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PERIODIC SOLUTIONS FOR SOME NONAUTONOMOUS $p(t)$-LAPLACIAN HAMILTONIAN SYSTEMS

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Abstract. In this paper, we deal with the existence of periodic solutions of the $p(t)$-Laplacian Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) = \nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0. \end{cases}$$

Some new existence theorems are obtained by using the least action principle and minimax methods in critical point theory, and our results generalize and improve some existence theorems.

Keywords: periodic solution, Hamiltonian system, $p(t)$-Laplacian system, critical point, minimax principle, least action principle

MSC 2010: 34C25, 58E50

1. Introduction

Consider the $p(t)$-Laplacian Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p(t)-2}\dot{u}(t)) = \nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases} \tag{1.1}$$

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1([0,T],\mathbb{R}^+)$ such that

$$|F(t,x)| \leqslant a(|x|)b(t), \quad |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

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Moreover, we suppose that $p(t) \in C([0,T],\mathbb{R}^+)$ satisfies the following assumption:

(A$'$) $p^+ := \max\limits_{0\leqslant t\leqslant T} p(t)$, $p^- := \min\limits_{0\leqslant t\leqslant T} p(t) > 1$, and $q^+ > 1$ satisfies $1/p^- + 1/q^+ = 1$.

When $p(t) = p$ is a constant, system (1.1) reduces to the ordinary p-Laplacian system. In recent years, the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated via the variational methods and many results were obtained based on various hypotheses on the potential functions, see, e.g., [10], [13], [14], [25] and references therein.

If $p = 2$, system (1.1) reduces to

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

The corresponding functional ψ on H_T^1 given by

$$\psi(u) := \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \,\mathrm{d}t + \int_0^T F(t,u(t))\,\mathrm{d}t$$

is continuously differentiable and weakly lower semicontinuous on H_T^1 (see [12]), where

$$H_T^1 := \{u\colon\ [0,T] \to \mathbb{R}^N,\ u \text{ is absolutely continuous}, \\ u(0) = u(T),\ \dot{u} \in L^2([0,T];\mathbb{R}^N)\}$$

is a Hilbert space with a norm defined by

$$\|u\|_{H_T^1} := \Big(\int_0^T |u(t)|^2\,\mathrm{d}t + \int_0^T |\dot{u}(t)|^2\,\mathrm{d}t\Big)^{1/2}$$

for $u \in H_T^1$.

Considerable attention has been paid to the periodic solutions for system (1.2) in recent years. It has been proved that system (1.2) has at least one solution which minimizes H_T^1 by the least action principle (see [1], [11], [12], [15], [16], [17], [22], [24]). Many solvability conditions were given, such as the coercivity condition (see [1]), the periodicity condition (see [24]), the convexity condition (see [11]), the boundedness condition (see [12], [23]), the subadditive condition (see [15]), and the sublinear condition (see [16]). Meanwhile, using the minimax methods, [5], [8], [20] considered the superquadratic second order Hamilton systems. The periodic potential (see [6], [9], [19]) and the subquadratic potential (see [6], [7], [16], [18]) have been also considered.

Specifically, when $F(t,x) = G(x) + H(t,x)$, H is sublinear, that is, there exist $f,g \in L^1(0,T,R^+)$ and $\alpha \in [0,1)$ such that

$$|\nabla H(t,x)| \leqslant f(t)|x|^\alpha + g(t) \tag{1.3}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$, assuming that there exists $r < 4\pi^2/T^2$ such that

$$(\nabla G(x) - \nabla G(y), x - y) \geqslant -r|x - y|^2 \tag{1.4}$$

for all $x, y \in \mathbb{R}^n$, and there exists a subset E of $[0, T]$ with $\operatorname{meas}(E) > 0$ such that

$$F(t, x) \to -\infty \quad \text{as } |x| \to \infty \tag{1.5}$$

for a.e. $t \in E$, under the condition of (1.3), (1.4), and (1.5), Ye and Tang (see [26]) proved the existence of periodic solutions for system (1.2).

The study of elliptic partial differential equations and variational problems with nonstandard growth conditions has been an interesting topic in recent years (see, for example, [3], [4], [27], [28]). The ordinary $p(t)$-Laplacian system (1.1) was studied by Fan (see [2]), then Wang (see [21]) obtained the existence and mulplicity of periodic solutions for ordinary $p(t)$-Laplacian system (1.1) under the generalized Ambrosetti-Rabinowitz conditions.

