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JULIA LINES OF GENERAL RANDOM DIRICHLET SERIES

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Abstract. In this paper, we consider a random entire function $f(s,\omega)$ defined by a random Dirichlet series $\sum_{n=1}^{\infty} X_n(\omega)\mathrm{e}^{-\lambda_n s}$ where X_n are independent and complex valued variables, $0 \leqslant \lambda_n \nearrow +\infty$. We prove that under natural conditions, for some random entire functions of order (R) zero $f(s,\omega)$ almost surely every horizontal line is a Julia line without an exceptional value. The result improve a theorem of J. R. Yu: Julia lines of random Dirichlet series. Bull. Sci. Math. 128 (2004), 341–353, by relaxing condition on the distribution of X_n for such function $f(s,\omega)$ of order (R) zero, almost surely.

Keywords: random Dirichlet series, order (R), Julia lines, entire function

MSC 2010: 30D35

1. INTRODUCTION

In the field of mathematical analysis, the Dirichlet series is an infinite series that takes the form of

$$f(s) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n s}, \tag{1}$$

where $\{a_n\} \subset \mathbb{C}$, $0 \leqslant \lambda_n \nearrow +\infty$, and $s = \sigma + \mathrm{i}t$ is a complex number. The Taylor series

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \tag{2}$$

is obtained when $\lambda_n = n$ and $z = \mathrm{e}^{-s}$. If the Dirichlet series (1) converges for any $s \in \mathbb{C}$, it is an entire function. The abscissa of convergence of a Dirichlet series can

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be defined as

$$\sigma_c = \inf\{\sigma \in \mathbb{R}\colon\ f(s) \text{ converges for any } s \text{ where } \operatorname{Re}(s) > \sigma\}.$$

The line $\sigma = \sigma_c$ is called the line of convergence. The half-plane of convergence is defined as

$$\mathbb{C}_{\sigma_c} = \{s \in \mathbb{C}\colon\ \operatorname{Re}(s) > \sigma_c\}.$$

The random Dirichlet series and random Taylor series can be defined by replacing the sequence of complex a_n by a sequence of complex random variables $X_n(\omega)$. We then have the random Dirichlet series

$$f(s,\omega) = \sum_{n=1}^{\infty} X_n(\omega)\mathrm{e}^{-\lambda_n s}, \tag{3}$$

and random Taylor series

$$g(z,\omega) = \sum_{n=1}^{\infty} X_n(\omega)z^n. \tag{4}$$

Dirichlet series and in particular the growth and the value distribution of entire functions defined by Dirichlet series are important branches of mathematical analysis. In [12], [13], the author obtained some important results on Dirichlet series convergent or convergent almost surely (*a.s.*) only in a half-plane. The related research has been extended to more general random Dirichlet series and more accurate results have been given in recent years (see [2], [8], [9], [10], [16]). For random entire functions defined by random Dirichlet series, there are some better results for functions a.s. of infinite order (R) [14]. Recently, J. R. Yu [15] gives a general result on some random entire functions a.s. of zero order (R) which does not contain a theorem of P. L. Davies [1].

In this paper, the conditions of the theorems in [15] are found to be too strong. By improving the method of the proof in [15], we discard the same distribution condition of the theorems in [15], and obtain the same results for a random entire function a.s. of zero order (R).

The paper is organized as follows. Our theorem and corollary will be given in Section 2. In Section 3, the proof of the main result appears.

2. Statement of the results

In this paper, we study the Julia line of Dirichlet series and the Julia direction of Taylor series. A line $\operatorname{Im} s = t_0$ is called a Julia line for entire f if in any open strip containing the line, f takes on all values, with at most one exception, infinitely often.

We first recall some results of [15]. The author of [15] introduces a special random Dirichlet series

$$f_1(s,\omega) = \sum_{n=1}^{+\infty} a_n X_n(\omega) \mathrm{e}^{\lambda_n s}, \tag{5}$$

where $\{a_n\} \subset \mathbb{C}$, $0 \leqslant \lambda_n \nearrow +\infty$, $s = \sigma + \mathrm{i}t$,

$$\varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n} < +\infty, \tag{6}$$

$$\varlimsup_{n\to+\infty} \frac{\ln |a_n|}{\lambda_n} = -\infty, \tag{7}$$

and in the probability space $(\Omega, \mathcal{A}, P)$ $(\omega \in \Omega)$, $\{X_n(\omega)\}$ is a sequence of non-degenerate, symmetric and independent complex random variables of the same distribution verifying

$$0 < \mathbb{E}(|X_n(\omega)|^2) = d^2 < +\infty. \tag{8}$$

For $f_1(s,\omega)$, the author of [15] has given the following results.

Proposition 1 ([15]). *If*

$$\varliminf_{n\to+\infty} \frac{\ln^+ \ln^+ |a_n|^{-1}}{\ln \lambda_n} = \frac{c}{c-1} \quad (1 < c < +\infty), \tag{9}$$

then almost surely (a.s.) $f_1(s,\omega)$ is an entire function verifying

$$\varlimsup_{\sigma\to+\infty} \frac{\ln^+ \ln^+ M(\sigma,\omega,f_1)}{\ln \sigma} = c \tag{10}$$

and a.s. every horizontal line in the s-plane is a Julia line without an exceptional value of $f_1(s,\omega)$, i.e. there exists $\mathbb{A} \in \mathcal{A}$ $(P(\mathbb{A}) = 1)$ such that for any $\omega \in \mathbb{A}$, (10) holds and for any $\omega \in \mathbb{A}$, $t_0 \in \mathbb{R}$, $\eta > 0$, and $\alpha \in \mathbb{C}$,

$$\lim_{\sigma\to+\infty} n(\sigma, t_0, \eta, \omega, f_1 = \alpha) = +\infty, \tag{11}$$

where

$$M(\sigma,\omega,f_1)=\sup(\{|f_1(\sigma+\mathrm{i}t,\omega)|\colon\ t\in\mathbb{R}\})$$

and

$$n(\sigma,t_0,\eta,\omega,f_1=\alpha)=\sharp\{s\colon\ f_1(s,\omega)=\alpha,\ \operatorname{Re}s\leqslant\sigma,\ |\operatorname{Im}s-t_0|<\eta\}.$$

And for random Taylor series

$$g_1(z,\omega)=\sum_{n=0}^{+\infty}a_nX_n(\omega)z^n, \tag{12}$$

where $\{a_n\}\subset\mathbb{C}$,

$$\varlimsup_{n\to+\infty}\frac{\ln|a_n|}{n}=-\infty \tag{13}$$

and $\{X_n(\omega)\}$ is a sequence of non-degenerate, symmetric and independent complex random variables of the same distribution verifying (8), we get

