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NUMERICAL RANGE OF OPERATORS ACTING ON BANACH SPACES

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Abstract. The aim of the paper is to propose a definition of numerical range of an operator on reflexive Banach spaces. Under this definition the numerical range will possess the basic properties of a canonical numerical range. We will determine necessary and sufficient conditions under which the numerical range of a composition operator on a weighted Hardy space is closed. We will also give some necessary conditions to show that when the closure of the numerical range of a composition operator on a small weighted Hardy space has zero.

Keywords: numerical range, weighted Hardy space, compact operator, composition operator

MSC 2010: 47B37, 47A12

1. Introduction

The concept of the numerical range of an operator on a Hilbert space was presented by O. Toeplitz in 1918 ([13]). The numerical range of a bounded operator T on a Hilbert space H is the set of complex numbers

$$V(T)=\{\langle Th,h\rangle :\ h\in H,\ \|h\|=1\}.$$

The Toeplitz-Hausdorff Theorem ([6], [13]) establishes the convexity of the numerical range for any operator on a Hilbert space. Some properties and further developments of the numerical range of a bounded linear operator on a Hilbert space can be found in [4], [5]. The concept of the numerical range on a Banach space X, extended by Baure and Lumer in [1], [8], is not necessarily convex, see [2, Example 21.6].

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Let X be a Banach space. Every $x \in X$ defines an element $\hat{x} \in X^{**}$ (second dual of X) by $\hat{x}(x^*) = x^*(x)$, where $x^* \in X^*$ and $\|x\| = \|\hat{x}\|$. Recall that X is a reflexive Banach space if $X^{**} = \{\hat{x}\colon\ x \in X\}$.

In this paper, by giving a definition of numerical range for operators on Banach spaces, we want to investigate some properties that are consistent for the Hilbert case.

A holomorphic function φ that takes the open unit disc $\mathbb{D}$ into itself induces a linear composition operator C_φ on the space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ as follows:

$$C_\varphi(f) =: fo\varphi \quad (f \in H(\mathbb{D})).$$

A lot of work has been done in studying composition operators on Hardy spaces and weighted Hardy spaces (see for example [3], [9]–[12], [14]).

We recall that the classical weighted Hardy space $H^2(\beta)$, with a weight sequence $\beta(n)$, is the function Hilbert space with a complete orthogonal set of nonzero monomials $1, z, z^2, \ldots$ in $H^2(\beta)$. Writing $\beta(n) = \|z^n\|$, the orthogonality implies that the norm of the formal power series $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n)z^n$ in $H^2(\beta)$ is given by

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2.$$

Also, recall that the inner product is given by

$$\left\langle \sum_{n=0}^{\infty} \hat{f}(n)z^n, \sum_{n=0}^{\infty} \hat{g}(n)z^n \right\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2.$$

We can extend the definitions to Banach spaces $H^p(\beta)$ as follows. Let $\{\beta(n)\}$ be a sequence of nonzero complex numbers with $\beta(0) = 1$ and $1 \leqslant p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p < \infty.$$

The notation $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n)z^n$ will be used wether or not the series converges for any value of $z \in \mathbb{D}$. These are called formal power series. Let $H^p(\beta)$ denote the space of such formal power series. It is called a weighted Hardy space. For $1 < p < \infty$, $H^p(\beta) \cong L^p(\mu)$, where μ is the σ-finite measure defined on the set of positive integers by $\mu(K) = \sum\limits_{n \in K} \beta(n)^p, K \subseteq \mathbb{N} \cup \{0\}$. Let $f_k(z) = z^k$, then $\{f_k\}_k$ forms a basis for the reflexive Banach space $H^p(\beta)$ ([12], [15]). Note that the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$, where $1 < p < \infty$, $1/p+1/q = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_{n=0}^{\infty}$. If $\{\beta(n)\}_n$ is a sequence of positive numbers with $\beta(0) = 1$ and $\underline{\lim}\, \beta(n)^{1/n} = 1$, then $H^p(\beta)$ is a Banach space of analytic functions on $\mathbb{D}$. The Hardy, Bergman and Dirichlet spaces can be viewed as such if we set respectively $\beta(n) \equiv 1$, $\beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{-1/2}$.