The ordinary $p(t)$-Laplacian system can be applied to describe the physical phenomena with "pointwise different properties" which arise from the nonlinear elasticity theory (see [27]). The $p(t)$-Laplacian system possesses more complicated nonlinearity than that of the p-Laplacian, for example, it is not homogeneous, which causes many troubles, and some classical theories and methods, such as the theory of Sobolev spaces are not applicable.

Inspired and motivated by the results mentioned above, in this paper we suppose that $H(t, x)$ is p^--sublinear, that is, there exist $f, g \in L^1([0, T], \mathbb{R}^+)$ and $\alpha \in [0, p^- - 1)$ such that

$$|\nabla H(t, x)| \leqslant f(t)|x|^\alpha + g(t) \tag{1.6}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$, and there exist $0 \leqslant r < 1/(p^+ T^{p^-})$ and $1 \leqslant \beta \leqslant p^-$ such that

$$(\nabla G(x) - \nabla G(y), x - y) \geqslant -r|x - y|^\beta \tag{1.7}$$

for all $x, y \in \mathbb{R}^n$. Under the condition of (1.5), (1.6), and (1.7), the existence of periodic solutions for system (1.1), which generalizes Ye-Tang's results, is obtained by the minimax methods in the critical point theory. Moreover, we consider system (1.1) with a potential which is the sum of a subconvex function and another function under suitable conditions by the least action principle, and the new solvability conditions develop and generalize the corresponding results in [24].

2. Preliminaries

In this section we recall some known results in the critical point theory, and the properties of the space $W_T^{1,p(t)}$ are listed for the convenience of readers.

Definition 2.1 ([21]). Let $p(t)$ satisfy the condition (A′), and define

$$L^{p(t)}([0,T],\mathbb{R}^N)=\left\{u\in L^1([0,T],\mathbb{R}^N)\colon\ \int_0^T |u|^{p(t)}\,\mathrm{d}t<\infty\right\}$$

with the norm

$$|u|_{p(t)}:=\inf\left\{\lambda>0\colon\ \int_0^T \left|\frac{u}{\lambda}\right|^{p(t)}\,\mathrm{d}t\leqslant 1\right\}.$$

For $u\in L^1_{\rm loc}([0,T],\mathbb{R}^N)$, let u' denote the weak derivative of u, if $u'\in L^1_{\rm loc}([0,T],\mathbb{R}^N)$ and satisfies

$$\int_0^T u'\phi\,\mathrm{d}t=-\int_0^T u\phi'\,\mathrm{d}t,\quad \forall\phi\in C_0^\infty([0,T],\mathbb{R}^N).$$

Define

$$W^{1,p(t)}([0,T],\mathbb{R}^N)=\{u\in L^{p(t)}([0,T],\mathbb{R}^N)\colon\ u'\in L^{p(t)}([0,T],\mathbb{R}^N)\}$$

with the norm $\|u\|_{W^{1,p(t)}}:=|u|_{p(t)}+|u'|_{p(t)}$.

Remark 2.1. If $p(t)=p$, where $p\in[1,\infty)$ is a constant, by the definition of $|u|_{p(t)}$ it is easy to get $|u|_p=\left(\int_0^T |u(t)|^p\,\mathrm{d}t\right)^{1/p}$, which is the same with the usual norm in the space L^p.

The space $L^{p(t)}$ is a generalized Lebesgue space, and the space $W^{1,p(t)}$ is a generalized Sobolev space. Because most of the following lemmas have appeared in [2], [4], [12], [21], we omit their proofs.

Lemma 2.1 ([2]). *Both $L^{p(t)}$ and $W^{1,p(t)}$ are Banach spaces with the norms defined above; when $p^->1$, they are reflexive.*

Definition 2.2 ([12]).

$$C_T^\infty=C_T^\infty(\mathbb{R},\mathbb{R}^N):=\{u\in C^\infty(\mathbb{R},\mathbb{R}^N)\colon\ u \text{ is } T\text{-periodic}\}$$

with the norm $\|u\|_\infty:=\max\limits_{t\in[0,T]}|u(t)|$.

For a constant $p\in[1,\infty)$, using another concept of weak derivative which is called the T-weak derivative, Mawhin and Willem gave the definition of the space $W_T^{1,p}$ in the following way.