Proposition 2 ([15])**.** *If*

$$\varliminf_{n\to+\infty}\frac{\ln^+\ln^+|a_n|^{-1}}{\ln n}=\frac{c}{c-1}\quad(1<c<+\infty), \tag{14}$$

then a.s. the random Taylor series (12) *is an entire function verifying*

$$\varlimsup_{n\to+\infty}\frac{\ln^+\ln^+M(r,\omega,g_1)}{\ln\ln r}=c, \tag{15}$$

and a.s. every ray from the origin in the z-plane is a Julia direction without an exceptional value of $g_1(z,\omega)$*, i.e. there exists* $\mathbb{A}\in\mathcal{A}$ $(P(\mathbb{A})=1)$ *such that for any* $\omega\in\mathbb{A}$, (15) *holds, and for any* $\omega\in\mathbb{A}$, $\theta_0\in[0,2\pi]$, $\eta>0$, *and* $\alpha\in\mathbb{C}$

$$\lim_{n\to+\infty}n(r,\theta_0,\eta,\omega,g_1=\alpha)=+\infty,$$

where

$$M(r,\omega,g_1)=\max\{|g_1(z,\omega)|\colon\ |z|=r\}$$

and

$$n(r,\theta_0,\eta,\omega,g_1=\alpha)=\sharp\{z\colon\ g_1(z,\omega)=\alpha,\ |z|\leqslant r,\ |\arg z-\theta_0|<\eta\}.$$

The random Dirichlet series (5) is so special in that $\{X_n(\omega)\}$ is a sequence of non-degenerate, symmetric and independent complex random variables of the same distribution and subject to (8). In this paper, we study the random Dirichlet series (3) and the random Taylor series (4). We give the results on some random entire functions a.s. of zero order (R) under a weaker condition on the complex random variables. Suppose that $\{X_n(\omega)\}$ is a sequence of non-degenerate, symmetric and independent complex random variables verifying the following conditions: $\mathbb{E}X_n = 0$,

$$\overline{\lim_{n\to+\infty}} \frac{\ln \mathbb{E}|X_n|^2}{\lambda_n} = -\infty, \tag{16}$$

$$d^2\mathbb{E}|X_n|^2 \leqslant \mathbb{E}^2|X_n| < +\infty, \tag{17}$$

where $d > 0$ is a constant. Conditions (6) and (3) mean that the series (3) is an entire function a.s. of zero order (R). For convenience, we denote $\sqrt{\mathbb{E}|X_n|^2}$ by $\mathrm{d}\Delta_n$. Then, the condition (16) will be equivalent to

$$\overline{\lim_{n\to+\infty}} \frac{\ln \Delta_n}{\lambda_n} = -\infty. \tag{18}$$

For $f(s,\omega)$ we have the following theorem.

Theorem 3. *If*

$$\underline{\lim_{n\to+\infty}} \frac{\ln^+ \ln^+ \Delta_n^{-1}}{\ln \lambda_n} = \frac{c}{c-1} \quad (1 < c < +\infty), \tag{19}$$

then almost surely $(a.s.)$ $f(s,\omega)$ *is an entire function verifying*

$$\overline{\lim_{\sigma\to+\infty}} \frac{\ln^+ \ln^+ M(\sigma,\omega,f)}{\ln \sigma} = c \tag{20}$$

and a.s. every horizontal line in the s-plane is a Julia line without an exceptional value of $f(s,\omega)$, *i.e. there exists* $\mathbb{A} \in \mathcal{A}$ $(P(\mathbb{A}) = 1)$ *such that for any* $\omega \in \mathbb{A}$, (20) *holds and for any* $\omega \in \mathbb{A}$, $t_0 \in \mathbb{R}$, $\eta > 0$, *and* $\alpha \in \mathbb{C}$,

$$\lim_{\sigma\to+\infty} n(\sigma, t_0, \eta, \omega, f = \alpha) = +\infty, \tag{21}$$

where

$$M(\sigma,\omega,f) = \sup(\{|f(\sigma + \mathrm{i}t,\omega)|\colon\ t \in \mathbb{R}\})$$

and

$$n(\sigma, t_0, \eta, \omega, f = \alpha) = \sharp\{s\colon\ f(s,\omega) = \alpha,\ \operatorname{Re} s \leqslant \sigma,\ |\operatorname{Im} s - t_0| < \eta\}.$$

In particular, the random Taylor series (4) verifies the following conditions: $\{X_n(\omega)\}$ is a sequence of non-degenerate, symmetric and independent complex random variables verifying the following conditions: $\mathbb{E}X_n = 0$,

$$\overline{\lim_{n\to+\infty}} \frac{\ln \Delta_n}{n} = -\infty. \tag{22}$$

We have the following corollary.

Corollary 4. *If*

$$\lim_{n\to+\infty} \frac{\ln^+ \ln^+ \Delta_n^{-1}}{\ln n} = \frac{c}{c-1} \quad (1 < c < +\infty), \tag{23}$$

then a.s. the random Taylor series (4) *is an entire function verifying*

$$\overline{\lim_{n\to+\infty}} \frac{\ln^+ \ln^+ M(r,\omega,g)}{\ln \ln r} = c, \tag{24}$$

and a.s. every ray from the origin in the z-plane is a Julia direction without an exceptional value of $g(z,\omega)$, *i.e. there exists* $\mathbb{A} \in \mathcal{A}(P(\mathbb{A}) = 1)$ *such that for any* $\omega \in \mathbb{A}$, (24) *holds, and for any* $\omega \in \mathbb{A}$, $\theta_0 \in [0, 2\pi]$, $\eta > 0$, *and* $\alpha \in \mathbb{C}$

$$\lim_{n\to+\infty} n(r, \theta_0, \eta, \omega, g = \alpha) = +\infty,$$

where

$$M(r,\omega,g) = \max\{|g(z,\omega)| \colon\ |z| = r\}$$

and

$$n(r, \theta_0, \eta, \omega, g = \alpha) = \sharp\{z \colon\ g(z,\omega) = \alpha,\ |z| \leqslant r,\ |\arg z - \theta_0| < \eta\}.$$

Letting $\lambda_n = n$ and $z = \mathrm{e}^{-s}$, Theorem 3 becomes Corollary 4 immediately. Our results are general and contain the results in [15]. Furthermore, let $X_n'(\omega) = X(\omega)/\Delta_n$ which are complex random variables of the same distribution, and $|a_n| = \Delta_n$. Then $X_n'(\omega)$ verify (8), and Theorem 3 and Corollary 4 reduce to Proposition 1 and Proposition 2 respectively, with $X_n(\omega)$ replacing $a_n X_n'(\omega)$.

