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IMPULSIVE BOUNDARY VALUE PROBLEMS FOR $p(t)$-LAPLACIAN'S VIA CRITICAL POINT THEORY

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Abstract. In this paper we investigate the existence of solutions to impulsive problems with a $p(t)$-Laplacian and Dirichlet boundary value conditions. We introduce two types of solutions, namely a weak and a classical one which coincide because of the fundamental lemma of the calculus of variations. Firstly we investigate the existence of solution to the linear problem, i.e. a problem with a fixed rigth hand side. Then we use a direct variational method and next a mountain pass approach in order to get the existence of at least one weak solution to the nonlinear problem.

Keywords: $p(t)$-Laplacian, impulsive condition, critical point, variational method, Dirichlet problem

MSC 2010: 34B37, 47J30

1. Introduction

The study of impulsive boundary value problems is important due to its various applications in which abrupt changes at certain times in the evolution process appear. The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of "impulses". Such problems arise in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, see [1].

Another vital area of research within boundary value problems is the investigation of the so called $p(x)$-Laplacian problems which began in [3] and was later developed by many authors, see [7] for an up to date study of such boundary value problems. Such problems model various phenomena arising in the study of elastic mechanics (see [16]), electrorheological fluids (see [11]) or image restoration (see [2]).

Therefore we believe it is important to investigate a Dirichlet problem with a $p(t)$-Laplacian, i.e. a one—dimensional counterpart of the $p(x)$-Laplacian, subject to some impulsive changes. In our research we mainly follow the approach applied in [10] with one significant difference. The existence results for problems with a fixed right hand side in [10] were proved via the Lax-Milgram Lemma and in our paper we apply a direct method of the calculus of variations together with the Fundamental Lemma of the calculus of variations which we prove in the case of functions from relevant Orlicz-Sobolev spaces. It is the variational approach for boundary value problems with a $p(x)$-Laplacian that prevails in the literature, see again [7], while for impulsive problems, the variational approach has only recently begun and most results have been obtained by other methods, see [5], [8].

The variational investigation of impulsive problems inspired by [10] has received a lot of attention recently. In [6] another variational framework for the Sturm-Liuville boundary value problem is developed in the case of second order impulsive ordinary differential equation of p-Laplacian type. Boundary value problems with dependence on a first order derivative are developed in [12], while [15] considers problems similar to those of [10] but for a slightly more general problem. Periodic solutions with impulses are considered via critical point theory in [14] within the framework sketched in [10].

The paper is organized as follows. Firstly, we consider the Fundamental Lemma of calculus of variations for the Orlicz-Sobolev spaces. Next, we investigate the problem with a fixed right hand side. Later we investigate nonlinear problems by the direct method of calculus of variations and by Mountain Pass Geometry.

2. Mathematical preliminaries

Let $p, q \in C([0,\pi], \mathbb{R}^+)$, $1/p(t) + 1/q(t) = 1$ for $t \in [0,\pi]$. In this paper we assume $p^- = \inf_{t\in[0,\pi]} p(t) > 1$, $p^+ = \sup_{t\in[0,\pi]} p(t) \leqslant 2$. By $L^{p(t)}(0,\pi)$ we mean the space

$$L^{p(t)}([0,\pi]) = \left\{u;\ u\colon\ [0,\pi] \to \mathbb{R} \quad \text{is measurable,}\ \int_0^\pi |u(t)|^{p(t)}\,\mathrm{d}t < +\infty\right\}$$

equipped with the norm

$$\|u\|_{L^{p(t)}} = \inf\left\{\lambda > 0;\ \int_0^\pi \left|\frac{u(t)}{\lambda}\right|^{p(t)}\mathrm{d}t \leqslant 1\right\}$$

and this is a Banach space called the generalized Lebesgue space, see [4].

$C_0^\infty(0,\pi)$ denotes the space of infinitely many times differentiable functions with compact support on $[0,\pi]$. $W^{1,p(t)}(0,\pi)$ is the generalized Orlicz-Sobolev space, namely

$$W^{1,p(t)}(0,\pi)=\Big\{u;\ u\in L^{p(t)}(0,\pi),\ \int_0^\pi\Big|\frac{\mathrm d}{\mathrm dt}u(t)\Big|^{p(t)}\,\mathrm dt<+\infty\Big\},$$

where the derivative $\mathrm d/\mathrm dt$ stands for the weak one, i.e. $(\mathrm d/\mathrm dt)u$ is an element of $L^{p(t)}(0,\pi)$ which satisfies

$$\int_0^\pi\frac{\mathrm d}{\mathrm dt}u(t)v(t)\,\mathrm dt=-\int_0^\pi u(t)\frac{\mathrm d}{\mathrm dt}v(t)\,\mathrm dt \tag{1}$$

for all $v\in C_0^\infty(0,\pi)$. It is apparent that any function belonging to $W^{1,p(t)}(0,\pi)$ is in fact absolutely continuous and so the weak derivative can be considered as an a.e. derivative which is what we understand for the remainder of this paper. Consider $W^{1,p(t)}(0,\pi)$ with the following norm

$$\|u\|_{W^{1,p(t)}}=\sqrt{\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}+\|u\|_{L^{p(t)}}}. \tag{2}$$

Now $W_0^{1,p(t)}(0,\pi)$ is the closure of $C_0^\infty(0,\pi)$ in $W^{1,p(t)}(0,\pi)$, see [4]. The norm in $W_0^{1,p(t)}(0,\pi)$ is

$$\|u\|_{W_0^{1,p(t)}}=\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}$$

which is equivalent to (2). Moreover, from [4] we see that there exist constants $C_1,C_2>0$ such that (the Poincaré inequality)

$$\|u\|_{L^{p(t)}}\leqslant C_1\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}\quad\text{for all } u\in W_0^{1,p(t)}(0,\pi)$$

and

$$\max_{t\in[0,\pi]}|u(t)|\leqslant C_2\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}\quad\text{for all } u\in W_0^{1,p(t)}(0,\pi).$$