Definition 2.3 ([12]). Let $u \in L^1([0,T],\mathbb{R}^N)$ and $v \in L^1([0,T],\mathbb{R}^N)$. If

$$\int_0^T v\phi \,\mathrm{d}t = -\int_0^T u\phi' \,\mathrm{d}t \quad \forall\, \phi \in C_T^\infty,$$

then v is called the T-weak derivative of u and is denoted by $\dot{u}$.

Definition 2.4 ([12]). Define

$$W_T^{1,p}([0,T],\mathbb{R}^N) = \{u \in L^p([0,T],\mathbb{R}^N) \colon\ \dot{u} \in L^p([0,T],\mathbb{R}^N)\}$$

with the norm $\|u\|_{W_T^{1,p}} = (|u|_p^p + |\dot{u}|_p^p)^{1/p}$.

Definition 2.5 ([2]). Define

$$W_T^{1,p(t)}([0,T],\mathbb{R}^N) = \{u \in L^{p(t)}([0,T],\mathbb{R}^N) \colon\ \dot{u} \in L^{p(t)}([0,T],\mathbb{R}^N)\}$$

and $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ to be the closure of C_T^∞ in $W^{1,p(t)}([0,T],\mathbb{R}^N)$.

Remark 2.2. From Definition 2.4, if $u \in W_T^{1,p(t)}([0,T],\mathbb{R}^N)$, it is easy to conclude that $u \in W_T^{1,p^-}([0,T],\mathbb{R}^N)$.

Lemma 2.2 ([12]). *For $u \in W_T^{1,p^-}$, let*

$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t \quad \textit{and} \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then

$$\int_0^T |\tilde{u}(t)|^{p^-}\,\mathrm{d}t \leqslant T^{p^-}\int_0^T |\dot{u}(t)|^{p^-}\,\mathrm{d}t.$$

Lemma 2.3 ([2]).

(i) *$C_T^\infty([0,T],\mathbb{R}^N)$ is dense in $W_T^{1,p(t)}([0,T],\mathbb{R}^N)$,*

(ii) *$W_T^{1,p(t)}([0,T],\mathbb{R}^N) = H_T^{1,p(t)}([0,T],\mathbb{R}^N) := \{u \in W^{1,p(t)}([0,T],\mathbb{R}^N) \colon\ u(0) = u(T)\}$,*

(iii) *if $u \in H_T^{1,1}$, then the derivative u' is also the T-weak derivative $\dot{u}$, i.e. $u' = \dot{u}$.*

Remark 2.3. In what follows, we use $\|u\|$ instead of $\|u\|_{W_T^{1,p(t)}}$ for convenience without clear indications.

Lemma 2.4 ([12]). *Assume that* $u \in W_T^{1,1}$. *Then*

(i) $\int_0^T \dot{u}\,\mathrm{d}t = 0$,

(ii) *u has its continuous representation, which is still denoted by* $u(t) = \int_0^t \dot{u}(s)\,\mathrm{d}s + u(0)$, $u(0) = u(T)$,

(iii) *$\dot{u}$ is the classical derivative of u, if* $\dot{u} \in C([0,T],\mathbb{R}^N)$.

Since every closed linear subspace of a reflexive Banach space is also reflexive, we have

Lemma 2.5 ([2]). $H_T^{1,p(t)}([0,T],\mathbb{R}^N)$ *is a reflexive Banach space if* $p^- > 1$.

Obviously, there are continuous embeddings $L^{p(t)} \hookrightarrow L^{p^-}$, $W^{1,p(t)} \hookrightarrow W^{1,p^-}$, and $H_T^{1,p(t)} \hookrightarrow H_T^{1,p^-}$. By the classical Sobolev embedding theorem we obtain

Lemma 2.6 ([2]). *There is a continuous embedding*

$$W^{1,p(t)} \text{ (or } H_T^{1,p(t)}) \hookrightarrow C([0,T],\mathbb{R}^N),$$

when $p^- > 1$, *the embedding is compact.*

Lemma 2.7. *Denoting* $W_T^{1,p(t)} = \widetilde{W}_T^{1,p(t)} \oplus \mathbb{R}^N$, *where*

$$\widetilde{W}_T^{1,p(t)} = \Big\{u \in W_T^{1,p(t)} \colon \int_0^T u(t)\,\mathrm{d}t = 0\Big\},$$

there exists $C_0 > 0$ *such that, if* $u \in \widetilde{W}_T^{1,p(t)}$,

$$\|u\|_\infty \leqslant 2C_0\Big(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} + 2C_0T^{1/p^-}.$$