3. The proof of Theorem 3

3.1. Some lemmas for the proof. We begin with some preliminary results. First, we need the following lemma.

Lemma 5 ([2],[14]). *Let* $\{X_n(\omega)\}$ *be a sequence of non-degenerate, symmetric, and independent complex random variables and satisfy the conditions* (17) *and* (18). *Then*

(i) *for any* $\omega \in \Omega$, *a.s. there exists* $N(\omega) \in \mathbb{N}$,

$$|X_n(\omega)| \leqslant n\Delta_n, \quad n > N(\omega);$$

(ii) *for any subsequence* $\{X_{n_k}\}$ *of* $\{X_n\}$,

$$P\Big(\overline{\lim_{k\to+\infty}} \Big\{|X_{n_k}| \geqslant \frac{d}{2}\Delta_{n_k}\Big\}\Big) = 1,$$

where d *and* Δ_{n_k} *are subject to the condition* (17);

(iii) *there exists* $\beta \in (0,1)$, *such that* $\sup\{P(X_n(\omega) = c)\colon\ c \in \mathbb{C},\ n \in \mathbb{N}\} < \beta$.

Lemma 6 (Paley-Zygmund). *Let* $\{Z_n(\omega)\}$ *be a sequence of independent complex random variables in* $(\Omega, \mathcal{A}, P)$ *verifying*

$$\mathbb{E}(X_n(\omega)) = 0, \quad \mathbb{E}(|X_n(\omega)|^2) = \Delta_n^2 > 0, \quad d = \inf_n \Big\{\mathbb{E}\Big(\Big|\frac{X_n(\omega)}{\Delta_n}\Big|\Big)\Big\} > 0.$$

Then for any $\mathbb{H} \in \mathcal{A}$, $(\mathbb{P}(\mathbb{H}) > 0)$, *there exist* $B = B(d, \mathbb{H})$ *and* $k = k(\mathbb{H}, \{X_n\}) \in N$ *such that for any sequence* $\{b_k\} \in \mathbb{C}$, *and any* p, q *verifying* $q > p > k$,

$$\int_{\mathbb{H}} \Big|\sum_{n=p}^{q} b_n X_n(\omega)\Big|^2 \mathbb{P}(\mathrm{d}\omega) \geqslant B \sum_{n=p}^{q} |b_n|^2 \Delta_n^2. \tag{25}$$

Lemma 6 is a generalization [10], [15] of the Paley-Zygmund lemma (see [6], [7]). The proof can be found in [16]. Another similar version of this well-known lemma can be found in [3].

Lemma 7. *Let* λ_n $(0 \leqslant n \nearrow +\infty)$ *verify* (6), Δ_n *be subject to* (18), *and* $f(s)$ *be defined by* (1). *Then*

$$\overline{\lim_{\sigma\to+\infty}} \frac{\ln^+ \ln^+ M(\sigma, F)}{\ln \sigma} = c \Leftrightarrow \overline{\lim_{n\to+\infty}} \frac{\ln^+ \ln^+ |a_n|^{-1}}{\ln \lambda_n} = \frac{c}{c-1}, \tag{26}$$

where $M(\sigma, F) = \sup\{|F(\sigma + \mathrm{i}t)|\colon\ t \in \mathbb{R}\}$ *and* $1 < c < +\infty$. *Under the condition* (26), *we have*

$$\overline{\lim_{\sigma\to+\infty}} \frac{\ln^+ \ln^+ m(\sigma, F)}{\ln \sigma} = \overline{\lim_{\sigma\to+\infty}} \frac{\ln^+ \ln^+ M(\sigma, F)}{\ln \sigma} = c,$$

where $m(\sigma, F) = \max\{|a_n|\mathrm{e}^{\lambda_n \sigma}\colon\ n \in \mathbb{N}\}$.

The proof of this lemma is similar to the proof of the results on order (R) of Dirichlet series in [11], [12].

J. R. Yu [15] introduces the following mappings,

$$z=\varphi_1(s)=\exp\left\{\frac{\pi}{2\eta}(s-\mathrm{i}t_0)\right\}\quad\text{and}\quad\omega=\varphi_2(z)=\frac{z-1}{z+1},\tag{27}$$

where $t_0\in R$ and $\eta>0$. Denote the inverse mappings by

$$s=\Phi_1(z)=\varphi_1^{-1}(z)\text{ and }z=\Phi_2(\omega)=\varphi_2^{-1}(\omega).\tag{28}$$

Let

$$\omega=\varphi(s)=\varphi_2\circ\varphi_1(s)\text{ and }s=\Phi(\omega)=\varphi^{-1}(\omega)=\Phi_1\Phi_2(\omega).\tag{29}$$

We introduce some sets as follows:

$$\mathbb{B}(t_0,\eta)=\{s\colon\ |\operatorname{Im}s-t_0|<\eta\},\tag{30}$$
$$\mathbb{B}^*(\sigma,t_0,\eta)=\{s\colon\ \operatorname{Re}s\leqslant\sigma\}\cap\mathbb{B}(t_0,\eta),\tag{31}$$
$$\mathbb{H}_1=\left\{z\colon\ |\arg z|<\frac{\pi}{2}\right\},\tag{32}$$
$$\mathbb{H}_2=\left\{z\colon\ |\arg z|<\frac{\pi}{4}\right\},\tag{33}$$
$$\mathbb{H}_k^*(r)=\{z\colon\ |z|\leqslant r\}\cap\mathbb{H}_k,\quad(k=1,2),\tag{34}$$
$$\mathbb{D}(r)=\{\omega\colon\ |\omega|<R\},\quad(0<R\leqslant1).\tag{35}$$

It is easy to see that

$$\varphi_1(\mathbb{B}(t_0,\eta))=\mathbb{H}_1,\quad\text{and}\quad\varphi_2(\mathbb{H}_1)=\mathbb{D}(1).$$

Thus by (29) we get

$$\varphi(\mathbb{B}(t_0,\eta))=\mathbb{D}(1),\quad\Phi(\mathbb{D}(1))=\mathbb{B}(t_0,\eta).\tag{36}$$

We have the following lemma.

