omitting a_1, we get a final contradiction with our choice of a. This proves $k = 1$ and completes the proof of the claim. $\square$

Claim 9. *Let M and $N \neq M'$ be two different maximal nets in L. Then either $M \cap N \neq \emptyset$ or $M \cap N' \neq \emptyset$.*

Proof. Take an arbitrary $d \in N$. If $d \notin M$ and $d \notin M'$, then there are $a, b \in M$ such that $a \leq b \vee d$. By (A2) we get $d \leq a \vee b'$. Take an arbitrary $e \in N$. If $e = a$ or $e = b'$, then we are done. If $a \neq a, b'$, then we find $f \in At(L)$ such that $d \leq e \vee f$. By (A4), e is related either to a or to b'. If a is related to e, then a is related to both d and e, hence $a \in N(d, e) = N$. It proves $M \cap N \neq \emptyset$. If b' is related to e, then we get similarly $b' \in M' \cap N$. $\square$

Claim 10. *Every atom $a \in At(L)$ is contained in exactly two maximal nets.*

Proof. There is a fundamental inequality $a \leq b \vee c$. Then $N(a, b)$ and $N(a, c)$ are two different maximal nets containing a (they are different since b is not related to c). Now take an atom $d \neq b, c$ such that a is related to d. It means there is $e \in At(L)$ such that $a \leq d \vee e$. By (A4), d is related either to b or to c. In the first case $N(a, d) = N(a, b)$, in the second one $N(a, d) = N(a, c)$. So every maximal net containing a is either $N(a, b)$ or $N(a, c)$. $\square$

Proof of Theorem 2. Define a set I as the set of all pairs $\{M, M'\}$ of apposite maximal nets in L. Suppose that $\{M, M'\}$ and $\{N, N'\}$ are two different elements of I. By Claims 7 and 9, either $M \cap N \neq \emptyset \neq M' \cap N'$ and $M \cap N' = \emptyset = M' \cap N$, or $M \cap N' \neq \emptyset \neq M' \cap N$ and $M \cap N = \emptyset = M' \cap N'$. We may assume that $M \cap N \neq \emptyset$ and denote the only element of $M \cap Y$ by a. We define a mapping $Z : At(L) \to At(\Omega(I))$ in the following way:

(i) We set $Z(a) = (\{M, M'\}, \{N, N'\})$, and $Z(a') = (\{N, N'\}, \{M, M'\})$.

(ii) If $b \in M$, then there is a unique maximal net $P \neq M$ such that $b \in M \cap P$, by Claim 10. Then we define $Z(b) = (\{M, M'\}, \{P, P'\})$ and $Z(b') = (\{P, P'\}, \{M, M'\})$.

(iii) If $d \notin M \cup M'$, then there are, by Claim 8, $b, c \in M$ such that $b \leq c \vee d$. If $c \in M \cap P$ and $b \in M \cap R$, then we define $Z(d) = (\{P, P'\}, \{R, R'\})$ and $Z(d') = (\{R, R'), \{P, P'\})$.

Note that the definition of $Z(d')$ is consistent with the inequality $c \leq b \vee d'$.

Since every atom of L is contained in exactly two maximal nets the mapping Z is well defined. Since every two maximal nets intersect in at most one point, by Claim 6, the mapping Z is injective. And it is also surjective, by Claim 7. Hence Z is a bijection between $At(L)$ and $At(\Omega(I))$.

Next we have to prove that if $b \leq c \vee d$, then $Z(b) \leq Z(c) \vee Z(d)$. Note that $Z(b) \leq Z(c) \vee Z(d)$ if and only if $Z(b') \leq Z(c') \vee Z(d')$ since every pair of opposite elements in L is mapped by Z to a pair of opposite edges in the complete loopless graph on I. Consider the maximal nets $P = N(b, d)$ and $R = N(c', d)$.

We distinguish two cases.

Case (a). One of the nets P, Q, R is either M or M'. Without loss of generality we may assume that one of P, Q, R is M, otherwise we could replace the inequality $b \leq c \vee d$ by $b' \leq c' \vee d'$ and use the remark at the beginning of the previous paragraph. If $N(b, c) = M$, then we have $b, c \in M$. Moreover, $b \in M \cap Q$ and $c \in M \cap R'$. By the definition of Z, $Z(b) = (\{M, M'\}, \{Q, Q'\})$, $Z(c) = (\{M, M'\}, \{R, R'\})$, and $Z(d) = (\{R, R'\}, \{Q, Q'\})$. Hence $Z(b) \leq Z(c) \vee Z(d)$. The case $N(b, d) = M$ is analogous, we can replace c by d. And if $M = N(c', d)$, then we consider the inequality $d \leq b \vee c'$. By what we have just proved, $Z(d) \leq Z(b) \vee Z(c')$ and since $\Omega(I)$ satisfies (A2), we get $Z(b) \leq Z(c) \vee Z(d)$.