Proof. Let $A = \{t \in [0,T] \colon |\dot{u}(t)| \geqslant 1\}$. From Remark 2.2 we obtain $u \in W_T^{1,p^-}$, and by virtue of the inequality in the classical Sobolev space there exists a positive constant $C_0 > 0$ such that

$$\begin{aligned}
\|u\|_\infty &\leqslant C_0\Big(\int_0^T |\dot{u}(t)|^{p^-}\,\mathrm{d}t\Big)^{1/p^-} = C_0\Big(\int_A |\dot{u}(t)|^{p^-}\,\mathrm{d}t + \int_{[0,T]\setminus A} |\dot{u}(t)|^{p^-}\,\mathrm{d}t\Big)^{1/p^-}\\
&\leqslant C_0\Big(\int_A |\dot{u}(t)|^{p(t)}\,\mathrm{d}t + \operatorname{meas}[0,T]\setminus A\Big)^{1/p^-}\\
&\leqslant C_0\Big(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t + T\Big)^{1/p^-}\\
&\leqslant 2C_0\Big(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} + 2C_0T^{1/p^-}.
\end{aligned}$$

This completes the proof of Lemma 2.7. □

Lemma 2.8 ([2]). *Each of the following two norms is equivalent to the norm in $W_T^{1,p(t)}$:*

(i) $|\dot{u}|_{p(t)} + |u|_q$, $1 \leqslant q \leqslant \infty$,

(ii) $|\dot{u}|_{p(t)} + |\bar{u}|$, *where* $\bar{u} = (1/T)\int_0^T u(t)\,\mathrm{d}t$.

Lemma 2.9 ([21]). *If we denote $\varrho(u) = \int_0^T |u|^{p(t)}\,\mathrm{d}t$, $\forall\, u \in L^{p(t)}$, then*

(i) $|u|_{p(t)} < 1\ (= 1; > 1) \iff \varrho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(t)} > 1 \Longrightarrow |u|_{p(t)}^{p^-} \leqslant \varrho(u) \leqslant |u|_{p(t)}^{p^+}$, $|u|_{p(t)} < 1 \Longrightarrow |u|_{p(t)}^{p^+} \leqslant \varrho(u) \leqslant |u|_{p(t)}^{p^-}$;

(iii) $|u|_{p(t)} \to 0 \iff \varrho(u) \to 0$; $|u|_{p(t)} \to \infty \iff \varrho(u) \to \infty$.

Proposition 2.1. *In the space $W_T^{1,p(t)}$, the implication*

$$\|u\| \to \infty \Longrightarrow \int_0^T |\dot{u}|^{p(t)}\,\mathrm{d}t + |\bar{u}| \to \infty$$

holds.

Proof. By Lemma 2.8 there exists a positive constant C_1 such that

$$\|u\| \leqslant C_1(|\dot{u}|_{p(t)} + |\bar{u}|).$$

If $|\dot{u}|_{p(t)} < 1$, it is easy to get

$$|\dot{u}|_{p(t)} < \int_0^T |\dot{u}|^{p(t)}\,\mathrm{d}t + 1. \tag{2.1}$$

When $|\dot{u}|_{p(t)} \geqslant 1$, we conclude that

$$|\dot{u}|_{p(t)} \leqslant \Big(\int_0^T |\dot{u}|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} \tag{2.2}$$

and by Lemma 2.9, (2.1) and (2.2) it follows that

$$\|u\| \leqslant C_1\bigg(\Big(\int_0^T |\dot{u}|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} + 1 + |\bar{u}|\bigg), \tag{2.3}$$

which implies that

$$\|u\| \to \infty \Longrightarrow \int_0^T |\dot{u}|^{p(t)}\,\mathrm{d}t + |\bar{u}| \to \infty.$$

Lemma 2.10 ([21]). *If $u, u_n \in L^{p(t)}$ $(n = 1, 2, \ldots)$, then the following statements are equivalent to each other:*

(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(t)} = 0$,

(ii) $\lim\limits_{n\to\infty} \varrho(u_n - u) = 0$,

(iii) *u_n converges to u in measure in $[0, T]$ and* $\lim\limits_{n\to\infty} \varrho(u_n) = \varrho(u)$.