Lemma 8 ([4], [15]). *Let* $R \in (0,1)$,

$$r = \frac{1+R}{1-R} \quad \text{and} \quad \sigma = \frac{2\eta}{\pi} \ln r.$$

Then

$$\begin{aligned}(37)\qquad \mathbb{B}^*\Bigl(\sigma + \frac{2\eta}{\pi}\ln k_1, t_0, \frac{\eta}{2}\Bigr) \cap \Bigl\{s\colon \operatorname{Re} s = \sigma + \frac{2\eta}{\pi}\ln r\Bigr\} \\ \subset \Phi(\mathbb{D}(R)) \subset \mathbb{B}^*(\sigma, t_0, \eta), \quad (\tfrac{1}{4} < k_1 < \tfrac{1}{2}),\end{aligned}$$

and

$$(38)\qquad \frac{\pi}{2\eta}\sigma - \ln 2 < -\ln(1-R) < \frac{\pi}{2\eta}\sigma.$$

Lemma 9 ([15]). *Let* $G(\omega)$ *be holomorphic in* $\mathbb{D}(1)$. *The following affirmations hold*

(i)

$$\frac{1-R}{4}\ln^+ M(R,G) \leqslant T\Bigl(\frac{R+1}{2}, G\Bigr) \leqslant \ln^+ M\Bigl(\frac{R+1}{2}, G\Bigr) \quad (0 < R < 1),$$

where $M(R,G) = \max\{|G(\omega)\colon\ |\omega| = R\}$ *and*

$$T(R,G) = (2\pi)^{-1}\int_0^{2\pi} \ln^+ |G(Re^{i\theta})|\, d\theta.$$

(ii) *If* g_1 *and* g_2 *are any two different complex numbers or* $g_1(z)$ *and* $g_2(z)$ *are any two different holomorphic function in* $\mathbb{D}(1)$ *verifying*

$$T(R, g_j(\omega)) = O(T(R, G(\omega))) \quad (R \nearrow 1,\ j = 1, 2),$$

then

$$T(R, G(\omega)) \leqslant 3\sum_{j=1}^{2} N(R, G = g_j) + O(\ln(1-R)^{-1}) \quad (R \nearrow 1),$$

or

$$\begin{aligned}T(R, G(\omega)) \leqslant 3\sum_{j=1}^{2} N\Bigl(\frac{R+1}{2}, G(\omega) = g_j(\omega)\Bigr) \\ + O\Bigl(N\Bigl(\frac{R+1}{2}, g_j(\omega)\Bigr) + \ln(1-R)^{-1}\Bigr) \quad (R \nearrow 1),\end{aligned}$$

where $n(R, G(\omega) - g_j \text{ or } g_j(\omega)) = \sharp\{\omega\colon\ G(\omega) = g_j \text{ or } g_j(\omega),\ |\omega| \leqslant R\}$ *and*

$$N(R, G(\omega) = g_j \quad \text{or} \quad g_j(\omega)) = \int_{R_0}^{R} \frac{n(u, G(\omega) = g_j \text{ or } g_j(\omega))}{u}\,\mathrm{d}u + O(1),$$

R_0 *being a fixed number in* $(0,1)$.

Lemma 9 is the Lemma 5 in [15].

3.2. Proof. Now we turn to the proof of Theorem 1. The proof is divided into the following nine steps.

(I) We first prove that $f(s,\omega)$ is a.s. an entire function.

Lemma 5 (i) implies that the abscissa of convergence of (3)

$$\sigma_c = \varlimsup_{n\to+\infty} \frac{\ln|X_n(\omega)|}{\lambda_n} \leqslant \varlimsup_{n\to+\infty} \frac{\ln n\Delta_n}{\lambda_n} \leqslant \varlimsup_{n\to+\infty} \frac{\ln \Delta_n}{\lambda_n} + \varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n} \ (a.s.). \tag{39}$$

By the conditions (6) and (22), the inequality (39) implies

$$\sigma_c = -\infty,$$

i.e. $f(s,\omega)$ is a.s. an entire function.

(II) We next prove that (20) holds.

By (6) and (18), there exists N such that for any $n > N$, $n < (\sqrt{\Delta_n})^{-1}$. Then, Lemma 5 (i) implies that

$$\begin{aligned}\varliminf_{n\to+\infty} \frac{\ln^+\ln^+|X_n(\omega)|^{-1}}{\ln\lambda_n} &\geqslant \varliminf_{n\to+\infty} \frac{\ln^+\ln^+(n\Delta_n)^{-1}}{\ln\lambda_n} \geqslant \varliminf_{n\to+\infty} \frac{\ln^+\ln^+(\sqrt{\Delta_n})^{-1}}{\ln\lambda_n}\\ &\geqslant \varliminf_{n\to+\infty} \frac{\ln^+\ln^+(\Delta_n)^{-1}}{\ln\lambda_n}.\end{aligned}$$

By (19), there is a sequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{k\to+\infty} \frac{\ln^+\ln^+(\frac{1}{2}d\Delta_{n_k})^{-1}}{\ln\lambda_{n_k}} = \varliminf_{n\to+\infty} \frac{\ln^+\ln^+(\Delta_n)^{-1}}{\ln\lambda_n}.$$

Combining this with Lemma 5 (ii) we get

$$\begin{aligned}\varliminf_{n\to+\infty} \frac{\ln^+\ln^+|X_n(\omega)|^{-1}}{\ln\lambda_n} &\leqslant \varliminf_{k\to+\infty} \frac{\ln^+\ln^+|X_{n_k}(\omega)|^{-1}}{\ln\lambda_{n_k}} \leqslant \varliminf_{k\to+\infty} \frac{\ln^+\ln^+(\frac{1}{2}d\Delta_{n_k})^{-1}}{\ln\lambda_{n_k}}\\ &\leqslant \lim_{k\to+\infty} \frac{\ln^+\ln^+(\frac{1}{2}d\Delta_{n_k})^{-1}}{\ln\lambda_{n_k}} = \varliminf_{n\to+\infty} \frac{\ln^+\ln^+(\Delta_n)^{-1}}{\ln\lambda_n}.\end{aligned}$$

On the other hand, for any $\omega \in \Omega$, the order of growth of $f_\omega(s)$ is c if and only if

$$\varliminf_{n\to+\infty} \frac{\ln^+ \ln^+ |X_n(\omega)|^{-1}}{\ln \lambda_n} = \frac{c}{c-1}.$$

Hence there exists $\mathbb{A} \in \mathcal{A}$ ($P(\mathbb{A}) = 1$) such that for any ω, (20) holds.