The functional $u\to\int_0^\pi|(\mathrm d/\mathrm dt)u(t)|^{p(t)}\,\mathrm dt$ is called the modular for $W_0^{1,p(t)}(0,\pi)$. We have the following relation between the modular and the norm

$$\begin{aligned}\min\Big\{\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}^{p^-},\ \Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}^{p^+}\Big\}&\leqslant\int_0^\pi\Big|\frac{\mathrm d}{\mathrm dt}u(t)\Big|^{p(t)}\,\mathrm dt\\&\leqslant\max\Big\{\Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}^{p^-},\ \Big\|\frac{\mathrm d}{\mathrm dt}u\Big\|_{L^{p(t)}}^{p^+}\Big\}.\end{aligned}$$

Let us consider an operator $L\colon W_0^{1,p(t)}(0,\pi)\to(W_0^{1,p(t)}(0,\pi))^*$ given by

$$\langle L(u),v\rangle=\int_0^\pi\Big|\frac{\mathrm d}{\mathrm dt}u(t)\Big|^{p(t)-2}\frac{\mathrm d}{\mathrm dt}u(t)\frac{\mathrm d}{\mathrm dt}v(t)\,\mathrm dt \tag{3}$$

for $u, v \in W_0^{1,p(t)}(0,\pi)$. Then L is a homeomorphism ([3], Theorem 3.1) and the Gâteaux derivative of $u \to \int_0^\pi |(\mathrm{d}/\mathrm{d}t)u(t)|^{p(t)}\,\mathrm{d}t$ is given by (3).

3. Remarks on the fundamental lemma of the calculus of variations

Now we provide a classical lemma of the calculus of variations which we formulate for generalized Orlicz-Sobolev spaces in order to get some regularity results for $p(t)$-Laplacian problems different from known results in the literature. Note that the proofs of subsequent lemmas are modifications of well known proofs in the area of the calculus of variations (see [13]). We note that in this paper we have assumed $p^+ \leqslant 2$ and $p^- > 1$ so $L^{q(t)}(0,\pi) \subset L^{p(t)}(0,\pi)$. Thus we see that $W_0^{1,q(t)}(0,\pi) \subset W_0^{1,p(t)}(0,\pi)$.

Lemma 1. *If $h \in L^{q(t)}(0,\pi)$ and*

$$\int_0^\pi h(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t = 0 \tag{4}$$

for all $v \in W_0^{1,p(t)}(0,\pi)$, then there exists a constant $c \in \mathbb{R}$ such that $h(t) = c$ a.e. on $[0,\pi]$.

Proof. Put $c = \pi^{-1}\int_0^\pi h(t)\,\mathrm{d}t$. For this c the function $v(t) = \int_0^t (h(\tau) - c)\,\mathrm{d}\tau$ is in $W_0^{1,p(t)}(0,\pi) \cap W_0^{1,q(t)}(0,\pi) = W_0^{1,q(t)}(0,\pi)$. Indeed, $(\mathrm{d}/\mathrm{d}t)v(t) = h(t) - c$ a.e. on $[0,\pi]$, $v(0) = v(\pi) = 0$ and the derivative is understood here as a classical a.e. derivative. From (4) we obtain that

$$\begin{aligned}0 \leqslant \int_0^\pi (h(t)-c)^2\,\mathrm{d}t &= \int_0^\pi (h(t)-c)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t \\ &= \int_0^\pi h(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + c\int_0^\pi \frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t = 0.\end{aligned}$$

Hence it follows that for almost all $t \in [0,\pi]$ we have $h(t) - c = 0$. □

Lemma 2. *If $g \in L^1(0,\pi)$, $h \in L^{q(t)}(0,\pi)$ and*

$$\int_0^\pi \Big(g(t)v(t) + h(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\Big)\,\mathrm{d}t = 0$$

for all $v \in W_0^{1,p(t)}(0,\pi)$, then $(\mathrm{d}/\mathrm{d}t)h = g$ a.e. on $[0,\pi]$ and $(\mathrm{d}/\mathrm{d}t)h \in L^1(0,\pi)$.

Proof. We define $G(t) = \int_0^t g(\tau)\,\mathrm{d}\tau$. Then G is absolutely continuous and $(\mathrm{d}/\mathrm{d}t)G(t) = g(t)$ for a.e. $t \in [0,\pi]$. Integrating by parts we obtain that

$$\begin{aligned}\int_0^\pi \Big(g(t)v(t) + h(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\Big)\,\mathrm{d}t &= \int_0^\pi (h(t) - G(t))\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + G(t)v(t)|_0^\pi \\ &= \int_0^\pi (h(t) - G(t))\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t = 0.\end{aligned}$$

Thus by Lemma 1 we obtain that $h(t)-G(t)=c$ and the assertion of the proposition follows. $\square$

4. Existence with fixed right hand side

Let the numbers $0<t_1<t_2<\ldots<t_m<\pi$ be fixed throughout the paper. Let us consider the following problem in $W_0^{1,p(t)}(0,\pi)$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big)=f(t) \qquad x(0)=x(\pi)=0 \tag{5}$$

with impulsive conditions

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+)-\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-)=d_j \quad \text{for } j=1,2,\ldots,m \tag{6}$$

and where $f\in L^1(0,\pi)$. In the condition (6) it is assumed that both limits $\lim_{t\to t_j^+}|(\mathrm{d}/\mathrm{d}t)x(t)|^{p(t)-2}(\mathrm{d}/\mathrm{d}t)x(t)$, $\lim_{t\to t_j^-}|(\mathrm{d}/\mathrm{d}t)x(t)|^{p(t)-2}(\mathrm{d}/\mathrm{d}t)x(t)$ exist and the required equality holds.

