Applications of Mathematics

Yongfeng Wu; Dingcheng Wang

Convergence properties for arrays of rowwise pairwise negatively quadrant dependent random variables

Applications of Mathematics, Vol. 57 (2012), No. 5, 463–476

Persistent URL: http://dml.cz/dmlcz/142911

CONVERGENCE PROPERTIES FOR ARRAYS OF ROWWISE PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES*

Yongfeng Wu, Tongling, Dingcheng Wang, Nanjing

(Received October 1, 2010)

Abstract. In this paper the authors study the convergence properties for arrays of rowwise pairwise negatively quadrant dependent random variables. The results extend and improve the corresponding theorems of *T. C. Hu, R. L. Taylor*: On the strong law for arrays and for the bootstrap mean and variance, Int. J. Math. Math. Sci *20* (1997), 375–382.

Keywords: complete convergence, complete moment convergence, L^q convergence, pairwise NQD random variables

MSC 2010: 60F15, 60F25

1. Introduction and main results

We first provide a series of definitions of dependence structures.

Definition 1.1. A finite family of random variables $\{X_k,\ 1 \leqslant k \leqslant n\}$ is said to be *negatively associated* (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on R^A and g on R^B,

$$\operatorname{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \leqslant 0,$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [8].

* This work was partially supported by Humanities and Social Sciences Foundation for The Youth Scholars of Ministry of Education of China (No. 12YJCZH217) and the Anhui Province College Excellent Young Talents Fund Project of China (No. S 2011SQRL143).

Definition 1.2. The random variables $X_1, \ldots, X_k$ are said to be *negatively upper orthant dependent* (NUOD) if for all real $x_1, \ldots, x_k$,

$$P(X_i > x_i,\ i - 1, \ldots, k) \leqslant \prod_{i=1}^{k} P(X_i > x_i),$$

and *negatively lower orthant dependent* (NLOD) if

$$P(X_i \leqslant x_i,\ i = 1, \ldots, k) \leqslant \prod_{i=1}^{k} P(X_i \leqslant x_i).$$

Random variables $X_1, \ldots, X_k$ are said to be *negatively orthant dependent* (NOD) if they are both NUOD and NLOD. This concept was introduced by Ebrahimi and Ghosh [4].

Definition 1.3. Two random variables X and Y are said to be *negatively quadrant dependent* (NQD) if

$$P(X \leqslant x, Y \leqslant y) \leqslant P(X \leqslant x)P(Y \leqslant y) \quad \text{for all } x \text{ and } y.$$

A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *pairwise NQD* if every pair of random variables in the sequence are NQD. This concept was introduced by Lehmann [9].

For a triangular array of rowwise independent random variables $\{X_{nk},\ 1 \leqslant k \leqslant n, n \geqslant 1\}$, let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$ and $\{\Psi(t)\}$ a positive, even function such that

$$\frac{\Psi(|t|)}{|t|^p} \uparrow \quad \text{and} \quad \frac{\Psi(|t|)}{|t|^{p+1}} \downarrow \quad \text{as } |t| \uparrow, \tag{1.1}$$

for some nonnegative integer p. We introduce conditions

$$EX_{nk} = 0, \quad 1 \leqslant k \leqslant n, \quad n \geqslant 1, \tag{1.2}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi(X_{nk})}{\Psi(a_n)} < \infty, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \Bigl(\sum_{k=1}^{n} E\Bigl(\frac{X_{nk}}{a_n}\Bigr)^2 \Bigr)^{2k} < \infty, \tag{1.4}$$

where k is a positive integer.

Hu and Taylor [7] proved the following theorems:

Theorem A. *Let* $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of rowwise independent random variables and let* $\{\Psi(t)\}$ *satisfy* (1.1) *for some integer* $p > 2$*. Then* (1.2), (1.3), *and* (1.4) *imply*

$$\frac{1}{a_n}\sum_{k=1}^{n} X_{nk} \to 0 \quad \text{a.s.} \tag{1.5}$$

Theorem B. *Let* $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of rowwise independent random variables and let* $\{\Psi(t)\}$ *satisfy* (1.1) *for* $p = 1$*. Then conditions* (1.2), (1.3) *imply* (1.5).