(III) We thirdly prove that there exists $\mathbb{A}^*(\subset \mathbb{A}) \in \mathcal{A}$ ($P(\mathbb{A}^*) = 1$) such that for any $\omega \in \mathbb{A}^*$, $t_0 \in R$ and $\eta > 0$,

$$\varlimsup_{\sigma\to+\infty} \frac{\ln^+ \ln^+ M(\sigma, t_0, \eta, \omega, f)}{\ln \sigma} = c, \tag{40}$$

where $M(\sigma, t_0, \eta, \omega, f) = \sup\{|f(\sigma + \mathrm{i}t, \omega)|\colon\ |t - t_0| < \eta\}$.

Suppose that for $t_j < R$, $\eta_k > 0$, there exists

$$\mathbb{H} = \Big\{ \varlimsup_{\sigma\to+\infty} \frac{\ln^+ \ln^+ M(\sigma, t_j, \eta_k, \omega, f)}{\ln \sigma} < c \Big\} (\subset \mathbb{A}) \in \mathcal{A},$$

such that $P(\mathbb{H}) > 0$. Then for any $\varepsilon \in (0, c-1)$ there exists $\mathbb{H}' \subset \mathbb{H}(P\ (\mathbb{H}') > 0)$ such that for any $\omega \in \mathbb{H}'$, and for σ sufficiently large,

$$M(\sigma, t_j, \eta_k, \omega, f) < \exp(\sigma^{c-\varepsilon}).$$

By Lemma 6, there exist B and $k > 0$ such that for any p, $q \in \mathbb{N}$ verifying $q > p > k$,

$$B \sum_{m=p}^{q} \Delta_n^2 \mathrm{e}^{2\lambda_n \sigma} \leqslant \int_{H_1} \Big| \sum_{n=p}^{q} X_n(\omega) \mathrm{e}^{\lambda_n \sigma} \Big|^2 P(\mathrm{d}\omega),$$

where $\operatorname{Re} s = \sigma$ and $|\operatorname{Im} s - t_j < \eta_k|$. By the Lebesgue theorem,

$$B \sum_{m=p}^{+\infty} \Delta_n^2 \mathrm{e}^{2\lambda_n \sigma} \leqslant \int_{H_1} \Big| \sum_{n=p}^{+\infty} X_n(\omega) \mathrm{e}^{\lambda_n \sigma} \Big|^2 P(\mathrm{d}\omega).$$

Hence for $\omega \in \mathbb{H}'$, and σ sufficiently large,

$$\Delta_n \mathrm{e}^{\lambda_n \sigma} \leqslant B' \exp(\sigma^{c-\varepsilon}),$$

where $n \geqslant p$ and B' is a positive constant.

We would have $\ln \Delta_n \leqslant \sigma^{c-\varepsilon} - \lambda_n \sigma + O(1)$ and by setting $\sigma = (\lambda_n/(c-\varepsilon))^{1/(c-\varepsilon-1)}$

$$\ln \Delta_n^{-1} > \lambda_n^{(c-\varepsilon)/(c-\varepsilon-1)} \Big(\frac{1}{c-\varepsilon}\Big)^{1/(c-\varepsilon-1)} \Big(1 - \frac{1}{c-\varepsilon}\Big) + O(1).$$

Then

$$\underline{\lim}_{n\to+\infty}\frac{\ln^+\ln^+\Delta_n^{-1}}{\ln\lambda_n}\geqslant\frac{c-\varepsilon}{c-\varepsilon-1}>\frac{c}{c-1},\tag{41}$$

which is in contradiction with (19). Hence $\mathbb{H}$ does not exist, so for $t_j\in\mathbb{R}$ and $\eta>0$, there exists $\mathbb{A}_{jk}(\subset\mathbb{A})\in\mathcal{A}$ such that $P(\mathbb{A}_{jk})=1$, where

$$\mathbb{A}_{jk}=\Big\{\omega\in\mathbb{A}\colon\ \overline{\lim_{\sigma\to+\infty}}\frac{\ln^+\ln^+M(\sigma,t_j,\eta_k,\omega,f)}{\ln\sigma}=c\Big\}.\tag{42}$$

Let $\{t_j\}$ be a sequence of rational numbers in $\mathbb{R}$ and $\eta_k\searrow 0$. Then for each pair (j,k), there exists $\mathbb{A}_{jk}$ verifying (42) and $P(\mathbb{A}_{jk})=1$. Let $\mathbb{A}^*=\bigcap\limits_{n=0}^{+\infty}\bigcap\limits_{j+k=0}^{n}\mathbb{A}_{jk}$. Then $P(\mathbb{A}^*)=1$ and for any $\omega\in\mathbb{A}^*$, $t_0\in\mathbb{R}$ and $\eta>0$, (40) holds.

(IV) We fourthly prove that if $\{s_m\}(\subset\mathbb{C})$ verifies

$$m^h\leqslant\operatorname{Re}s_m=\sigma_m\leqslant(m+1)^h\quad(m\in N,h>1),$$

then

$$\overline{\lim_{m\to+\infty}}\frac{\ln^+\ln^+|f(s_m,\omega)|}{\ln\sigma}=c\quad(a.s.).$$

We shall prove that $P(\mathbb{J})=0$, where

$$\mathbb{J}=\Big\{\omega\in\mathbb{A}^*\colon\ \overline{\lim_{m\to+\infty}}\frac{\ln^+\ln^+|f(s_m,\omega)|}{\ln\sigma}<c\Big\}.$$

Suppose that $P(\mathbb{J})>0$. Then there exist $\varepsilon\in(0,c-1)$ and $M_n>0$ such that $P(\mathbb{J}_1)>0$, where

$$\mathbb{J}_1=\Big\{\omega\in\mathbb{J}\colon\ m>M_n,\ \Big|\sum_{n=p}^{+\infty}X_n(\omega)\mathrm{e}^{\lambda_n\sigma_m}\Big|<\exp(\sigma_m^{c-\varepsilon})\Big\}.$$

By Lemma 6, for some $p\in\mathbb{N}$

$$\sum_{n=0}^{+\infty}\Delta_n^2\mathrm{e}^{2\lambda_n\sigma_m}\leqslant K\inf_{\mathbb{J}_1}\Big|\sum_{n=p}^{+\infty}X_n(\omega)\mathrm{e}^{\lambda_n\sigma_m}\Big|\mathbb{P}(\mathrm{d}\omega)<K'\exp(2\sigma_m^{c-\varepsilon}),$$

where K and K' are positive constants. Hence for $n\geqslant p$

$$\Delta_n\mathrm{e}^{\lambda_n\sigma_m}<\exp(\sigma_m^{c-\varepsilon})$$

and

$$\ln \Delta_n < -\lambda_n \sigma_m + \sigma_m^{c-\varepsilon} + K'',$$

where K'' is a positive constant.