Case (b). None of the nets P, Q, R is either M or M'. By Claim 8 there are $a_1, a_2 \in M$ such that $a_1 \leq a_2 \vee b$. Hence $b \leq a_1 \vee a_2'$. We have $b \leq c \vee d$, so by (A4), a_1 is related to c or to d. Again without loss of generality we may assume that a_1 is related to c. Hence there is $a_3 \in At(L)$ such that $a_1 \leq a_3 \vee c$. By (A5), $d \leq a_3 \vee a_2'$, hence $a_3 \leq a_2 \vee d$. It follows that a_1 is related to both b and c, hence $a_1 \in P = N(b, c)$. Similarly, a_3 is related to both d and c', which means $a_3 \in R = N(c', d)$. And a_2 is related to b' and d', which means $a_2 \in Q' = N(b', d')$. By part (iii) of the definition of Z we get $Z(b) = (\{Q, Q'\}, \{P, P'\})$, $Z(c) = (\{R, R'\}, \{P, P'\})$, and $Z(d) = (\{Q, Q'\}, \{R, R'\})$. This proves $Z(b) \leq Z(c) \vee Z(d)$.

Finally, we have to prove that $b \leq c \vee d$ if $Z(b) \leq Z(c) \vee Z(d)$. Let $Z(b) = (\{Q, Q'\}, \{P, P'\})$, $Z(c) = (\{R, R'\}, \{P, P'\})$, and $Z(d) = (\{Q, Q'\}, \{R, R'\})$. We may assume $b \in P \cap Q$, $c \in P \cap R$, and $d \in Q \cap R'$. The maximal nets P, Q, R must be different, so it cannot be $d \in Q \cap R$, since it would means $b, c, d \in P = Q = R$. We again distinguish two cases.

Case (c). One of the pairs $\{P, P'\}, \{Q, Q'\}, \{R, R'\}$ is equal to $\{M, M'\}$. Suppose, for example, $M = Q$. By the third part of the definition of $Z, b \leq c \vee d$. If $M = P'$ (it cannot be $M = P$, since both b and c are edges with the terminal vertex $\{P, P'\} = \{M, M'\}$), then $b', c' \in M$, and by part (iii) of the definition of

Z, $c' \leq b' \vee d$. Applying (A2) two times, we get $d \leq c' \vee b$ and $b \leq c \vee d$. The last possibility is $M = R'$, hence $c', d \in M$. In this case, part (iii) of the definition of Z gives $c' \leq b' \vee d$, hence $d \leq b \vee c'$, and $b \leq c \vee d$.

Case (d). Assume $M \neq P, P', Q, Q', R, R'$. Consider the atoms $a_1, a_2, a_3 \in At(L)$ such that $Z(a_1) = (\{M, M'\}, \{P, P'\})$, $Z(a_2) = (\{M, M'\}, \{Q, Q'\})$ and $Z(a_3) = (\{M, M'\}, \{R, R'\})$. Then $Z(a_1) \leq Z(a_2) \vee Z(b)$, $Z(a_1) \leq Z(a_3) \vee Z(c)$, $Z(a_3) \leq Z(a_2) \vee Z(d)$. By (c), $a_1 \leq a_2 \vee b$, $a_1 \leq a_3 \vee c$, and $a_3 \leq a_2 \vee d$. Applying (A5), we get $b \leq c \vee d$. $\square$

The group of automorphisms of $\Omega(I)$

There are two obvious types of automorphisms of $\Omega(I)$. If φ is a permutation on I, then it induces a permutation on the set of edges of the complete oriented loopless graph on I. If (i, j, k) is a triple corresponding to the fundamental inequality $(i, k) \leq (i, j) \vee (j, k)$, then the triple $(\varphi(i), \varphi(j), \varphi(k))$ corresponds to the fundamental inequality $(\varphi(i), \varphi(k)) \leq (\varphi(i), \varphi(j)) \leq (\varphi(j), \varphi(k))$. Hence by Lemma 1, the permutation on the edges can be extended to an automorphism $\bar{\varphi}$ of $\Omega(I)$. Obviously, $\overline{\varphi\psi} = \bar{\varphi}\bar{\psi}$, hence the group $Aut(\Omega(I))$ contains a subgroup isomorphic to the symmetric group $Sym(I)$ of all permutations of I.

If we map any atom of $\Omega(I)$ to its opposite, then we get another automorphism of $\Omega(I)$, by Claim 4 and Lemma 1. This automorphism will be denoted by τ. The automorphism τ is not of the form $\bar{\varphi}$ for any $\varphi \in Sym(I)$, provided $|I| > 2$. On the other hand, τ commutes with each $\bar{\varphi}$, hence $Aut(\Omega(I))$ contains a subgroup isomorphic to $Sym(I) \times Z_2$, if $|I| > 2$. In fact, there are no other automorphism of $\Omega(I)$.