Definition 2.6 ([24]). A function $G\colon \mathbb{R}^N \to \mathbb{R}$ is said to be (λ, μ)-subconvex if

$$G(\lambda(x + y)) \leqslant \mu(G(x) + G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in \mathbb{R}^N$. A function $G\colon \mathbb{R}^N \to \mathbb{R}$ is called γ-subadditive if it is $(1, \gamma)$-subconvex. A function $G\colon \mathbb{R}^N \to \mathbb{R}$ is called subadditive if it is 1-subadditive. The convex and subadditive function are special subconvex functions.

Lemma 2.11 ([26]). *Suppose that F satisfies the assumption* (A) *and E is a measurable subset of $[0, T]$. Assume that*

$$F(t, x) \to -\infty \quad \text{as } |x| \to \infty$$

for a.e. $t \in E$. Then for every $\delta > 0$ there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ *such that*

$$F(t, x) \to -\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$.

Lemma 2.12 ([21]). *The functional on $W_T^{1,p(t)}$ given by*

$$\varphi(u) = \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)}\,\mathrm{d}t + \int_0^T F(t, u(t))\,\mathrm{d}t \tag{2.4}$$

is continuously differentiable and weakly lower semicontinuous on $W_T^{1,p(t)}$. Moreover, we have

$$\langle \varphi'(u), v\rangle = \int_0^T [(|\dot{u}(t)|^{p(t)-2}\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))]\,\mathrm{d}t$$

for all $u, v \in W_T^{1,p(t)}$. It is well known that the critical points of φ correspond to the solutions of system (1.1).

Lemma 2.13. *The functional φ defined on $W_T^{1,p(t)}$ given by* (2.4) *is weakly lower semicontinuous on $W_T^{1,p(t)}$.*

Proof. We divide φ into two parts, $\varphi(u) = \varphi_1(u) + \varphi_2(u)$, where

$$\varphi_1(u) = \int_0^T \frac{1}{p(t)} |\dot{u}(t)|^{p(t)} \, \mathrm{d}t \quad \text{and} \quad \varphi_2(u) = \int_0^T F(t, u(t)) \, \mathrm{d}t.$$

It is obvious that φ_1 is convex and continuous by Lemma 2.10, then φ_1 is weakly lower semicontinuous by Theorem 1.2 in [12], and φ_2 is weakly continuous, that is, φ is the sum of two weakly lower semicontinuous functionals, which implies that φ is weakly lower semicontinuous. □

Lemma 2.14 ([21])**.** *J' is a bounded linear functional and a mapping of (S_+) on $W_T^{1,p(t)}$, that is, if $u_n \rightharpoonup u$ weakly in $W_T^{1,p(t)}$ and $\limsup\limits_{n\to\infty} (J'(u_n) - J'(u), u_n - u)) \leqslant 0$, then u_n has a convergent subsequence, where J' is given by*

$$\langle J'(u), v \rangle = \int_0^T (|\dot{u}(t)|^{p(t)-2} \dot{u}(t), \dot{v}(t)) \, \mathrm{d}t. \tag{2.5}$$

3. Main results and proofs of theorems

Our main results are the following theorems.

Theorem 3.1. *Suppose that $F(t,x)$ satisfies assumption* (A), (1.5), *and* (1.7)*. Assume that there exists $\gamma(t) \in L^1([0,T], \mathbb{R})$ such that*

$$F(t,x) \leqslant \gamma(t) \tag{3.1}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, and there exists $g \in L^1([0,T], \mathbb{R}^+)$ such that

$$|\nabla H(t,x)| \leqslant g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Then system (1.1) *has at least one solution in $W_T^{1,p(t)}$.*

Theorem 3.2. *Suppose that $F(t,x)$ satisfies assumption* (A), (1.6), *and* (1.7). *Assume that*

$$|x|^{-q^+\alpha}F(t,x) \to -\infty \quad \text{as } |x| \to \infty$$

uniformly for a.e. $t \in [0,T]$, where α is the same as in (1.6). *Then system* (1.1) *has at least one solution in $W_T^{1,p(t)}$.*

Remark 3.1. Theorem 3.1 and Theorem 3.2 generalize Theorem 1 and Theorem 2 in [26], respectively.

Theorem 3.3. *Suppose that $F(t,x)$ satisfies assumption* (A), (1.6), (1.7), *and* (3.1). *Assume that there exists a subset E of $[0,T]$ with $\operatorname{meas}(E) > 0$ such that*

$$|x|^{-q^+\alpha}F(t,x) \to -\infty \quad \text{as } |x| \to \infty \tag{3.2}$$

for a.e. $t \in E$. Then system (1.1) *has at least one solution in $W_T^{1,p(t)}$.*

Remark 3.2. Theorem 3.3 is a more general result than Theorem 3.1 and Theorem 3.2, which generalizes Theorem 3 in [26].