A sequence of random variables $\{U_n,\ n \geqslant 1\}$ is said to *converge completely to a constant* a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In this case we write $U_n \to a$ completely. This notion was used for the first time by Hsu and Robbins [6].

Let $\{Z_n,\ n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0, \tag{1.6}$$

then (1.6) was called the *complete moment convergence* by Chow [3].

Gan and Chen [5] extended and improved Theorem A and Theorem B to the case of NA random variables. They studied the complete convergence and convergence in probability under some general and weaker conditions. Wu and Zhu [12] extended and improved Theorem A and Theorem B to the case of NOD random variables. They studied the complete convergence, the complete moment convergence, and the L^1 convergence under the same conditions as Gan and Chen [5]. However, according to our knowledge, no one has discussed the subject for arrays of rowwise pairwise NQD random variables. The goal of this paper is to study the complete convergence, the complete moment convergence, and the L^q convergence for rowwise pairwise NQD random arrays.

Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ be a sequence of nonnegative even functions satisfying

$$\frac{\Psi_n(|t|)}{|t|^q} \uparrow \quad \text{and} \quad \frac{\Psi_n(|t|)}{|t|^p} \downarrow \quad \text{as } |t| \uparrow \tag{1.7}$$

for some $1 \leqslant q < p \leqslant 2$. We introduce conditions

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty}\log^2 n\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty. \tag{1.9}$$

Now we will present the main results of the paper. The proofs will be detailed in the next section.

Theorem 1.1. *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables, and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ be a sequence of nonnegative even functions satisfying* (1.7) *for $q = 1$. Then conditions* (1.2) *and* (1.8) *imply*

$$\frac{1}{a_n}\sum_{k=1}^{n}X_{nk} \to 0 \quad \textit{completely.} \tag{1.10}$$

Theorem 1.2. *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables, and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ be a sequence of nonnegative even functions satisfying* (1.7) *for $q = 1$. Then conditions* (1.2) *and* (1.9) *imply*

$$\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}X_{nk}\right| \to 0 \quad \textit{completely.} \tag{1.11}$$

Theorem 1.3. *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables, and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ be a sequence of nonnegative even functions satisfying* (1.7) *for $1 \leqslant q < p \leqslant 2$. Then conditions* (1.2) *and* (1.8) *imply*

$$\sum_{n=1}^{\infty}a_n^{-q}E\left\{\left|\sum_{k=1}^{n}X_{nk}\right| - \varepsilon a_n\right\}_+^q < \infty \quad \forall\,\varepsilon > 0. \tag{1.12}$$

Theorem 1.4. *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables, and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ is a sequence of nonnegative even functions satisfying* (1.7) *for* $1 \leqslant q < p \leqslant 2$. *Then Conditions* (1.2) *and* (1.9) *imply*

$$\sum_{n=1}^{\infty} a_n^{-q} E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_{nk}\right| - \varepsilon a_n\right\}_+^q < \infty \quad \forall\, \varepsilon > 0. \tag{1.13}$$

Theorem 1.5. *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise pairwise NQD random variables with* (1.2), *and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Let $\{\Psi_n(t),\ n \geqslant 1\}$ be a sequence of nonnegative even functions satisfying* (1.7) *for* $1 \leqslant q < p \leqslant 2$.