Theorem 3. *The automorphism group of $\Omega(I)$ is isomorphic to the direct product $Sym(I) \times Z_2$, if $|I| > 2$.*

Proof. We may identify the set I with the set of pairs of opposite maximal nets. Since every automorphism of $\Omega(I)$ preserves the relation of being related, it has to map maximal nets to maximal nets, and pairs of opposite maximal nets to pairs of opposite maximal nets. Hence every automorphism μ of $\Omega(I)$ induced a permutation φ on the set I. Now consider the automorphism $\bar{\varphi}^{-1}\mu$. This automorphism has to fix every pair of opposite maximal nets. So for every maximal net M in $\Omega(I)$, either $\bar{\varphi}^{-1}\mu(M) = M$ or $\bar{\varphi}^{-1}\mu(M) = M'$.

If there is a maximal net M such that $\bar{\varphi}^{-1}\mu(M) = M$, then $\bar{\varphi}^{-1}\mu(M') = M'$. If N is another maximal net which intersects M, then $\bar{\varphi}^{-1}\mu(M \cap N) \subseteq M$. Moreover, $M \cap N' = \emptyset$, by Claim 7, hence $\bar{\varphi}^{-1}\mu(M \cap N) = M \cap N$. However, $M \cap N$ contains exactly one element, by Claim 6. Since every atom in $\Omega(I)$ is contained in exactly two maximal nets, we see that $\bar{\varphi}^{-1}\mu$ fixes all elements of M. Similarly, we prove that $\bar{\varphi}^{-1}\mu$ fixes all elements of M'. For every atom $d \notin M \cup M'$ there are $a, b \in M$ such that $b \leq a \vee d$, by Claim 8. Since $\bar{\varphi}^{-1}\mu$ fixes both a and b, it has also to fix d by Claim 1(a). It proves that $\bar{\varphi}^{-1}\mu$ fixes all the atoms of $\Omega(I)$, hence it is the identity automorphism. Therefore $\mu = \bar{\varphi}$ in this case.

Now suppose that there is a maximal net M which is mapped by $\bar{\varphi}^{-1}\mu$ to M'. If N is another maximal net which intersects M in an atom a, then $M' \cap N' = \{a'\}$ and $M \cap N' = M' \cap N = \emptyset$ by Claim 7. Hence $\bar{\varphi}^{-1}\mu$ maps N to N' and a to a'. Since every $b \in M$ is contained in two maximal nets, we get that $\bar{\varphi}^{-1}\mu$ maps every $b \in M$ to b', and similarly, every $b' \in M'$ to b. Now consider $b \notin M \cup M'$ and $a, b \in M$ such that $a \leq b \vee d$. By Claim 4, $a' \leq b' \vee d'$. Applying Claim 1(a) again, we find that $\bar{\varphi}^{-1}\mu$ maps d to d'. Hence $\bar{\varphi}^{-1}\mu = \tau$, i.e. $\mu = \bar{\varphi}\tau$. This proves that there are now other automorphisms of $\Omega(I)$ except those contained in the subgroup isomorphic to $Sym(I) \times Z_2$. $\square$

Notes

The results presented in this paper are the first two steps towards understanding the structure of intervals in quasi-ordering lattices, at least in the finite case. The intervals are no longer atomistic, as we can find e.g. N_5 or the djal of D_1 as intervals in $\Omega(I)$ if $|I| > 2$. On the other hand, it can be shown that if a is a join-irreducible element in an interval $[\alpha, \beta]$ in $\Omega(I)$, then there is a sequence of join-irreducibles $a = a_0 > a_2 > \dots > a_k > a_{k+1} = \alpha$ such that every a_i convers a_{i+1} for every $i = 0, 1, \dots, k$. We can call an inequality $a > b$ between two join-irreducible elements in $[\alpha, \beta]$ *minimal*, if a covers b. Then every inequality between two join-irreducible elements is a consequence of minimal inequalities. Similarly, an inequality $a \leq b \vee c$, a, b, c join-irreducibles, can be called *minimal* if $a \leq b_1 \vee c_1$ whenever $b_1 \leq b$, $c_1 \leq c$, b_1, c_1 joint-irreducibles, and the sharp inequality holds in at least one case. It can be shown that every inequality $a \leq b_1 \vee \dots \vee b_k$ involving joint-irreducible elements in $[\alpha, \beta]$ is a consequence of minimal inequalities of these two types.

The axioms (A1)$-$(A6) are "completeness" axioms, stating always that some atoms exist or inequalities hold. Some weaker forms of these axioms hold in every interval, and the extent to which (A1)$-$(A6) fail is a measure how different β is from 1_I. On the other hand, the existence of comparable joint-irreducible elements in $[\alpha, \beta]$ is a direct consequence of the fact that $\alpha \neq 0_I$. Hence we find a way to obtain information about the quasi-orderings α, β from the shape of the interval $[\alpha, \beta]$. The structure of intervals in quasi-ordering lattices will be the topic of another paper.

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