Theorem 3.4. *Suppose that $F = F_1+F_2$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *$F_1(t,\cdot)$ is (λ,μ)-subconvex with $\lambda > 1/2$ and $1/2 \leqslant \mu < 2^{p^- -1}\lambda^{p^-}$ for a.e. $t \in [0,T]$, and there exist $\alpha \in [0, p^- - 1)$, $f, g \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|\nabla F_2(t,x)| \leqslant f(t)|x|^\alpha + g(t)$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

(ii)

$$\frac{1}{|x|^{q^+\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda x)\,\mathrm{d}t + \int_0^T F_2(t,x)\,\mathrm{d}t\right] \to \infty \quad \text{as } |x| \to \infty.$$

Then system (1.1) *has at least one solution which minimizes φ on $W_T^{1,p(t)}$.*

Corollary 3.1. *Assume that $F = F_1+F_2$, where F_1 and F_2 satisfy assumption* (A) *and the following conditions:*

(i) *$F_1(t,\cdot)$ is subadditive for a.e. $t \in [0,T]$, and there exist $\alpha \in [0, p^- - 1)$, $f, g \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|\nabla F_2(t,x)| \leqslant f(t)|x|^\alpha + g(t)$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

(ii)

$$|x|^{-q^+\alpha}\int_0^T F(t,x)\,\mathrm{d}t \to \infty \quad \text{as } |x|\to\infty.$$

Then system (1.1) *has at least one solution which minimizes* φ *on* $W_T^{1,p(t)}$.

Corollary 3.2. *Assume that* F *satisfies assumption* (A) *and the following conditions:*

(i) $F(t,\cdot)$ *is* (λ,μ)*-subconvex with* $\lambda > 1/2$ *and* $1/2 \leqslant \mu < 2^{p^- -1}\lambda^{p^-}$ *for a.e.* $t\in[0,T]$.

(ii)

$$\int_0^T F(t,x)\,\mathrm{d}t \to \infty \quad \text{as } |x|\to\infty.$$

Then system (1.1) *has at least one solution which minimizes* φ *on* $W_T^{1,p(t)}$.

Remark 3.3. Theorem 3.4, Corollary 3.1, and Corollary 3.2 generalize Theorem 1, Corollary 1, and Corollary 2 in [24], respectively.

Theorem 3.5. *Assume that* $F = F_1 + F_2$ *satisfies assumption* (A) *and the following conditions:*

(i) $F_1(t,\cdot)$ *is* (λ,μ)*-subconvex for a.e.* $t\in[0,T]$, *satisfies*

$$F_1(t,x) \geqslant (h(t),x) + \gamma_1(t)$$

for a.e. $t\in[0,T]$ *and all* $x\in\mathbb{R}^n$, *where* $\gamma_1\in L^1([0,T],\mathbb{R})$ *and* $h\in L^1([0,T],\mathbb{R}^N)$ *with* $\int_0^T h(t)\,\mathrm{d}t = 0$, *and there exists* $g\in L^1([0,T],\mathbb{R})$ *and* $D\in\mathbb{R}$ *such that*

$$|\nabla F_2(t,x)| \leqslant g(t)$$

for a.e. $t\in[0,T]$ *and* $x\in\mathbb{R}^n$, *and*

$$\int_0^T F_2(t,x)\,\mathrm{d}t \geqslant D$$

for all $x\in\mathbb{R}^N$.

(ii)

$$\frac{1}{\mu}\int_0^T F_1(t,\lambda x)\,\mathrm{d}t + \int_0^T F_2(t,x)\,\mathrm{d}t \to \infty \quad \text{as } |x|\to\infty.$$

Then system (1.1) *has at least one solution which minimizes* φ *on* $W_T^{1,p(t)}$.

Corollary 3.3. *Assume that F satisfies assumption* (A) *and the following conditions:*

(i) $F(t,\cdot)$ *is (λ,μ)-subconvex for a.e. $t\in[0,T]$ and satisfies*

$$F(t,x)\geqslant (h(t),x)+\gamma_1(t)$$

for a.e. $t\in[0,T]$ and all $x\in\mathbb{R}^N$, where $\gamma_1\in L^1([0,T],\mathbb{R})$ and $h\in L^1([0,T],\mathbb{R}^N)$ with $\int_0^T h(t)\,\mathrm{d}t=0$.