(1) *If*

$$\sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n \to \infty, \tag{1.14}$$

then

$$\frac{1}{a_n}\sum_{k=1}^{n} X_{nk} \xrightarrow{L^q} 0. \tag{1.15}$$

(2) *If*

$$\log^2 n \sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n \to \infty, \tag{1.16}$$

then

$$\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_{nk}\right| \xrightarrow{L^q} 0. \tag{1.17}$$

R e m a r k 1.1. Since an independent random variable sequence is a special pairwise NQD sequence, Theorem 1.1 and Theorem 1.2 hold for arrays of rowwise independent random variables. So Theorems 1.1 and 1.2 are extensions and improvements of Theorem B. It is worth pointing out that our conclusions are much stronger and conditions are more general and much weaker.

R e m a r k 1.2. Wang and Zhao [10] investigated the complete moment convergence for NA random variable sequences. Compared with the paper of Wang and Zhao [10], we consider pairwise NQD random variables instead of NA random variables. Since both NA and NOD imply pairwise NQD, Theorems 1.1–1.5 remain true for NA or NOD random variables.

R e m a r k 1.3. The proofs in this paper are based on the following famous inequality:

$$E\left(\sum_{k=1}^{n} X_k\right)^2 \leqslant C \sum_{k=1}^{n} EX_k^2, \tag{1.18}$$

which was established by Wu [11]. According to our knowledge, the following sequences of mean zero random variables satisfy (1.18) with the indicated value of C, such as Martingale difference ($C = 1$, Adler, Rosalsky, and Volodin [1]), φ-mixing random variables with $\sum\limits_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ ($C = 1 + 4 \sum\limits_{n=1}^{\infty} \varphi^{1/2}(n)$, Yang [13]), ϱ-mixing random variables with $\sum\limits_{n=1}^{\infty} \varrho(n) < \infty$ ($C = 1 + 4 \sum\limits_{n=1}^{\infty} \varrho(n)$, Yang [13]), ϱ^*-mixing random variables with $\varrho^* < 1$ for some integer $s \geqslant 1$ ($C = 64s(1 - \varrho^*(s))^{-2}$, Bryc and Smoleński [2]).

Therefore, following the methods of this paper, we can easily get similar results for the above sequences.

In this paper, the symbol C always stands for a generic positive constant which may differ from one place to another.

2. Proofs

We need the following lemmas (cf. Lehmann [9], Wu [11]).

Lemma 2.1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of pairwise NQD random variables. Let $\{f_n,\ n \geqslant 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n),\ n \geqslant 1\}$ is a sequence of pairwise NQD random variables.*

Lemma 2.2. *Let $\{X_n,\ n \geqslant 1\}$ be a pairwise NQD random variable sequence with mean zero and $EX_n^2 < \infty$, and let $T_j(k) = \sum\limits_{i=j+1}^{j+k} X_i$, $j \geqslant 0$. Then*

$$E(T_j(k))^2 \leqslant C \sum_{i=j+1}^{j+k} EX_i^2, \quad E \max_{1 \leqslant k \leqslant n} (T_j(k))^2 \leqslant C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2.$$

P r o o f of Theorem 1.1. For any $1 \leqslant k \leqslant n,\ n \geqslant 1$, let

$$\begin{aligned} Y_{nk} &= -a_n I(X_{nk} < -a_n) + X_{nk} I(|X_{nk}| \leqslant a_n) + a_n I(X_{nk} > a_n), \\ Z_{nk} &= X_{nk} - Y_{nk} = (X_{nk} + a_n) I(X_{nk} < -a_n) + (X_{nk} - a_n) I(X_{nk} > a_n). \end{aligned}$$

To prove (1.10), it suffices to show

$$\frac{1}{a_n} \sum_{k=1}^{n} Z_{nk} \to 0 \quad \text{completely,} \tag{2.1}$$

$$\frac{1}{a_n} \sum_{k=1}^{n} (Y_{nk} - EY_{nk}) \to 0 \quad \text{completely,} \tag{2.2}$$

$$\frac{1}{a_n} \sum_{k=1}^{n} EY_{nk} \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