Let λ be a complex number. The functional of evaluation at λ, e_λ, is defined by $e_\lambda(p) = p(\lambda)$ for all polynomials p. Also, λ is said to be a bounded point evaluation on $H^p(\beta)$ if the function e_λ extends to be a bounded linear functional on $H^p(\beta)$. In this case we have $e_\lambda(f) = f(\lambda)$, $f \in H^p(\beta)$.

Theorem 1.1 ([15]). *A complex number λ is a bounded point evaluation on $H^p(\beta)$ if and only if $\{\lambda^n/\beta(n)\}_n \in l^q$, where $1/p + 1/q = 1$.*

The functional of evaluation of the j-th derivative at λ is denoted by $e_\lambda^{(j)}$. Also, we note that $e_\lambda(z) = \sum\limits_{n=0}^{\infty} \bar{\lambda}^n \beta(n)^{-p} z^n$ and $e_\lambda^{(j)} = (\mathrm{d}^j/\mathrm{d}\bar{\lambda}^j)(e_\lambda)$. Thus

$$e_\lambda^{(j)}(z) = \sum_{n \geqslant j} n(n-1)\dots(n-j+1)\frac{(\bar{\lambda})^{n-j}}{\beta(n)^p} z^n.$$

For some sources on these topics one can refer to [3], [7], [9], [15]–[26].

2. Preliminaries

In this section we give a definition of numerical range for Banach spaces which extends the earlier definition to the case of Hilbert spaces. Then, using this definition we will investigate some well-known properties that are consistent for Hilbert spaces.

Definition 2.1. Let X be a reflexive Banach space and $T \in B(X)$. The numerical range of T is defined by

$$W(T) = \mathrm{co}(V(T)),$$

where $\mathrm{co}(V(T))$ is the convex hull of $V(T)$ and

$$V(T) = \{x^*(T(x))\colon\ x \in X,\ x^* \in X^*;\ \ \|x\| = \|x^*\| = x^*(x) = 1\}.$$

Obviously, W(T) is convex and contains V(T). We use the usual notation $\sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$ respectively for the spectrum, eigenspace, and approximate point spectrum of T. Note that $\sigma_p(T) \subseteq \sigma_{ap}(T)$.

Proposition 2.2. *Let X be a reflexive Banach space and $T \in B(X)$. Then*

(i) $W(T) = W(T^*)$;

(ii) $W(T)$ *is a convex subset of the complex plane that lies in the closed disc with radius $\|T\|$ centred at origin;*

(iii) $\sigma_p(T) \subseteq W(T)$;

(iv) $\sigma(T) \subseteq \overline{W(T)}$.

Proof. (i): Note that $W(T^*) = \mathrm{co}\{\hat{x}(T^*x^*);\ x^* \in X^*,\ \hat{x} \in X^{**},\ \|\hat{x}\| = \|x^*\| = \hat{x}(x^*) = 1\}$. Since X is a reflexive Banach space with $\hat{x}(x^*) = x^*(x)$, $\|x\| = \|\hat{x}\|$ for all $x \in X$, we have $W(T^*) = W(T)$.

(ii): Obviously $W(T)$ is convex and $|x^*T(x)| \leqslant \|T\|$ for all unit vectors $x \in X$ and $x^* \in X^*$. Thus (ii) holds.

(iii): Let $\lambda \in \sigma_p(T)$. Then there exists a nonzero vector $x \in X$ such that $\|x\| = 1$ and $Tx = \lambda x$. By the Hahn-Banach Theorem, there exists $x^* \in X^*$ such that $\|x\| = \|x^*\| = x^*(x) = 1$. Thus, $x^*(Tx) = x^*(\lambda x) = \lambda x^*(x) = \lambda$ and so $\lambda \in W(T)$.