We introduce two types of solution for the problem (5)–(6).

Weak solution: We call a function $x\in W_0^{1,p(t)}(0,\pi)$ a weak solution to (5)–(6) if it satisfies

$$\int_0^\pi\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\,\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t+\sum_{j=1}^m \mathrm{d}_j v(t_j)\,\mathrm{d}t-\int_0^\pi f(t)v(t)\,\mathrm{d}t=0 \tag{7}$$

for all $v\in W_0^{1,p(t)}(0,\pi)$.

Classical solution: A function $x\in W_0^{1,p(t)}(0,\pi)$ is called a classical solution to (5)–(6) if it is a weak solution such that the function $|(\mathrm{d}/\mathrm{d}t)x(\cdot)|^{p(\cdot)-2}(\mathrm{d}/\mathrm{d}t)x(\cdot)$ is absolutely continuous on $[0,\pi]$, the limits in (6) are defined and the relation (6) holds together with the boundary condition $x(0)=x(\pi)=0$ and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big)=f(t)$$

is satisfied for a.e. $t\in[0,\pi]$ and $(\mathrm{d}/\mathrm{d}t)(|(\mathrm{d}/\mathrm{d}t)x(t)|^{p(t)-2}(\mathrm{d}/\mathrm{d}t)x(t))\in L^1(0,\pi)$.

Lemma 3. *If $\bar{u}$ is a weak solution to* (5)–(6), *then it is also a classical one.*

P r o o f. Let the function $\bar{u}$ satisfy (7), i.e.

$$\int_0^{\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t+\sum_{j=1}^{m}\mathrm{d}_j v(t_j)-\int_0^{\pi}f(t)v(t)\,\mathrm{d}t=0 \tag{8}$$

for all $v\in W_0^{1,p(t)}(0,\pi)$.

Now, we shall show that $\bar{u}$ is a classical solution. Let us take any interval (t_j,t_{j+1}) and a function $v\in W_0^{1,p(t)}(t_j,t_{j+1})$. Extend function v to $W_0^{1,p(t)}(0,\pi)$ by taking 0 outside (t_j,t_{j+1}). Then we have from (8)

$$\int_{t_j}^{t_{j+1}}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t-\int_{t_j}^{t_{j+1}}f(t)v(t)\,\mathrm{d}t=0.$$

An application of Lemma 2 shows that the function $|(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot)|^{p(\cdot)-2}(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot)$ is absolutely continuous, its derivative exists for a.e. $t\in(t_j,t_{j+1})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(\cdot)\Big|^{p(\cdot)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(\cdot)\Big)\Big|_{(t_j,t_{j+1})}\in L^1(t_j,t_{j+1}).$$

Therefore both limits

$$\lim_{t\to t_j^+}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t),\quad \lim_{t\to t_{j+1}^-}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)$$

exist. Taking intervals (t_j,t_{j+1}) for $j=0,1,2,\ldots,m$ we see that (5) is satisfied a.e. on $[0,\pi]$ and that $(\mathrm{d}/\mathrm{d}t)(|(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot)|^{p(\cdot)-2}(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot))\in L^1(0,\pi)$. Hence, we may multiply (5) with $x=\bar{u}$ by any $v\in W_0^{1,p(t)}(0,\pi)$ and obtain

$$\int_0^{\pi}-\frac{\mathrm{d}}{\mathrm{d}t}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)v(t)\,\mathrm{d}t-\int_0^{\pi}f(t)v(t)\,\mathrm{d}t=0.$$

By integrating by parts it follows for a fixed interval $[t_j,t_{j+1}]$ that

$$\begin{aligned}\int_{t_j}^{t_{j+1}}&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big)v(t)\,\mathrm{d}t\\&=\int_{t_j}^{t_{j+1}}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\\&\quad+\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+)v(t_j^+)-\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)v(t_{j+1}^-).\end{aligned}\tag{9}$$

Note that

$$\sum_{j=0}^{m}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})v(t_{j+1}^{-}) \\ -\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})v(t_{j}^{+})=\sum_{j=1}^{m}\mathrm{d}_j v(t_j).$$

Summing (9) for $j=0,1,2,\ldots,m$ we get recalling that v is continuous

$$\int_0^{\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t-\int_0^{\pi}f(t)v(t)\,\mathrm{d}t \\ +\sum_{j=1}^{m}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})-\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big)v(t_j)=0. \tag{10}$$

Since $\bar{u}$ is a weak solutions we get equating (8) and (10) that

$$\sum_{j=1}^{m}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})-\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big)v(t_j)=\sum_{j=1}^{m}\mathrm{d}_j v(t_j). \tag{11}$$

Since (11) holds for arbitrary v it follows that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j}^{+})-\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^{-})=\mathrm{d}_j$$

for $j=1,2,\ldots,m$ and therefore the impulsive conditions (6) are also satisfied. $\square$

The action functional $J\colon W_0^{1,p(t)}(0,\pi)\to\mathbb{R}$ corresponding to (5)–(6) is as follows

$$J(u)=\int_0^{\pi}\frac{1}{p(t)}\Big|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big|^{p(t)}\,\mathrm{d}t+\sum_{j=1}^{m}\mathrm{d}_j u(t_j)-\int_0^{\pi}f(t)u(t)\,\mathrm{d}t.$$

Lemma 4. *J is a Gâteaux differentiable, weakly l.s.c. and coercive functional and its critical points correspond to the classical solutions of* (5)–(6).

P r o o f. Note that J is differentiable. Now we take an arbitrary $u\in W_0^{1,p(t)}(0,\pi)$ and fix an arbitrary $v\in W_0^{1,p(t)}(0,\pi)$. Then the Gâteaux derivative is

$$J'(u;v)=\int_0^{\pi}\Big|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}u(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t+\sum_{j=1}^{m}\mathrm{d}_j v(t_j)-\int_0^{\pi}f(t)v(t)\,\mathrm{d}t.$$

Therefore each critical point of J is a weak solution of (5)–(6).