First, we prove (2.1). If $X_{nk} > a_n$, $0 < Z_{nk} = X_{nk} - a_n < X_{nk}$. If $X_{nk} < -a_n$, $X_{nk} < Z_{nk} = X_{nk} + a_n \leqslant 0$. So $|Z_{nk}| \leqslant |X_{nk}| I(|X_{nk}| > a_n)$. Consequently, by (1.7) and (1.8), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P\biggl(\biggl|\sum_{k=1}^{n} Z_{nk}\biggr| > a_n \varepsilon\biggr) \\ &\qquad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E|Z_{nk}|}{a_n} \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E|X_{nk}| I(|X_{nk}| > a_n)}{a_n} \\ &\qquad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty. \end{aligned}$$

Secondly, we prove (2.2). By Lemma 2.1 we know that $\{Y_{nk} - EY_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ is an array of rowwise pairwise NQD mean zero random variables. Note that $|Y_{nk}| \leqslant |X_{nk}|$ and $1 < p \leqslant 2$. By the Markov inequality, Lemma 2.2, (1.7), and (1.8), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P\biggl(\frac{1}{a_n}\biggl|\sum_{k=1}^{n} (Y_{nk} - EY_{nk})\biggr| > \varepsilon\biggr) \\ &\qquad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-2} E(Y_{nk} - EY_{nk})^2 \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{EY_{nk}^2}{a_n^2} \\ &\qquad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E|Y_{nk}|^p}{a_n^p} \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi_k(|Y_{nk}|)}{\Psi_k(a_n)} \\ &\qquad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty. \end{aligned}$$

Finally, we prove (2.3). For $1 \leqslant k \leqslant n$, $n \geqslant 1$, $EX_{nk} = 0$, we have $EY_{nk} = -EZ_{nk}$. By an argument similar to that in the proof of (2.1), we have

$$\begin{aligned}\frac{1}{a_n}\Bigl|\sum_{k=1}^{n} EY_{nk}\Bigr| &= \frac{1}{a_n}\Bigl|\sum_{k=1}^{n} EZ_{nk}\Bigr| \\ &\leqslant \sum_{k=1}^{n}\frac{E|Z_{nk}|}{a_n} \leqslant \sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

The proof is complete. □

P r o o f of Theorem 1.2. Following the notation and by an argument similar to that in the proof of Theorem 1.1, we can easily prove Theorem 1.2. Therefore, we omit the details. □

P r o o f of Theorem 1.3. We have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^{-q} E\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| - \varepsilon a_n\Bigr\}_+^q &= \sum_{n=1}^{\infty} a_n^{-q}\int_0^{\infty} P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| - \varepsilon a_n > t^{1/q}\Bigr\}\,\mathrm{d}t \\ &= \sum_{n=1}^{\infty} a_n^{-q}\Bigl(\int_0^{a_n^q} P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > \varepsilon a_n + t^{1/q}\Bigr\}\,\mathrm{d}t \\ &\quad + \int_{a_n^q}^{\infty} P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > \varepsilon a_n + t^{1/q}\Bigr\}\,\mathrm{d}t\Bigr) \\ &\leqslant \sum_{n=1}^{\infty} P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > \varepsilon a_n\Bigr\} + \sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > t^{1/q}\Bigr\}\,\mathrm{d}t \\ &\mathrel{\widehat{=}} I_1 + I_2.\end{aligned}$$

By Theorem 1.1, we have $I_1 < \infty$. To prove (1.12), it suffices to prove $I_2 < \infty$. Let

$$\begin{aligned}Y_{nk} &= -t^{1/q} I(X_{nk} < -t^{1/q}) + X_{nk} I(|X_{nk}| \leqslant t^{1/q}) + t^{1/q} I(X_{nk} > t^{1/q}), \\ Z_{nk} &= X_{nk} - Y_{nk} = (X_{nk} + t^{1/q}) I(X_{nk} < -t^{1/q}) + (X_{nk} - t^{1/q}) I(X_{nk} > t^{1/q}).\end{aligned}$$