Choose $m = m_n$ and σ_{m_n} such that

$$m_n^h \leqslant \Big(\frac{\lambda_n}{c-\varepsilon}\Big)^{1/(c-\varepsilon-1)} \leqslant (m_n+1)^h \quad \text{and} \quad m_n^h \leqslant \sigma_{m_n} \leqslant (m_n+1)^h.$$

Then

$$\Big(\frac{m_n}{m_n+1}\Big)^h \Big(\frac{\lambda_n}{c-\varepsilon}\Big)^{1/(c-\varepsilon-1)} \leqslant \sigma_{m_n} \leqslant \Big(\frac{m_n+1}{m_n}\Big)^h \Big(\frac{\lambda_n}{c-\varepsilon}\Big)^{1/(c-\varepsilon-1)}.$$

We would have

$$\ln \delta_n < \lambda_n^{(c-\varepsilon)/(c-\varepsilon-1)} \Big(\frac{1}{c-\varepsilon}\Big)^{1/(c-\varepsilon-1)} \Big(-\Big(\frac{m_n}{m_n+1}\Big)^h + \Big(\frac{m_n+1}{m_n}\Big)^{h(c-\varepsilon)} \frac{1}{c-\varepsilon}\Big).$$

For n sufficiently large, the expression on the right-hand side of the last inequality would be negative and consequently

$$\varliminf_{n\to+\infty} \frac{\ln^+ \ln^+ \Delta_n^{-1}}{\ln \lambda_n} \neq \frac{c-\varepsilon}{c-\varepsilon-1} > \frac{c}{c-1},$$

which is in contradiction with (19). Hence $P(\mathbb{J}) = 0$.

(V) The equation (36) shows that Φ maps the unit disc $\mathbb{D}(1)$ in the w-plane into the strip $\mathbb{B}(t_0, \eta)$ in the s-plane. Consequently, the mapping Φ transforms the function (3) into a holomorphic function in $\mathbb{D}(1)$ as follows:

$$\psi(w,\omega) = \sum_{n=0}^{+\infty} X_n(\omega) \exp\{\lambda_n \Phi(w)\}. \tag{43}$$

In this step, we prove that for any $\omega \in \mathbb{A}'$ ($P(\mathbb{A}') = 1$) and $\alpha \in \mathbb{C}$, except perhaps for an exceptional value

$$\varlimsup_{R\to 1} \frac{N(R,\omega,\psi=\alpha)}{\ln 1/(1-R)} = +\infty. \tag{44}$$

By Lemma 8, for $\omega \in \mathbb{A}^*$, $0 < R < 1$, $r = (1+R)/(1-R)$ and for $\sigma = (2\eta/\pi)\ln r$ sufficiently large

$$\begin{aligned}\frac{\ln^+ \ln^+ M(\sigma + 2\eta/\pi, t_0, \eta/2, \omega, f)}{\ln \pi\sigma/(2\eta)} &\leqslant \frac{\ln^+ \ln^+ M(R,\omega,\psi)}{\ln \ln 1/(1-R)} \\ &\leqslant \frac{\ln^+ \ln^+ M(\sigma, t_0, \eta, \omega, f)}{\ln(\pi\sigma/(2\eta) - \ln 2)}.\end{aligned}$$

Hence by (41) and by Lemma 9 (i)

$$\varlimsup_{R\to 1}\frac{\ln^+\ln^+ M(R,\omega,\psi)}{\ln\ln 1/(1-R)}=c>1 \quad\text{and}\quad \varlimsup_{R\to 1}\frac{\ln^+\ln^+ T(R,\omega,\psi)}{\ln\ln 1/(1-R)}\leqslant c.$$

We shall prove $P(\mathbb{V})=0$, where

$$\mathbb{V}=\Big\{\omega\in\mathbb{A}^*-\mathbb{J}\colon\ \varlimsup_{R\to 1}\frac{\ln^+\ln^+ T(R,\omega,\psi)}{\ln\ln 1/(1-R)}<c\Big\}.$$

Suppose that $P(\mathbb{V})>0$. Then there exist $\varepsilon\in(0,c-1)$ and $R^*\in(0,1)$ such that $P(\mathbb{V}_1)>0$, where

$$\mathbb{V}_1=\Big\{\omega\in\mathbb{V}\colon\ T(R,\omega,\psi)<\Big(\ln\frac{1}{1-R}\Big)^{c-2\varepsilon},\ 1>R>R^*\Big\}.$$

Then for $\omega\in\mathbb{V}_1$ and $1>R>R^*$

$$\frac{1}{2\pi}\int_{-\pi/4}^{\pi/4}\ln^+|\psi(Re^{i\theta},\omega)|\,d\theta<\Big(\ln\frac{1}{1-R}\Big)^{c-2\varepsilon}.$$

Suppose $R_n>R^*$, $R_n\nearrow 1$ and $(\ln 1/(1-R_n))^{-\varepsilon}=1/(8n^2)$. Let $\upsilon=\sum\limits_{n=1}^{+\infty}1/n^2$. Consider

$$\mathbb{A}_n(\theta,\omega)=\Big\{(\theta,\omega)\colon\ \theta\in\Big[-\frac{\pi}{4},\frac{\pi}{4}\Big],\ \omega\in\mathbb{V}_1,\ |\psi(R_ne^{i\theta},\omega)|>\exp\Big(\ln\frac{1}{1-R_n}\Big)^{c-\varepsilon}\Big\}.$$

Then for a fixed $\omega\in V_1$, $\mathbf{m}\mathbb{A}_n(\theta,\omega)<2\pi(\ln 1/(1-R_n))^{-\varepsilon}$ and

$$m\Big(\bigcup_n\mathbb{A}_n(\theta,\omega)\Big)\leqslant\sum_{n=1}m\mathbb{A}_n(\theta,\omega)<2\pi\sum_n\Big(\ln\frac{1}{1-R_n}\Big)<2\pi\sum_n\frac{1}{8\upsilon n^2}<\frac{\pi}{4},$$

where m denotes the Lebesgue measure on $[-\pi/4,\pi/4]$.