Let us take any weakly convergent sequence $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,p(t)}(0,\pi)$. This means that $\{(\mathrm{d}/\mathrm{d}t)u_k\}_{k=1}^{\infty}$ is weakly convergent in $L^{p(t)}(0,\pi)$. Then $\{u_k\}_{k=1}^{\infty}$ has a subsequence $\{u_{k_n}\}_{n=1}^{\infty}$ which is strongly convergent in $L^{p(t)}(0,\pi)$ and convergent in $C[0,\pi]$. Denote by $\bar{u} \in W_0^{1,p(t)}(0,\pi)$ the weak limit of $\{u_k\}_{k=1}^{\infty}$. Hence,

$$\begin{aligned}\liminf_{n\to\infty} J(u_{k_n}) &\geqslant \liminf_{n\to\infty} \int_0^{\pi} \frac{1}{p(t)} \Big|\frac{\mathrm{d}}{\mathrm{d}t}u_{k_n}(t)\Big|^{p(t)}\,\mathrm{d}t \\ &\quad + \lim_{n\to\infty} \Big(\sum_{j=1}^{m} \mathrm{d}_j u_{k_n}(t_j)\,\mathrm{d}t - \int_0^{\pi} f(t)u_{k_n}(t)\,\mathrm{d}t\Big) \geqslant J(\bar{u}).\end{aligned}$$

Hence J is weakly l.s.c. on $W_0^{1,p(t)}(0,\pi)$. Moreover, we see that for any $u \in W_0^{1,p(t)}(0,\pi)$

$$\begin{aligned}\int_0^{\pi} \frac{1}{p(t)}\Big|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big|^{p(t)}\,\mathrm{d}t + \sum_{j=1}^{m} d_j u(t_j) - \int_0^{\pi} f(t)u(t)\,\mathrm{d}t \\ \geqslant \frac{1}{p^+}\min\Big\{\Big\|\frac{\mathrm{d}}{\mathrm{d}t}u\Big\|_{L^{p(t)}}^{p^-},\ \Big\|\frac{\mathrm{d}}{\mathrm{d}t}u\Big\|_{L^{p(t)}}^{p^+}\Big\} \\ -C_2\sum_{j=1}^{m}\mathrm{d}_j\|u\|_{W_0^{1,p(t)}} C_1\|u\|_{W_0^{1,p(t)}}\|f\|_{L^{q(t)}}.\end{aligned}$$

Since $p^- > 1$ we see that J is coercive.

Thus J is Gâteaux differentiable, weakly l.s.c. and coercive on $W_0^{1,p(t)}(0,\pi)$. Therefore there exists $\bar{u} \in W_0^{1,p(t)}(0,\pi)$ such that $J(\bar{u}) = \inf\limits_{u\in W_0^{1,p(t)}(0,\pi)} J(u)$ and thus $\bar{u}$ satisfies (8). An application of Lemma 3 finishes the proof. ☐

5. Existence of solutions for a nonlinear problem by a direct variational method

In this section we will apply a direct variational argument to a so called nonlinear problem.

Let $I_j\colon \mathbb{R} \to \mathbb{R}$ be continuous functions, $j = 1,2,\ldots,m$, and let $f\colon [0,\pi]\times\mathbb{R}$ be a Caratheodory function. Let $F(t,v) = \int_0^v f(t,\tau)\,\mathrm{d}\tau$. Assume that

(H1) *there exist constants M_j for $j = 1,2,\ldots,m$ such that*

$$|I_j(v)| \leqslant M_j \quad \text{for all } v \in \mathbb{R};$$

(H2) *there exist a constant $\alpha > 0$ and functions $l \in L^1(0,\pi)$, $s \in C[0,\pi]$, with $1 < s^- \leqslant s^+ < p^-$, such that for all $v \in \mathbb{R}$ and a.e. $t \in [0,\pi]$*

$$F(t,v) \geqslant -\alpha|v|^{s(t)} + l(t);$$

(H3) *for each* $r > 0$ *there exist functions* $g_r \in L^1(0,\pi)$ *and* $h_r \in L^1(0,\pi)$ *such that for all* $v \in W_0^{1,p(t)}(0,\pi)$ *satisfying* $\|v\|_{W_0^{1,p(t)}} \leqslant r$ *and a.e.* $t \in [0,\pi]$

$$|F(t,v(t))| \leqslant g_r(t) \text{ and } |f(t,v(t))| \leqslant h_r(t).$$

Now we consider in $W_0^{1,p(t)}(0,\pi)$ the following problem

$$\begin{gathered} -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big) + f(t,x(t)) = 0 \\ x(0) = x(\pi) = 0 \end{gathered} \tag{12}$$

with impulsive conditions

$$\begin{aligned} \Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+) - \Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-) \\ = I_j(x(t_j)) \quad \text{for } j = 1,2,\ldots,m. \end{aligned} \tag{13}$$

As before in the condition (13) it is assumed that both limits

$$\lim_{t\to t_j^+}\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t), \quad \lim_{t\to t_j^-}\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

exist and the given equality holds.

We call a function $x \in W_0^{1,p(t)}(0,\pi)$ a weak solution to (12)–(13) if it satisfies

$$\begin{aligned} \int_0^\pi \Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + \sum_{j=1}^m I_j(x(t_j))v(t_j)\,\mathrm{d}t \\ + \int_0^\pi f(t,x(t))v(t)\,\mathrm{d}t = 0 \end{aligned} \tag{14}$$

for all $v \in W_0^{1,p(t)}(0,\pi)$.