Since $X_{nk} = Y_{nk}$ if $|X_{nk}| \leqslant t^{1/q}$, we get

$$\begin{aligned}P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > t^{1/q}\Bigr\} &\leqslant P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > t^{1/q}, \bigcup_{k=1}^{n}\{|X_{nk}| > t^{1/q}\}\Bigr\} \\ &\quad + P\Bigl\{\Bigl|\sum_{k=1}^{n} X_{nk}\Bigr| > t^{1/q}, \bigcap_{k=1}^{n}\{|X_{nk}| \leqslant t^{1/q}\}\Bigr\} \\ &\leqslant \sum_{k=1}^{n} P\{|X_{nk}| > t^{1/q}\} + P\Bigl\{\Bigl|\sum_{k=1}^{n} Y_{nk}\Bigr| > t^{1/q}\Bigr\}.\end{aligned}$$

Hence,

$$\begin{aligned} I_2 &\leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} P\{|X_{nk}|>t^{1/q}\}\,\mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} P\Big\{\Big|\sum_{k=1}^{n} Y_{nk}\Big|>t^{1/q}\Big\}\,\mathrm{d}t \\ &\mathrel{\widehat{=}} I_3+I_4. \end{aligned}$$

Clearly, for $t \geqslant a_n^q$ we have

$$\begin{aligned} I_3 &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q}^{\infty} P\{|X_{nk}|I(|X_{nk}|>a_n)>t^{1/q}\}\,\mathrm{d}t \\ &\leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{0}^{\infty} P\{|X_{nk}|I(|X_{nk}|>a_n)>t^{1/q}\}\,\mathrm{d}t \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^q I(|X_{nk}|>a_n)}{a_n^q} \\ &\leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty. \end{aligned}$$

Now we prove $I_4<\infty$. By (1.2), (1.7), and (1.8), we have

$$\begin{aligned} &\max_{t\geqslant a_n^q}\Big|t^{-1/q}\sum_{k=1}^{n} EY_{nk}\Big| \\ &\quad = \max_{t\geqslant a_n^q}\Big|t^{-1/q}\sum_{k=1}^{n} EZ_{nk}\Big| \leqslant \max_{t\geqslant a_n^q} t^{-1/q}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}|>t^{1/q}) \\ &\quad \leqslant \sum_{k=1}^{n}\frac{E|X_{nk}|I(|X_{nk}|>a_n)}{a_n} \leqslant \sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0. \end{aligned}$$

Therefore, while n is sufficiently large, for $t \geqslant a_n^q$,

$$\Big|\sum_{k=1}^{n} EY_{nk}\Big| \leqslant t^{1/q}/2.$$

Then

$$P\Big\{\Big|\sum_{k=1}^{n} Y_{nk}\Big|>t^{1/q}\Big\} \leqslant P\Big\{\Big|\sum_{k=1}^{n}(Y_{nk}-EY_{nk})\Big|>t^{1/q}/2\Big\}. \tag{2.4}$$

Let $d_n = [a_n] + 1$, by (2.4), Lemma 2.2, and C_r-inequality, we have

$$\begin{aligned}
I_4 &\leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q} E Y_{nk}^2 \, \mathrm{d}t \\
&= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q} E X_{nk}^2 I(|X_{nk}| \leqslant d_n) \, \mathrm{d}t \\
&\quad + C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q} E X_{nk}^2 I(d_n < |X_{nk}| \leqslant t^{1/q}) \, \mathrm{d}t \\
&\quad + C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} P(|X_{nk}| > t^{1/q}) \, \mathrm{d}t \\
&\,\widehat{=}\, I_{41} + I_{42} + I_{43}.
\end{aligned}$$