Let $\mathbb{B}(\theta,\omega)=[-\pi/4,\pi/4]\times\mathbb{V}_1-\bigcup\limits_n\mathbb{A}_n(\theta,\omega)$. Then

$$(m\times P)\mathbb{B}(\theta,\omega)\geqslant\frac{1}{3}\pi P(\mathbb{V}_1)-\frac{1}{4}\pi P(\mathbb{V}_1)=\frac{1}{4}\pi P(\mathbb{V}_1). \tag{45}$$

On the other hand, if for any $\theta\in[-\pi/4,\pi/4]$, $P(\mathbb{B}(\theta,\omega))<\frac{1}{2}P(\mathbb{V}_1)$, we would have

$$(m\times P)\mathbb{B}(\theta,\omega)=\int_{-\pi/4}^{\pi/4}d\theta\int_{\mathbb{V}_1}\mathbf{1}_{\mathbb{B}(\theta,\omega)}P(d\omega)<\frac{1}{4}\pi P(\mathbb{V}_1),$$

which is in contradiction with (45). Hence there exists $\theta_0 \in [-\pi/4, \pi/4]$ such that $P(\mathbb{B}(\theta_0, \omega)) \geqslant \frac{1}{2} P(\mathbb{V}_1) > 0$ i.e.

$$P\Big(\Big\{\omega \in \mathbb{V}_1\colon\ |\psi(R_n e^{i\theta_0}, \omega)| \leqslant \exp\Big\{\Big(\ln \frac{1}{1-R_n}\Big)^{c-\varepsilon}\Big\},\ n \in \mathbb{N}\Big\}\Big) > 0.$$

Applying the mapping in (27), we obtain

$$s_n = \Psi(R_n e^{i\theta_0}) \quad (\operatorname{Re} s_n = \sigma_n),$$

and

$$\ln \frac{1}{1-R_n} = O(1)(\sigma_n + O(1)) \quad (n \to +\infty).$$

Hence for $\omega \in \mathbb{V}_1$

$$\varlimsup_{n\to+\infty} \frac{\ln^+ \ln^+ |f(s_n, \omega)|}{\ln \sigma_n} \leqslant c - \varepsilon,$$

which is contrary to the result in (18). Hence $P(\mathbb{V}) = 0$ and for any $\omega \in \mathbb{A}^* - \mathbb{J} - \mathbb{V}$, $(P(\mathbb{A}^* - \mathbb{J} - \mathbb{V}) = 1)$,

$$\varlimsup_{R\to 1} \frac{\ln T(R, \omega, \psi)}{\ln \ln 1/(1-R)} = c \quad \text{and} \quad \varlimsup_{R\to 1} \frac{T(R, \omega, \psi)}{\ln 1/(1-R)} = +\infty.$$

Consequently for any $\omega \in \mathbb{A}' = \mathbb{A}^* - \mathbb{J} - \mathbb{V}$ and $\alpha \in \mathbb{C}$, (44) holds except perhaps for an exceptional value.

(VI) Consider now some non-random holomorphic function in $\mathbb{D}(1)$. For any $M \in \mathbb{N}$, let $\{\Delta_n\}$ be as in (17) and $\{c_j\}_{j=M+1}^{+\infty} \subset \mathbb{C}$ be such that

$$\varliminf_{n\to+\infty} \frac{\ln^+ \ln^+ |\Delta_n c_n|^{-1}}{\ln \lambda_n} = \frac{c}{c-1}. \tag{46}$$

Then by Lemma 5 (i) and (19), as in the above proof,

$$G_M(w) = \sum_{m=M+1}^{+\infty} \Delta_n c_n \exp(\lambda_n \Phi(w)) \tag{47}$$

is holomorphic in $\mathbb{D}(1)$. By (19), we can choose c_n such that (47) and

$$\varlimsup_{R\to 1} \frac{T(R, G_M)}{\ln 1/(1-R)} = +\infty \tag{48}$$

hold.

(VII) In the seventh step, we prove that there exist a point $(c_0, c_1, \ldots, c_M) \in \mathbb{C}^{M+1}$ and a number $\alpha \in \mathbb{C}$ such that

$$\overline{\lim_{R\to 1}} \frac{N(R, G^*(w,c) = \alpha)}{\ln 1/(1-R)} < +\infty \tag{49}$$

and

$$\overline{\lim_{R\to 1}} \frac{T(R, G^*(w,c))}{\ln 1/(1-R)} < +\infty, \tag{50}$$

where

$$G^*(w, \mathbf{c}) = \sum_{n=0}^{M} \Delta_n c_n \exp(\lambda_n \Phi(w)) + G_M(w), \tag{51}$$

and $\mathbf{c} = (c_0, c_1, \ldots, c_M, c_{M+1}, c_{M+2}, \ldots) \in \mathbb{C}^{+\infty}$.

We just prove the case $\lambda_0 > 0$ (the case $\lambda_0 = 0$ can be deduced). In fact, if (49) holds, we can not find a point $(c_0', c_1', \ldots, c_M', \alpha') \neq (c_0, c_1, \ldots, c_M, \alpha) \in \mathbb{C}^{M+1}$ such that (23) $\sim$ (25) hold with $c_0, c_1, \ldots, c_M$ and α replaced by $c_0', c_1', \ldots, c_M'$ and α' respectively. Otherwise, the condition $\lambda_0 > 0$ implies that there exist two different holomorphic functions in $\mathbb{D}(1)$,

$$g_1(w, \mathbf{c}) = \alpha - \sum_{n=0}^{M} \Delta_n c_n \exp(\lambda_n \Phi(w)) \quad \text{and} \quad g_2(w, \mathbf{c}) = \alpha' - \sum_{n=0}^{M} \Delta_n c_n' \exp(\lambda_n \Phi(w)),$$

such that

$$\overline{\lim_{R\to 1}} \frac{N(R, G_M) = g_j}{\ln 1/(1-R)} < +\infty \quad (j = 1, 2),$$

which is in contradiction with Lemma 9 (ii).