(iv): It is well known that

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*).$$

By (ii) and (iii) we have $\sigma_p(T^*) \subseteq W(T^*) = W(T)$. Let $\lambda \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}$ in $\mathrm{Ball}(X)$ such that $(T - \lambda)x_n \to 0$ as $n \to \infty$. Now by the Hahn Banach Theorem, there exists $x_n^* \in X^*$ such that

$$x_n^*(x_n) = \|x_n\| = 1;\ \ \|x_n^*\| = 1,$$

for all n. Thus, $x_n^*(Tx_n) \to \lambda$ as $n \to \infty$ and $\lambda \in \overline{W(T)}$. So $\sigma_{ap}(T) \subseteq \overline{W(T)}$. This proves that $\sigma(T) \subseteq \overline{W(T)}$. □

3. Main results

Note that in [26] we used a definition of the numerical range of an operator acting on $H^p(\beta)$ that is not necessarily convex. Here by the general Definition 2.1. for Banach spaces, the numerical range is convex and we can extend some earlier results. In this section, we prove some results about the numerical range of a compact operator acting on the Banach space $H^p(\beta)$.

Theorem 3.1. *If T is a compact operator on a reflexive Banach space X, then*

$$\overline{V(T)} \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in V(T)\}.$$

Proof. Let $\beta \in \overline{V(T)}$. Then there exists a sequence $\{x_n^*(Tx_n)\}_n$ in $V(T)$ such that $x_n^* T x_n \to \beta$ as $n \to \infty$, where $\|x_n^*\| = \|x_n\| = x_n^*(x_n) = 1$. Note that $\operatorname{Ball} X$ is weakly compact, thus there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $x \in \operatorname{Ball} X$ such that $x_{n_k} \to x$ weakly. Also

$$x^*{}_{n_k}(Tx_{n_k}) \to \beta,\ \{x^*{}_{n_k}\}_k \subseteq \operatorname{Ball} X^*. \tag{1}$$

But $\operatorname{Ball} X^*$ is ωk^* compact and $\{x^*{}_{n_k}\}_k \subseteq \operatorname{Ball} X^*$, thus there exists a subsequence $\{x^*{}_{n_{k_m}}\}_m$ of $\{x^*{}_{n_k}\}_k$ such that $x^*{}_{n_{k_m}} \to x^*$ weak star for some $x^* \in X^*$ with $\|x^*\| \leqslant 1$. Note that (1) holds also for the subsequence $\{x^*{}_{n_{k_m}}\}_m$. For simplicity, we suppose that $x_n \overset{\omega k}{\to} x$, $x_n^* \overset{\omega k^*}{\to} x^*$ and $x_n^*(Tx_n) \to \beta$ as $n \to \infty$. Since T is compact, it is completely continuous and so $Tx_n \to Tx$. We have $(x_n^* - x^*)Tx \to 0$, since $x_n^* - x^* \overset{\omega k^*}{\to} 0$. Also note that

$$|x_n^*(Tx_n) - x^*(Tx)| \leqslant \|T(x_n - x)\|\ \|x_n^*\| + |(x_n^* - x^*)Tx|. \tag{2}$$

Hence, $x_n^*(Tx_n) \to x^*(Tx)$ as $n \to \infty$ and so indeed $\beta = x^* T x$.

If $\beta \neq 0$, then x and x^* are nonzero and we get

$$\beta = \|x\|\|x^*\|\Big(\frac{x^*}{\|x^*\|}T\Big(\frac{x}{\|x\|}\Big)\Big).$$

But by the Hahn-Banach Theorem, there exists $y^* \in X^*$ such that $y^*(x/\|x\|) = 1$ and $\|y^*\| = 1$. Put $t = x^*(x)/\|x\|\|x^*\|$. Then $\|t\| \leqslant 1$ and we have

$$T^*\Big(ty^*\Big(\frac{x}{\|x\|}\Big)\Big) = T^*\Big(\frac{x^*(x)}{\|x\|\ \|x^*\|}\Big).$$

Thus,

$$ty^*\Big(T\Big(\frac{x}{\|x\|}\Big)\Big) = \frac{x^*}{\|x^*\|}\Big(T\Big(\frac{x}{\|x\|}\Big)\Big).$$

Hence, $\beta = \|x^*\|\|x\| t y^*(Tx/\|x\|)$ where $y^*(x/\|x\|) = \|y^*\| = 1$.