By an argument similar to that in the proof of $I_3 < \infty$, we can prove $I_{43} < \infty$. For I_{41}, by $q < 2$, $(a_n + 1)/a_n \to 1$ as $n \to \infty$, and (1.8), we have

$$\begin{aligned}
I_{41} &= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} E X_{nk}^2 I(|X_{nk}| \leqslant d_n) \int_{a_n^q}^{\infty} t^{-2/q} \, \mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E X_{nk}^2 I(|X_{nk}| \leqslant d_n)}{a_n^2} \\
&\leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \Big(\frac{a_n + 1}{a_n}\Big)^2 \frac{E X_{nk}^2 I(|X_{nk}| \leqslant d_n)}{d_n^2} \\
&\leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E |X_{nk}|^p I(|X_{nk}| \leqslant d_n)}{d_n^p} \\
&\leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E \Psi_k(X_{nk})}{\Psi_k(d_n)} \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E \Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty.
\end{aligned}$$

For I_{42}, since

$$C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{d_n^q} t^{-2/q} E X_{nk}^2 I(d_n < |X_{nk}| \leqslant t^{1/q}) \, \mathrm{d}t = 0,$$

we have

$$I_{42} = C \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_n^{-q} \int_{d_n^q}^{\infty} t^{-2/q} E X_{nk}^2 I(d_n < |X_{nk}| \leqslant t^{1/q}) \, \mathrm{d}t.$$

Let $t = u^q$. By $q > 1$, (1.7), and (1.8), we have

$$\begin{aligned}
I_{42} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{d_n}^{\infty} u^{q-3} EX_{nk}^2 I(d_n < |X_{nk}| \leqslant u)\,\mathrm{d}u \\
&= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\sum_{m=d_n}^{\infty}\int_{m}^{m+1} u^{q-3} EX_{nk}^2 I(d_n < |X_{nk}| \leqslant u)\,\mathrm{d}u \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\sum_{m=d_n}^{\infty} m^{q-3} EX_{nk}^2 I(d_n < |X_{nk}| \leqslant m+1) \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\sum_{m=d_n}^{\infty} m^{q-3}\sum_{s=d_n}^{m} EX_{nk}^2 I(s < |X_{nk}| \leqslant s+1) \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\sum_{s=d_n}^{\infty} EX_{nk}^2 I(s < |X_{nk}| \leqslant s+1)\sum_{m=s}^{\infty} m^{q-3} \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\sum_{s=d_n}^{\infty} s^{q-2} EX_{nk}^2 I(s < |X_{nk}| \leqslant s+1) \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} E|X_{nk}|^q I(|X_{nk}| > d_n) \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty.
\end{aligned}$$

The proof is complete. □

P r o o f of Theorem 1.4. Following the notation and by an argument similar to that in the proof of Theorem 1.3, we can easily prove Theorem 1.4. Therefore, we omit the details. □

P r o o f of Theorem 1.5. We follow the notation in the proof in Theorem 1.3. To start with, we prove (1.15). For all $\varepsilon > 0$,

$$\begin{aligned}
E\biggl(a_n^{-1}\biggl|\sum_{k=1}^{n} X_{nk}\biggr|\biggr)^q &= a_n^{-q}\int_0^{\infty} P\biggl(\biggl|\sum_{k=1}^{n} X_{nk}\biggr| > t^{1/q}\biggr)\,\mathrm{d}t \\
&\leqslant \varepsilon + a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} P\biggl(\biggl|\sum_{k=1}^{n} X_{nk}\biggr| > t^{1/q}\biggr)\,\mathrm{d}t \\
&\leqslant \varepsilon + a_n^{-q}\int_{a_n^q\varepsilon}^{\infty}\sum_{k=1}^{n} P\{|X_{nk}| > t^{1/q}\}\,\mathrm{d}t \\
&\quad + a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} P\biggl\{\biggl|\sum_{k=1}^{n} Y_{nk}\biggr| > t^{1/q}\biggr\}\,\mathrm{d}t \\
&\mathrel{\widehat{=}} \varepsilon + I_5 + I_6.
\end{aligned}$$