A function $x \in W_0^{1,p(t)}(0,\pi)$ is called a classical solution to (12)–(13) if it is a weak solution such that the function $|(\mathrm{d}/\mathrm{d}t)x(\cdot)|^{p(\cdot)-2}(\mathrm{d}/\mathrm{d}t)x(\cdot)$ is absolutely continuous on $[0,\pi]$, the limits in (13) are defined and the relation (13) holds together with the boundary condition $x(0) = x(\pi) = 0$ and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big) + f(t,x(t)) = 0$$

is satisfied for a.e. $t \in [0,\pi]$ and $(\mathrm{d}/\mathrm{d}t)(|(\mathrm{d}/\mathrm{d}t)x(t)|^{p(t)-2}(\mathrm{d}/\mathrm{d}t)x(t)) \in L^1(0,\pi)$.

The action functional $J\colon W_0^{1,p(t)}(0,\pi) \to \mathbb{R}$ corresponding to (12)–(13) is

$$J(u) = \int_0^\pi \frac{1}{p(t)}\Big|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big|^{p(t)}\,\mathrm{d}t + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t + \int_0^\pi F(t,u(t))\,\mathrm{d}t. \tag{15}$$

Lemma 5. *If $\bar{u}$ is a weak solution to* (12)–(13), *then it is also a classical one.*

P r o o f. Let a function $\bar{u}$ satisfy (14). As in the proof of Lemma 3 we take any interval (t_j, t_{j+1}) and a function $v \in W_0^{1,p(t)}(t_j, t_{j+1})$ extended to $W_0^{1,p(t)}(0, \pi)$ by taking 0 outside (t_j, t_{j+1}). Then we have in (14)

$$\int_{t_j}^{t_{j+1}} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2} \frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + \int_{t_j}^{t_{j+1}} f(t, \bar{u}(t))v(t)\,\mathrm{d}t = 0.$$

It follows from (H3) that $f(\cdot, \bar{u}(\cdot))|_{(t_j,t_{j+1})} \in L^1(t_j, t_{j+1})$. Then, by Lemma 2 $|(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot)|^{p(\cdot)-2}(\mathrm{d}/\mathrm{d}t)\bar{u}(\cdot)$ is absolutely continuous on $[t_j, t_{j+1}]$, its derivative exists for a.e. $t \in (t_j, t_{j+1})$ and belongs to $L^1(t_j, t_{j+1})$. Next we obtain

$$\begin{aligned}\int_0^{\pi} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)-2} \frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + \int_0^{\pi} f(t, \bar{u}(t))v(t)\,\mathrm{d}t \\ + \sum_{j=1}^{m}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+) - \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)\Big)v(t_j) = 0.\end{aligned}$$

Since $\bar{u}$ is a weak solutions we get equating the above relation and (14) that

$$\begin{aligned}\sum_{j=1}^{m}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+) - \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_{j+1}^-)\Big)v(t_j) \\ = \sum_{j=1}^{m} I_j(\bar{u})v(t_j).\end{aligned}$$

Hence for all $j = 1, 2, \ldots, m$ we have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^+) - \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t_j^-) = I_j(\bar{u}).$$

□

Lemma 6. *J is a Gâteaux differentiable, weakly l.s.c. and coercive functional and its critical points correspond to the classical solutions of* (12)–(13).

P r o o f. By the assumption (H3) we see that J is well defined on $W_0^{1,p(t)}(0, \pi)$. Again by (H3) we see that J is Gâteaux differentiable. Indeed, it suffices to show that $u \to \int_0^{\pi} F(t, u(t))\,\mathrm{d}t$ is differentiable in the sense of Gâteaux. Let us fix $u \in W_0^{1,p(t)}(0, \pi)$. Now we fix an arbitrary $v \in W_0^{1,p(t)}(0, \pi)$ and take any $\varepsilon \in (-1, 1)$. Then there exists a constant $r > 0$ and functions $g_r \in L^1(0, \pi)$ and $h_r \in L^1(0, \pi)$ such that $\|u + \varepsilon v\|_{W_0^{1,p(t)}} \leqslant r$ and for a.e. $t \in [0, \pi]$

$$|F(t, v(t))| \leqslant g_r(t) \text{ and } |f(t, v(t))| \leqslant h_r(t).$$

Then we can differentiate the auxiliary function $g(\varepsilon) = \int_0^\pi F(t, u(t) + \varepsilon v(t))\,\mathrm{d}t$ at 0. This proves the Gâteaux differentiability.

Let us take an arbitrary $u \in W_0^{1,p(t)}(0,\pi)$ and fix $v \in W_0^{1,p(t)}(0,\pi)$. Then the Gâteaux derivative is

$$J'(u;v) = \int_0^\pi \Bigl|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Bigr|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}u(t)\frac{\mathrm{d}}{\mathrm{d}t}v(t)\,\mathrm{d}t + \sum_{j=1}^m I_j(u)v(t_j) + \int_0^\pi f(t,u(t))v(t)\,\mathrm{d}t.$$

Therefore each critical point of J is a weak solution of (12)–(13).

Let us take any weakly convergent sequence $\{u_k\}_{k=1}^\infty \subset W_0^{1,p(t)}(0,\pi)$. This means that $\{(\mathrm{d}/\mathrm{d}t)u_k\}_{k=1}^\infty$ is weakly convergent in $L^{p(t)}(0,\pi)$. Then $\{u_k\}_{k=1}^\infty$ has a subsequence $\{u_{k_n}\}_{n=1}^\infty$ which is strongly convergent in $L^{p(t)}(0,\pi)$ and convergent in $C[0,\pi]$. Denote by $\bar{u} \in W_0^{1,p(t)}(0,\pi)$ the weak limit of $\{u_k\}_{k=1}^\infty$. Then by continuity we have that

$$\sum_{j=1}^m \int_0^{u_{k_n}(t_j)} I_j(t)\,\mathrm{d}t \to \sum_{j=1}^m \int_0^{\bar{u}(t_j)} I_j(t)\,\mathrm{d}t.$$