(ii)

$$\int_0^T F(t,x)\,\mathrm{d}t\to\infty\quad \text{as } |x|\to\infty.$$

Then system (1.1) *has at least one solution which minimizes φ on $W_T^{1,p(t)}$.*

Theorem 3.6. *Assume that $F=F_1+F_2$ satisfies assumption* (A) *and the following conditions:*

(i) *there exist $\gamma_1\in L^1([0,T],\mathbb{R})$ and $h\in L^1([0,T],\mathbb{R}^N)$ with $\int_0^T h(t)\,\mathrm{d}t=0$ such that*

$$F_1(t,x)\geqslant (h(t),x)+\gamma_1(t)$$

for a.e. $t\in[0,T]$ and all $x\in\mathbb{R}^N$, and there exist $g\in L^1([0,T],\mathbb{R})$ and $D\in\mathbb{R}$ such that

$$|\nabla F_2(t,x)|\leqslant f(t)|x|^{\alpha}+g(t)$$

for a.e. $t\in[0,T]$ and all $x\in\mathbb{R}^n$.

(ii)

$$|x|^{-q^+\alpha}\int_0^T F_2(t,x)\,\mathrm{d}t\to\infty\quad \text{as } |x|\to\infty.$$

Then system (1.1) *has at least one solution which minimizes φ on $W_T^{1,p(t)}$.*

Remark 3.4. Theorem 3.5, Corollary 3.3, and Theorem 3.6 generalize Theorem 2, Corollary 3, and Theorem 3 in [24], respectively.

Because Theorem 3.3 is a more general result than Theorem 3.1 and Theorem 3.2, we only need to prove Theorem 3.3, and our steps to prove Theorem 3.3 are organized as follows. First, we show that the functional φ satisfies the conditions (PS), that is, $\{u_n\}$ has a convergent subsequence, whenever it satisfies that $\varphi(u_n)$ is bounded and $\varphi'(u_n)\to 0$; second, we prove that φ satisfies the other conditions of the Saddle Point Theorem (see Theorem 4.6 in [12]); after these two steps, by the Saddle Point Theorem we know that φ has at least one critical point, which is a periodic solution for system (1.1).

P r o o f of Theorem 3.3. From (1.7) and Lemma 2.2 we obtain

$$\begin{aligned}
(3.3)\qquad &\int_0^T (\nabla G(u(t)), \tilde u(t))\,\mathrm{d}t \\
&= \int_0^T (\nabla G(u(t)) - \nabla G(\bar u), \tilde u(t))\,\mathrm{d}t \\
&\geqslant -r\int_0^T |\tilde u(t)|^{\beta}\,\mathrm{d}t \geqslant -r\Big(\int_0^T |\tilde u(t)|^{p^-}\,\mathrm{d}t + T\Big) \\
&\geqslant -r\Big(T^{p^-}\int_0^T |\dot u(t)|^{p^-}\,\mathrm{d}t + T\Big) \\
&\geqslant -rT^{p^-}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t + T\Big) - rT \\
&\geqslant -\frac{1}{p^+}\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t - C_2
\end{aligned}$$

for all $u(t) \in W_T^{1,p(t)}$, and C_2 is a positive constant.

It follows from (1.6), Lemma 2.7, and by the Young inequality that

$$\begin{aligned}
(3.4)\qquad &\Big|\int_0^T (\nabla H(t, u(t)), \tilde u(t))\,\mathrm{d}t\Big| \\
&\leqslant \int_0^T f(t)|\bar u + \tilde u(t)|^{\alpha}|\tilde u(t)|\,\mathrm{d}t + \int_0^T g(t)|\tilde u(t)|\,\mathrm{d}t \\
&\leqslant 2^{p^- -1}\int_0^T f(t)(|\bar u|^{\alpha} + |\tilde u(t)|^{\alpha})|\tilde u(t)|\,\mathrm{d}t + \int_0^T g(t)|\tilde u(t)|\,\mathrm{d}t \\
&\leqslant 2^{p^- -1}(|\bar u|^{\alpha} + \|\tilde u\|_\infty^{\alpha})\|\tilde u\|_\infty\int_0^T f(t)\,\mathrm{d}t + \|\tilde u\|_\infty\int_0^T g(t)\,\mathrm{d}t \\
&= \Big((C_3)^{1/p^-}\frac{\|\tilde u\|_\infty}{4C_0}\Big)\Big((2^{p^- +1})(C_3)^{-1/p^-}C_0\int_0^T f(t)\,\mathrm{d}t\Big)|\bar u|^{\alpha} \\
&\quad + 2^{p^- -1}\|\tilde u\|_\infty^{1+\alpha}\int_0^T f(t)\,\mathrm{d}t + \|\tilde u\|_\infty\int_0^T g(t)\,\mathrm{d}t \\
&\leqslant \frac{p^+ - 1}{2p^+}\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t + C_4|\bar u|^{q^+\alpha} \\
&\quad + C_5\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{(\alpha+1)/p^-} \\
&\quad + C_6\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} + C_7
\end{aligned}$$