(VIII) In the eighth step, we calculate the probability of the event

$$\mathbb{S} = \Big\{\omega \in \mathbb{A}' \colon \text{ there exists } \alpha \in \mathbb{C} \text{ such that } \overline{\lim_{R\to 1}} \frac{N(R, G^*(w,\omega) = \alpha)}{\ln 1/(1-R)} < +\infty\Big\}.$$

Let

$$\mathbb{E}_\infty = \{\mathbf{c} \in \mathbb{C}^\infty \colon\ G^*(w,\omega) \text{ verifies (46), (49) and (50)}\},$$
$$\mathbb{E}_{\infty,M} = \{(c_{M+1}, c_{M+2}, \ldots,) \colon\ \mathbf{c} \in \mathbb{C}^\infty\}$$

and

$$\mathbb{S}_\infty = \{(X_0(\omega), X_1(\omega), \ldots) \colon\ \omega \in \mathbb{S}\} \subset \mathbb{E}_\infty.$$

Consider the probability space $(\mathbb{C}, \mathcal{B}, \mu_n)$ generated by the random variables $X_n(\omega)$, and for any fixed M, let

$$\mu_\infty = \prod_{n=0}^{+\infty}, \quad \tilde{\mu}_M = \prod_{n=0}^{M} \mu_n, \quad \mu_{\infty,M} = \prod_{n=M+1}^{+\infty} \mu_n,$$
$$z = (z_0, z_1, \ldots), \quad \tilde{z}_M = (z_0, z_1, \ldots, z_M) \quad \text{and} \quad z_{\infty,M} = (z_{M+1}, z_{M+2}, \ldots).$$

We have, by Lemma 5 (iii)

$$\begin{aligned} P(\mathbb{S}) &= \int_\Omega \mathbf{1}\mathbb{P}(\mathrm{d}\omega) = \int_{\mathbb{C}^\infty} \mathbf{1}_{\mathbb{S}_\infty} \mu_\infty(\mathrm{d}z) \leqslant \int_{\mathbb{C}^\infty} \mathbf{1}_{\mathbb{E}_\infty} \mu_\infty(\mathrm{d}z) \\ &= \int_{\mathbb{E}_\infty} \mu_{\infty,M}(\mathrm{d}z_{\infty,M}) \int_{\mathbb{C}^{M+1}} \mathbf{1}_{\{z_0 = \Delta_0 c_0, \ldots, z_1 = \Delta_M c_M\}}(\mathrm{d}\tilde{z}_m) \\ &= \int_{\mathbb{E}_{\infty,M}} \prod_{n=0}^{M} \mathbb{P}(\{X_n(\omega) = \Delta_n c_n\}) \mu_{\infty,M}(\mathrm{d}z_{\infty,M}) < \beta^{M+1}. \end{aligned}$$

Letting $M \nearrow +\infty$, we see that $P(\mathbb{S}) = 0$, i.e. for $\omega \in \mathbb{A}' - \mathbb{S}$ $(P(\mathbb{A}' - \mathbb{S}) = 1)$ and $\alpha \in \mathbb{C}$

$$\overline{\lim_{R\to 1}} \frac{N(R, \psi(w,\omega) = \alpha)}{\ln 1/(1-R)} = +\infty,$$
$$\lim_{R\to 1} n(R, \psi(w,\omega) = \alpha) = +\infty$$

and (21) holds for fixed pair (t_0, η).

(IX) We finally complete the proof of Theorem 1.

As in (17), for each pair (t_j, η_k) we can obtain the sets $\mathbb{V}_{jk}$ and $\mathbb{S}_{jk}$ as we have obtained $\mathbb{V}$ and $\mathbb{S}$ in (19) and in (21), and construct the event $\mathbb{A}^{**} = \bigcap_{n=0}^{+\infty} \bigcap_{j+k=0}^{+\infty} (\mathbb{A}_{jk} - \mathbb{J} - \mathbb{V}_{jk} - \mathbb{S}_{jk})$. Then $P(\mathbb{A}^{**}) = 1$ and for any $\omega \in \mathbb{A}^{**}$, $\{s\colon \operatorname{Im} s = t_0\}$ is a Julia line without an exceptional value of $f(s,\omega)$. Hence $\mathbb{A}$ in Theorem 3 is replaced by $\mathbb{A}^{**}$. $\square$

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References

[1] *P. L. Davies*: Some results on the distribution of zeros of random entire functions. Proc. Lond. Math. Soc., III. Ser. *26* (1973), 99–141.

[2] *X. Ding, J. Yu*: Picard points of random Dirichlet series. Bull. Sci. Math. *124* (2000), 225–238.

[3] *J.-P. Kahane*: Some Random Series of Functions (2nd, ed.). Cambridge University Press, Cambridge, 1985.

[4] *J. E. Littlewood, A. C. Offord*: On the distribution of zeros and a-values of a random integral function. Ann. Math. *49* (1948), 885–952; Errata. Ann. Math. *50* (1949), 990–991.
[5] *R. Nevanlinna*: Le Théorème de Picard-Borel et la Théorie des Functions Méromorphes. Gauthier-Villiars, Paris, 1929. (In French.)
[6] *R. E. A. C. Paley, A. Zygmund*: On some series of functions. I, II. Proceedings Cambridge Philos. Soc. *26* (1930), 337–357, 458–474.
[7] *R. E. A. C. Paley, A. Zygmund*: On some series of functions. III. Proc. Camb. Philos. Soc. *28* (1932), 190–205.
[8] *D. C. Sun, J. R. Yu*: Sur la distribution des valeurs de certaines séries aléatoires de Dirichlet. II. C. R. Acad. Sci., Paris, Sér. I *308* (1989), 205–207. (In French.)
[9] *D. C. Sun, J. R. Yu*: On the distribution of values of random Dirichlet series. II. Chin. Ann. Math., Ser. B *11* (1990), 33–44.
[10] *F. J. Tian, D. C. Sun, J. R. Yu*: On random Dirichlet series. (Sur les séries aléatoires de Dirichlet). C. R. Acad. Sci., Paris, Sér. I, Math. *326* (1998), 427–431. (In French. Abridged English version.)
[11] *C.-Y. Yu*: Sur les droites de Borel de certaines fonctions entières. Ann. Sci. Éc. Norm. Supér., III. Sér. *68* (1951), 65–104. (In French.)
[12] *J. R. Yu*: Some properties of random Dirichlet series. Acta Math. Sin. *21* (1978), 97–118. (In Chinese.)
[13] *J. R. Yu*: Sur quelques séries gaussiennes de Dirichlet. (On some gaussian Dirichlet series). C. R. Acad. Sci., Paris, Sér. I *300* (1985), 521–522. (In French.)
[14] *J. R. Yu*: Borel lines of random Dirichlet series. Acta Math. Sci., Ser. B, Engl. Ed. *22* (2002), 1–8.
[15] *J. R. Yu*: Julia lines of random Dirichlet series. Bull. Sci. Math. *128* (2004), 341–353.
[16] *J. R. Yu, D. C. Sun*: On the distribution of values of random Dirichlet series. I. Lectures on complex analysis, Proc. Symp., Xian/China 1987 (1988), 67–95.

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