Since $\{u_{k_n}\}_{n=1}^\infty$ is weakly convergent in $W_0^{1,p(t)}(0,\pi)$ there exist a constant $r > 0$ such that

$$\Bigl\|\frac{\mathrm{d}}{\mathrm{d}t}u_{k_n}\Bigr\|_{L^{p(t)}} \leqslant r$$

for all $n \in \mathbb{N}$. Now from (H3) there exists a function $g_r \in L^1(0,\pi)$ such that $|F(t, u_{k_n}(t))| \leqslant g_r(t)$ for a.e. $t \in [0,\pi]$. Then by the Lebesgue Dominated Convergence Theorem we get

$$\int_0^\pi F(t, u_{k_n}(t))\,\mathrm{d}t \to \int_0^\pi F(t, \bar{u}(t))\,\mathrm{d}t.$$

Therefore, J is weakly l.s.c. on $W_0^{1,p(t)}(0,\pi)$.

Moreover, we see that for any $u \in W_0^{1,p(t)}(0,\pi)$

$$\begin{aligned}(16)\qquad J(u) \geqslant{} & \frac{1}{p^+}\min\Bigl\{\Bigl\|\frac{\mathrm{d}}{\mathrm{d}t}u\Bigr\|_{L^{p(t)}}^{p^-}, \Bigl\|\frac{\mathrm{d}}{\mathrm{d}t}u\Bigr\|_{L^{p(t)}}^{p^+}\Bigr\} - \alpha\int_0^\pi |u(t)|^{s(t)}\,\mathrm{d}t \\ & + \int_0^\pi l(t)\,\mathrm{d}t - C_2\sum_{j=1}^m M_j\Bigl\|\frac{\mathrm{d}}{\mathrm{d}t}u\Bigr\|_{L^{p(t)}}.\end{aligned}$$

Since $s^+ < p^-$ it follows from (16) that J is coercive.

Thus J is Gâteaux differentiable, weakly l.s.c. and coercive on $W_0^{1,p(t)}(0,\pi)$. Therefore there exists $\bar{u} \in W_0^{1,p(t)}(0,\pi)$ such that $J(\bar{u}) = \inf\limits_{u \in W_0^{1,p(t)}(0,\pi)} J(u)$ and thus $\bar{u}$ satisfies (14). An application of Lemma 3 finishes the proof. □

6. Existence of mountain pass solutions

In this section we consider the problem

$$(17)\qquad -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big) = f(t,x(t))$$
$$x(0) = x(\pi) = 0$$

with impulsive conditions for $j = 1,2,\ldots,m$

$$(18)\qquad \Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^+) - \Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-)\Big|^{p(t)-2}\frac{\mathrm{d}}{\mathrm{d}t}x(t_j^-) + I_j(x(t_j)) = 0,$$

with assumptions which do not yield the coercivity of the action functional $J\colon W_0^{1,p(t)}(0,\pi) \to \mathbb{R}$

$$(19)\qquad J(u) = \int_0^\pi \frac{1}{p(t)}\Big|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\Big|^{p(t)}\,\mathrm{d}t - \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t - \int_0^\pi F(t,u(t))\,\mathrm{d}t.$$

Now (H1) reads: Let $I_j\colon \mathbb{R}\to\mathbb{R}$ be continuous functions, $j = 1,2,\ldots,m$, and let $f\colon [0,\pi]\times\mathbb{R}$ be a Caratheodory function. We assume that (H1) holds and additionally

(H4) *there exists a constant* $\theta > p^+$ *such that for* $v\in\mathbb{R}$, $v\neq 0$ *and a.e.* $t\in[0,\pi]$ *and all* $j = 1,2,\ldots,m$

$$0 < \theta F(t,v) \leqslant v f(t,v),$$
$$0 < \theta\int_0^v I_j(s)\,\mathrm{d}s \leqslant v I_j(v);$$

(H5) *there exist constants* $\beta_1,\beta_2,\beta_1^j,\beta_2^j,\alpha,\alpha_j > 0$ *with* $\alpha > p^+$, $\alpha_j > p^+$ *for* $j = 1,2,\ldots,m$ *and such that for all* $v\in\mathbb{R}$ *and a.e.* $t\in[0,\pi]$

$$|f(t,v)| \leqslant \beta_1|v|^{\alpha-1} + \beta_2,$$
$$|I_j(v)| \leqslant \beta_1^j|v|^{\alpha_j-1} + \beta_2^i;$$

(H6) $\lim\limits_{v\to 0}|f(t,v)|/|v|^{p^+-1} = 0$ uniformly for $t\in[0,\pi]$ and $\lim\limits_{v\to 0}|I_j(v)|/|v|^{p^+-1} = 0$ for $j = 1,2,\ldots,m$.

Note that with (H1), (H5) the functional J is well defined. The assumption which we employ were already used in [3] in the context of boundary value problems for partial differential equations connected with the so called $p(x)$-Laplacian. Let us recall some preliminaries. Let E be a Banach space. For any sequence $\{u_n\}\subset E$, if $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$ as $n\to\infty$ possesses a convergent subsequence, then we say J satisfies the Palais-Smale condition—(PS) condition for short.

Lemma 7 (Mountain Pass Lemma [9]). *Let $J \in C^1(E, \mathbb{R})$ satisfy the (PS) condition. Suppose that*

1. $J(0) = 0$;
2. *there exist* $\varrho > 0$ *and* $\alpha > 0$ *such that* $J(u) \geqslant \alpha$ *for all* $u \in E$ *with* $\|u\| = \varrho$;
3. *there exist* u_1 *in* E *with* $\|u_1\| \geqslant \varrho$ *such that* $J(u_1) < \alpha$.