Thus, $y^*(T(x/\|x\|)) \in V(T)$. So $\beta \in cV(T)$ for some $0 \leqslant c \leqslant 1$. Hence,

$$\overline{V(T)} \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in V(T)\}.$$

□

Corollary 3.2. *If T is a compact operator on a reflexive Banach space, then*

$$\overline{W(T)} \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in W(T)\}.$$

Proof. By Theorem 3.1, we get

$$\overline{V(T)} \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in V(T)\} \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in W(T)\}.$$

Note that $W(T)$ is convex and so $\{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in W(T)\}$ is convex, thus

$$\operatorname{co}(\overline{V(T)}) \subseteq \{c\alpha\colon\ 0 \leqslant c \leqslant 1,\ \alpha \in W(T)\}.$$

On the other hand, $\operatorname{co}(\overline{V(T)}) = \overline{\operatorname{co}(V(T))} = \overline{W(T)}$. Hence, $\overline{W(T)} \subseteq \{c\alpha,\ 0 \leqslant c \leqslant 1,\ \alpha \in W(T)\}$. □

Theorem 3.3. *Let $1/p + 1/q = 1$ and $\sum\limits_{n\geqslant 0} n^{qj}/\beta(n)^q = \infty$ for some j. If C_φ is bounded on $H^p(\beta)$, then $0 \in \overline{W(C_\varphi)}$.*

Proof. Let $\{z_k\}_{k\geqslant 1}$ be any sequence in $\mathbb{D}$ with $z_k \to z_0$ for some $z_0 \in \partial\mathbb{D}$. Also, let j be the least non-negative integer such that $\sum\limits_{n=1}^{\infty} n^{qj}/\beta(n)^q = \infty$. Set $e_k = e_{z_k}^{(j)}/\|e_{z_k}^{(j)}\|$. Then $\|e_k\| = 1$ and for $j = 0$ we have

$$\lim_k \|e_{z_k}\|^q = \lim_k \sum_{n\geqslant 0} \frac{|z_k|^{nq}}{\beta(n)^q} = \sum_{n\geqslant 0} \frac{1}{\beta(n)^q} = +\infty.$$

So if s is a polynomial, then $\lim\limits_k e_k(s) = \lim\limits_k s(z_k)/\|e_{z_k}\| = 0$. But polynomials are dense in $H^p(\beta)$. Thus, $e_k \to 0$ weakly as $k \to \infty$. If $j > 0$, then since $|z_k| \to 1$ and $\sum\limits_{n\geqslant 0} n^{qj}/\beta(n)^q = \infty$, we have

$$\begin{aligned}
\lim_k \|e_{z_k}^{(j)}\|^q &= \lim_k \left\| \sum_{n\geqslant 0} n(n-1)\ldots(n-j+1)\frac{(\bar{z_k})^{n-j}}{\beta(n)^p} z^n \right\|^q \\
&= \lim_k \sum [n(n-1)\ldots(n-j+1)]^q \frac{|z_k|^{(n-j)q}}{\beta(n)^q} \\
&= \infty.
\end{aligned}$$

Since polynomials are dense in $H^p(\beta)$, by using the same argument as in the previous case, we can see that $e_k \to 0$ weakly as $k \to \infty$. Thus $C_\varphi^*(e_k) \to 0$ weakly as $k \to \infty$. Now by the Hahn-Banach Theorem, there exists $F \in H^p(\beta)$ such that $e_k(F) = \|e_k\| = 1$ and $\|F\| = 1$. Thus, $F(C_\varphi^* e_k) \in V(C_\varphi^*) \subseteq W(C_\varphi^*) = W(C_\varphi)$ and $F(C_\varphi^* e_k) \to 0$ weakly as $k \to \infty$. Thus indeed $0 \in \overline{W(C_\varphi)}$ and so the proof is complete. □

Corollary 3.4. *Under the conditions of Theorem* 3.3, *if* C_φ *is compact, then* $0 \in W(C_\varphi)$ *if and only if* $W(C_\varphi)$ *is closed.*