Without loss of generality we may assume $0 < \varepsilon < 1$. By the Markov inequality, (1.7), and (1.14), we have

$$
\begin{aligned}
I_5 &\leqslant \sum_{k=1}^n a_n^{-q} \int_{a_n^q \varepsilon}^{\infty} P\{|X_{nk}| I(a_n \varepsilon^{1/q} < |X_{nk}| \leqslant a_n) > t^{1/q}\} \, \mathrm{d}t \\
&\quad + \sum_{k=1}^n a_n^{-q} \int_{a_n^q \varepsilon}^{\infty} P\{|X_{nk}| I(|X_{nk}| > a_n) > t^{1/q}\} \, \mathrm{d}t \\
&\leqslant \sum_{k=1}^n a_n^{-q} E|X_{nk}|^p I(|X_{nk}| \leqslant a_n) \int_{a_n^q \varepsilon}^{\infty} t^{-p/q} \, \mathrm{d}t \\
&\quad + \sum_{k=1}^n a_n^{-q} \int_0^{\infty} P\{|X_{nk}| I(|X_{nk}| > a_n) > t^{1/q}\} \, \mathrm{d}t \\
&= C\varepsilon^{1-p/q} \sum_{k=1}^n \frac{E|X_{nk}|^p}{a_n^p} I(|X_{nk}| \leqslant a_n) + \sum_{k=1}^n a_n^{-q} E|X_{nk}|^q I(|X_{nk}| > a_n) \\
&\leqslant (C\varepsilon^{1-p/q} + 1) \sum_{k=1}^n \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Then we prove $I_6 < \infty$. By an argument similar to that in the proof of (2.4), we have

$$
\begin{aligned}
&\max_{t \geqslant a_n^q \varepsilon} \left| t^{-1/q} \sum_{k=1}^n EY_{nk} \right| \\
&\quad = \max_{t \geqslant a_n^q \varepsilon} \left| t^{-1/q} \sum_{k=1}^n EZ_{nk} \right| \leqslant a_n^{-1} \varepsilon^{-1/q} \sum_{k=1}^n E|X_{nk}| I(|X_{nk}| > a_n \varepsilon^{1/q}) \\
&\quad \leqslant \varepsilon^{-1/q} \sum_{k=1}^n \frac{E|X_{nk}|}{a_n} I(|X_{nk}| > a_n) \\
&\qquad + \varepsilon^{-p/q} \sum_{k=1}^n \frac{E|X_{nk}|^p}{a_n^p} I(a_n \varepsilon^{1/q} < |X_{nk}| \leqslant a_n) \\
&\quad \leqslant (\varepsilon^{-1/q} + \varepsilon^{-p/q}) \sum_{k=1}^n \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, while n is sufficiently large, for $t \geqslant a_n^q \varepsilon$, we have

$$
P\left\{ \left| \sum_{k=1}^n Y_{nk} \right| > t^{1/q} \right\} \leqslant P\left\{ \left| \sum_{k=1}^n (Y_{nk} - EY_{nk}) \right| > t^{1/q}/2 \right\}. \tag{2.5}
$$

Let $d_n = [a_n] + 1$. By (2.5), Lemma 2.2, and the C_r-inequality, we have

$$\begin{aligned}
I_6 &\leqslant C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} t^{-2/q}E(Y_{nk}-EY_{nk})^2\,\mathrm{d}t\\
&\leqslant C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} t^{-2/q}EY_{nk}^2\,\mathrm{d}t\\
&= C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} t^{-2/q}EX_{nk}^2I(|X_{nk}|\leqslant d_n)\,\mathrm{d}t\\
&\quad + C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} t^{-2/q}EX_{nk}^2I(d_n<|X_{nk}|\leqslant t^{1/q})\,\mathrm{d}t\\
&\quad + C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{\infty} P(|X_{nk}|>t^{1/q})\,\mathrm{d}t\\
&\widehat{=}\ \varepsilon + I_7 + I_8 + I_9.
\end{aligned}$$