Then J has a critical value $c \geqslant \alpha$. Moreover, c can be characterized as

$$\inf_{g \in \Gamma} \max_{u \in g([1,0])} J(u),$$

where $\Gamma = \{g \in C([1,0], E) \colon\ g(0) = 0,\ g(1) = u_1\}$.

Lemma 8. *Suppose* (H1), (H4), (H5) *hold. Then the functional J given by* (19) *satisfies the (PS) condition.*

Proof. Let us take a sequence $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,p(t)}(0,\pi)$ such that $\{J(u_k)\}_{k=1}^{\infty}$ is bounded and $\|J'(u_k)\|_{W_0^{1,q(t)}(0,\pi)} \to 0$ as $k \to \infty$. We shall show that $\{u_k\}_{k=1}^{\infty}$ has a convergent subsequence.

Since $\|J'(u_k)\|_{W_0^{1,q(t)}(0,\pi)} \to 0$, we see that for some $\varepsilon > 0$ there exists k_0 with $\|J'(u_k)\|_{W_0^{1,q(t)}(0,\pi)} \leqslant \varepsilon$ for $k \geqslant k_0$. Note that for $k \geqslant k_0$

$$\langle J'(u_k), u_k \rangle \leqslant \varepsilon \Big\| \frac{\mathrm{d}}{\mathrm{d}t} u_k \Big\|_{L^{p(t)}}$$

and

$$\langle J'(u_k), u_k \rangle = \int_0^{\pi} \Big| \frac{\mathrm{d}}{\mathrm{d}t} u_k(t) \Big|^{p(t)} - \sum_{j=1}^{m} I_j(u_k(t_j)) u_k(t_j) - \int_0^{\pi} f(t, u_k(t)) u_k(t)\, \mathrm{d}t.$$

Then, we see that

$$\begin{aligned}
&- \int_0^{\pi} F(t, u_k(t))\, \mathrm{d}t - \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(t)\, \mathrm{d}t \\
&\quad \geqslant -\frac{1}{\theta} \Big(\int_0^{\pi} f(t, u_k(t)) u_k(t)\, \mathrm{d}t + \sum_{j=1}^{m} I_j(u_k(t_j)) u_k(t_j) \Big) \\
&\quad = \frac{1}{\theta} J'(u_k; u_k) - \frac{1}{\theta} \int_0^{\pi} \Big| \frac{\mathrm{d}}{\mathrm{d}t} u_k(t) \Big|^{p(t)} \\
&\quad \geqslant -\frac{\varepsilon}{\theta} \Big\| \frac{\mathrm{d}}{\mathrm{d}t} u_k \Big\|_{L^{p(t)}} - \frac{1}{\theta} \max \Big\{ \Big\| \frac{\mathrm{d}}{\mathrm{d}t} u_k \Big\|_{L^{p(t)}}^{p^-},\ \Big\| \frac{\mathrm{d}}{\mathrm{d}t} u_k \Big\|_{L^{p(t)}}^{p^+} \Big\}.
\end{aligned}$$

Since $\{J(u_k)\}_{k=1}^{\infty}$ is bounded, there exists a constant C such that $C \geqslant J(u_k)$. Using the above estimates we obtain

$$C \geqslant J(u_k) \geqslant \Big(\frac{1}{p^+} - \frac{1}{\theta}\Big) \max\Big\{\Big\|\frac{\mathrm{d}}{\mathrm{d}t}u_k\Big\|_{L^{p(t)}}^{p^-}, \Big\|\frac{\mathrm{d}}{\mathrm{d}t}u_k\Big\|_{L^{p(t)}}^{p^+}\Big\} - \frac{\varepsilon}{\theta}\Big\|\frac{\mathrm{d}}{\mathrm{d}t}u\Big\|_{L^{p(t)}}.$$

Hence, it follows that $\{(\mathrm{d}/\mathrm{d}t)u_k\}_{k=1}^{\infty}$ is bounded in $L^{p(t)}(0,\pi)$. Hence $\{u_k\}_{k=1}^{\infty}$ has a weakly convergent subsequence in $W_0^{1,p(t)}(0,\pi)$. This means that $\{(\mathrm{d}/\mathrm{d}t)u_k\}_{k=1}^{\infty}$ is weakly convergent in $L^{p(t)}(0,\pi)$. Then $\{u_k\}_{k=1}^{\infty}$ has a subsequence $\{u_{k_n}\}_{n=1}^{\infty}$ which is strongly convergent in $L^{p(t)}(0,\pi)$ and convergent in $C[0,\pi]$. Denote by $\bar{u} \in W_0^{1,p(t)}(0,\pi)$ the weak limit of $\{u_k\}_{k=1}^{\infty}$. We see that as $k \to \infty$

$$\langle J'(u_k) - J'(\bar{u}), u_k - \bar{u}\rangle \to 0 \tag{20}$$

and further it follows by a direct calculation that as $k \to \infty$

$$\int_0^{\pi} \Big|\frac{\mathrm{d}}{\mathrm{d}t}u_k(t)\Big|^{p(t)} \,\mathrm{d}t \to \int_0^{\pi} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)\Big|^{p(t)} \,\mathrm{d}t. \tag{21}$$

Indeed, as $k \to \infty$ we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} I_j(u_k(t_j))u_k(t_j) - \int_0^{\pi} f(t, u_k(t))u_k(t)\,\mathrm{d}t \\
&\qquad \to \sum_{j=1}^{m} I_j(\bar{u}(t_j))\bar{u}(t_j) - \int_0^{\pi} f(t, \bar{u}(t))\bar{u}(t)\,\mathrm{d}t, \\
&\sum_{j=1}^{m} I_j(u_k(t_j))\bar{u}(t_j) - \int_0^{\pi} f(t, u_k(t))\bar{u}(t)\,\mathrm{d}t \\
&\qquad \to \sum_{j=1}^{m} I_j(\bar{u}(t_j))\bar{u}(t_j) - \int_0^{\pi} f(t, \bar{u}(t))\bar{u}(t)\,\mathrm{d}t, \\
&\sum_{j=1}^{m} I_j(\bar{u}(t_j))u_k(t_j) - \int_0^{\pi} f(t, \bar{u}(t))u_k(t)\,\mathrm{d}t \\
&\qquad \to \sum_{j=1}^{m} I_j(\bar{u}(t_j))\bar{u}(t_j) - \int_0^{\pi} f(t, \bar{u}(t))\bar{u}(t)\,\mathrm{d}t.
\end{aligned}$$