Proof. If $W(c_\varphi)$ is closed, then by Theorem 3.3, $0 \in W(c_\varphi)$. Let $\alpha \in \overline{W(C_\varphi)}$. Thus, by Corollary 3.2, there exists $0 \leqslant c \leqslant 1$ such that $\alpha \in cW(C_\varphi)$. If $\alpha \neq 0$ and $c \neq 0$, then $\alpha/c \in W(C_\varphi)$. Since $W(C_\varphi)$ is convex and $0 \in W(C_\varphi)$, we have $\alpha \in W(C_\varphi)$. Therefore, $\overline{W(C_\varphi)} \subseteq W(C_\varphi) \subseteq \overline{W(C_\varphi)}$. This implies that $W(C_\varphi)$ is closed □

Remark 3.5. We have denoted the numerical range of an operator T acting on a Banach space by $W(T)$ and defined it by $W(T) = \mathrm{co}(V(T))$, and we noted that the set $V(T)$ is not necessarily convex ([2, Example 21.6]). Our idea of this definition is to obtain the same properties as those holding under the earlier definition of numerical range of an operator acting on a Hilbert space, such as the properties stated in Proposition 2.2. In particular, the property of convexity of $W(T)$ is useful and it is used in the proof of Theorem 3.3. In the proof of Theorem 3.3, we should show that $W(C_\varphi)$ is normed closed, but we could only prove that $W(C_\varphi)$ is weakly closed. Then its convexity and boundedness imply that it should be also normed closed.

Recall that $H^p(\beta)$ is *a small weighted Hardy space*, if $H^p(\beta)$ is contained in the disc algebra $A(\mathbb{D})$. Let $1/p + 1/q = 1$ and $\sum\limits_{n\geqslant 0} 1/\beta(n)^q < \infty$. Then the weighted Hardy space $H^p(\beta)$ is small, since if $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n)z^n$ is in $H^p(\beta)$, then by the Hölder inequality we have

$$\sum_{n=0}^{\infty} |\hat{f}(n)| \leqslant \Big(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p\Big)^{1/p} \Big(\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q}\Big)^{1/q} < \infty.$$

Some additional conditions which have been used to define a small weighted Hardy space $H^p(\beta)$ include:

1) $\sum\limits_{n=0}^{\infty} n^q/\beta(n)^q < \infty$, where $1/p + 1/q = 1$.

2) $\sum\limits_{n=0}^{\infty} \beta[\alpha n]/\beta(n)^q < \infty$ for some $0 < \alpha < 1$ and $\{\beta(n)\}_n$ is increasing.

Condition (1) shows that the space $H^p(\beta)$ has derivative which extends continuously to $\overline{\mathbb{D}}$ ([3]).

In condition (2), since $\{\beta(n)\}$ is an increasing sequence and $\beta(0) = 1$, we have $\sum\limits_{n=0}^{\infty} 1/\beta(n)^q \leqslant \sum\limits_{n=0}^{\infty} \beta[\alpha n]/\beta(n)^q < \infty$.

Theorem 3.6. *Suppose that $\{\beta(n)\}_n$ is an increasing sequence such that $\sum\limits_{n=0}^{\infty} \beta[\alpha n]/\beta(n)^q$ converges for some $0 < \alpha < 1$. If $\varphi_r(z) = (1-r) + rz$, then $0 \in \overline{W(C_{\varphi_r})}$ for all $0 < r < \alpha$.*

P r o o f. It is obvious that φ_r maps the open unit disc $\mathbb{D}$ into itself with $\|\varphi_r\|_\infty = 1$. Define $T_N\colon H^p(\beta) \longrightarrow H^p(\beta)$ by $T_N f = \sum\limits_{n=1}^{N} \hat{f}(n)\varphi_r^n$ for all $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$ in $H^p(\beta)$. Notice that φ_r^n, being a polynomial, belongs to $H^p(\beta)$. Thus

$$\|T_N - C_{\varphi_r}\| = \sup_{\|f\|_p=1} \|T_N(f) - C_{\varphi_r} f\|_p = \sup_{\|f\|_p=1} \left\| \sum_N^{\infty} \hat{f}(n)\varphi_r^n \right\|_p ,$$

hence $\lim\limits_N \|T_N - C_{\varphi_r}\| = 0$. Therefore C_{φ_r} is compact. On the other hand, a compact operator on an infinite dimensional space $H^p(\beta)$ is not invertible . Hence 0 is in the spectrum of C_{φ_r} and therefore $0 \in \overline{W(C_{\varphi_r})}$. □

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