By an argument similar to that in the proof of $I_5 \to 0$, we can prove $I_9 \to 0$. By an argument similar to that in the proof of $I_{41} < \infty$, we can prove

$$I_7 \leqslant C\varepsilon^{1-2/q}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n\to\infty.$$

For I_8, since

$$C\sum_{k=1}^{n} a_n^{-q}\int_{a_n^q\varepsilon}^{d_n^q} t^{-2/q}EX_{nk}^2I(d_n<|X_{nk}|\leqslant t^{1/q})\,\mathrm{d}t = 0,$$

we have

$$I_8 = C\sum_{k=1}^{n} a_n^{-q}\int_{d_n^q}^{\infty} t^{-2/q}EX_{nk}^2I(d_n<|X_{nk}|\leqslant t^{1/q})\,\mathrm{d}t.$$

Therefore, by an argument similar to that in the proof of $I_{42} < \infty$, we can prove

$$I_8 \leqslant C\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} \to 0 \quad \text{as } n\to\infty.$$

The proof of (1.17) is similar to that of (1.15), so we omit it. The proof is complete. □

Acknowledgment. The author would like to thank the referee for a detailed list of valuable suggestions and comments which greatly improved the presentation of the work.

References

[1] *A. Adler, A. Rosalsky, A. Volodin*: A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces. Stat. Probab. Lett. *32* (1997), 167–174.

[2] *W. Bryc, W. Smoleński*: Moment conditions for almost sure convergence of weakly correlated random variables. Proc. Am. Math. Soc. *119* (1993), 629–635.

[3] *Y. S. Chow*: On the rate of moment complete convergence of sample sums and extremes. Bull. Inst. Math. Acad. Sin. *16* (1988), 177–201.

[4] *N. Ebrahimi, M. Ghosh*: Multivariate negative dependence. Commun. Stat., Theory Methods *10* (1981), 307–337.

[5] *S. X. Gan, P. Y. Chen*: On the limiting behavior of the maximum partial sums for arrays of rowwise NA random variables. Acta Math. Sci., Ser. B, Engl. Ed. *27* (2007), 283–290.

[6] *P. L. Hsu, H. Robbins*: Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. *33* (1947), 25–31.

[7] *T. C. Hu, R. L. Taylor*: On the strong law for arrays and for the bootstrap mean and variance. Int. J. Math. Math. Sci. *20* (1997), 375–382.

[8] *K. Joag-Dev, F. Proschan*: Negative association of random variables with applications. Ann. Stat. *11* (1983), 286–295.

[9] *E. L. Lehmann*: Some concepts of dependence. Ann. Math. Stat. *37* (1966), 1137–1153.

[10] *D. C. Wang, W. Zhao*: Moment complete convergence for sums of a sequence of NA random variables. Appl. Math., Ser. A (Chin. Ed.) *21* (2006), 445–450. (In Chinese.)

[11] *Q. Y. Wu*: Convergence properties of pairwise NQD random sequences. Acta Math. Sin. *45* (2002), 617–624. (In Chinese.)

[12] *Y. F. Wu, D. J. Zhu*: Convergence properties of partial sums for arrays of rowwise negatively orthant dependent random variables. J. Korean Stat. Soc. *39* (2010), 189–197.

[13] *S. C. Yang*: Almost sure convergence of weighted sums of mixing sequences. J. Syst. Sci. Math. Sci. *15* (1995), 254–265. (In Chinese.)

Author's address: *Y. Wu*, Department of Mathematics and Computer Science, Tongling University, Tongling 244000, P. R. China, e-mail: wyfwyf@126.com; *D. Wang* (corresponding author), Institute of Financial Engineering, School of Finance, School of Applied Mathematics, Nanjing Audit University, Nanjing 211815, P. R. China, e-mail: wangdc@uestc.edu.cn.