Hence (20) implies (21) and thus $\{u_{k_n}\}_{n=1}^{\infty}$ is strongly convergent. $\square$

Lemma 9. *Suppose that* (H1), (H4)–(H6) *hold. Then there exist numbers* η, $\xi > 0$ *such that* $J(u) \geqslant \xi$ *for all* $u \in W_0^{1,p(t)}(0,\pi)$ *such that* $\|u\|_{W_0^{1,p(t)}} = \eta$. *Moreover, there exists an element* $v \in W_0^{1,p(t)}(0,\pi)$ *with* $\|u\|_{W_0^{1,p(t)}} > \eta$ *and such that* $J(v) < 0$.

Proof. From [4] there exist constants $C_3, C_4, C_4^1, \ldots, C_4^m > 0$ with

$$\|u\|_{L^{p^+}} \leqslant C_3\|u\|_{W_0^{1,p(t)}} \quad \text{and } \|u\|_{L^\alpha} \leqslant C_4\|u\|_{W_0^{1,p(t)}},$$
$$\|u\|_{L^{\alpha_j}} \leqslant C_4^j\|u\|_{W_0^{1,p(t)}}$$

for all $u \in W_0^{1,p(t)}(0,\pi)$. Let us take a small $\varepsilon > 0$. By (H5), (H6) we see for some $A(\varepsilon), A_1(\varepsilon), \ldots, A_m(\varepsilon) > 0$ that

$$F(t,v) \leqslant \varepsilon|v|^{p^+} + A(\varepsilon)|v|^\alpha \quad \text{for a.e. } t \in [0,\pi],\ v \in \mathbb{R}$$

and for $j = 1, 2, \ldots, m$

$$\int_0^v I_j(s)\,\mathrm{d}s \leqslant \varepsilon|v|^{p^+} + A_j(\varepsilon)|v|^{\alpha_j} \quad \text{for } (t,v) \in [0,\pi] \times \mathbb{R}.$$

Note

$$\int_0^\pi F(t,u)\,\mathrm{d}t + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t$$
$$\leqslant 2\varepsilon C_3\|u\|_{W_0^{1,p(t)}}^{p^+} + A(\varepsilon)C_4\|u\|_{W_0^{1,p(t)}}^\alpha + \sum_{j=1}^m A_j(\varepsilon)C_4^j\|u\|_{W_0^{1,p(t)}}^{\alpha_j}$$

and as a result we see that

$$J(u) \geqslant \Big(\frac{1}{p^+} - 2\varepsilon C_3\Big)\|u\|_{W_0^{1,p(t)}}^{p^+} - A(\varepsilon)C_4\|u\|_{W_0^{1,p(t)}}^\alpha - \sum_{j=1}^m A_j(\varepsilon)C_4^j\|u\|_{W_0^{1,p(t)}}^{\alpha_j}.$$

Taking ε small enough, we see for some $\eta, \xi > 0$, $\eta < 1$ that $J(u) \geqslant \xi$ for all $u \in W_0^{1,p(t)}(0,\pi)$ such that $\|u\|_{W_0^{1,p(t)}} = \eta$.

Notice for $v \in \mathbb{R}$ that

$$-\sum_{j=1}^m \int_0^v I_j(s)\,\mathrm{d}s \leqslant 0.$$

From (H4) if $u \in \mathbb{R}$ it is easy to see that there exists a constant $C_5 > 0$ such that

$$F(t,u) \geqslant C_5|u|^\theta.$$

Fix $v \in W_0^{1,p(t)}(0,\pi)$ with $v \neq 0$. For $s \in \mathbb{R}_+$ note that

$$\begin{aligned} J(sv) &= \int_0^{\pi} \frac{1}{p(t)} \Big| s \frac{\mathrm{d}}{\mathrm{d}t} v(t) \Big|^{p(t)} \, \mathrm{d}t - \sum_{j=1}^{m} \int_0^{sv(t_j)} I_j(t) \, \mathrm{d}t - \int_0^{\pi} F(t, sv(t)) \, \mathrm{d}t \\ &\leqslant \frac{1}{p^-} s^{p^+} \Big\| \frac{\mathrm{d}}{\mathrm{d}t} v \Big\|_{L^{p(t)}}^{p^+} - C_5 s^{\theta} \int_0^{\pi} |v(t)|^{\theta} \, \mathrm{d}t. \end{aligned}$$

Since $\theta > p^+$ we see that

$$\lim_{s \to \infty} J(sv) = -\infty$$

and the condition 3. of Lemma 7 is satisfied. □

Theorem 10. *Suppose that* (H1), (H4)–(H6) *hold. Then the problem* (17)–(18) *admits at least one nontrivial classical solution.*

P r o o f. By Lemmas 8 and 9 we see that we can apply Lemma 7 to obtain the existence of at least one nontrivial weak solution. Applying Lemma 5 we see that the solution is in fact a classical one. □

Remark 1. From the proof above it is easy to see that one could replace

$$|f(t,v)| \leqslant \beta_1 |v|^{\alpha - 1} + \beta_2$$

in (H5) with

$$|f(t,v)| \leqslant \beta_1 |v|^{\alpha(t) - 1} + \beta_2,$$

for some function $\alpha \in C[0,\pi]$ with $\alpha^- > p^+$.

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