for all $u \in W_T^{1,p(t)}$ where C_0 is the same as in Lemma 2.7 and $C_3 := (p^+ - 1)/2p^+$, and for some positive constants C_4, C_5, and C_6.

From (3.3) and (3.4) we have

$$\begin{aligned}
\|\tilde{u}_n\| &\geqslant |\langle \varphi'(u_n), \tilde{u}_n\rangle| \\
&\geqslant \int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t \\
&\geqslant \int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t + \int_0^T (\nabla G(u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t \\
&\quad + \int_0^T (\nabla H(t,u_n(t)), \tilde{u}_n(t))\,\mathrm{d}t \\
&\geqslant \frac{p^+-1}{2p^+}\int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t - C_4|\bar{u}_n|^{q^+\alpha} \\
&\quad - C_5\Bigl(\int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{(\alpha+1)/p^-} \\
&\quad - C_6\Bigl(\int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/p^-} - C_8
\end{aligned} \tag{3.5}$$

for all large n.

It follows from the Proposition 2.1 that there exists a constant $C_1 > 0$ such that

$$\|\tilde{u}_n\| \leqslant C_1\Bigl(\Bigl(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/p^-} + 1\Bigr). \tag{3.6}$$

By (3.5) and (3.6) we have

$$\Bigl(\int_0^T |\dot{u}_n(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/q^+} \leqslant C_9|\bar{u}_n|^{\alpha} + C_{10} \tag{3.7}$$

for some positive constants C_9, C_{10} and all large n, which implies that

$$\|\tilde{u}_n\|_\infty \leqslant C_{11}(|\bar{u}_n|^{q^+\alpha/p^-} + 1)$$

for all large n and some positive constant C_{11} by Lemma 2.7.

If $(|\bar{u}_n|)$ is unbounded, we may assume that going to a subsequence if necessary

$$|\bar{u}_n| \to \infty \quad \text{as } n \to \infty. \tag{3.8}$$

Then we have

$$|u_n(t)| \geqslant |\bar{u}_n| - |\tilde{u}_n(t)| \geqslant |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geqslant |\bar{u}_n| - C_9(|\bar{u}_n|^{q^+\alpha/p^-} + 1)$$

for all large n and every $t \in [0,T]$ because of $q^+\alpha/p^- < 1$, which implies that

$$|u_n(t)| \geqslant \frac{1}{2}|\bar{u}_n| \tag{3.9}$$

for all large n and every $t \in [0,T]$.

Set $\delta = (\operatorname{meas} E)/2$. It follows from (3.2) and Lemma 2.11 that there exists a subset E_δ of E with $\operatorname{meas}(E \setminus E_\delta) < \delta$ such that

$$|x|^{-q^+\alpha} F(t,x) \to -\infty \quad \text{as } |x| \to \infty$$

uniformly for all $t \in E_\delta$, which implies that

$$\operatorname{meas} E_\delta = \operatorname{meas} E - \operatorname{meas}(E \setminus E_\delta) > \delta > 0 \tag{3.10}$$

and for every $N > 0$, there exists $M \geqslant 1$ such that

$$|x|^{-q^+\alpha} F(t,x) \leqslant -N \tag{3.11}$$

for all $|x| \geqslant M$ and all $t \in E_\delta$. By (3.8) and (3.9) we have

$$|u_n(t)| \geqslant M \tag{3.12}$$

for large n and every $t \in [0,T]$. It follows from (3.1), (3.7), (3.10), (3.11), (3.12) that

$$\begin{aligned} \varphi(u_n) &\leqslant (C_9|\bar{u}_n|^\alpha + C_{10})^{q^+} + \int_{[0,T]\setminus E_\delta} \gamma(t)\,\mathrm{d}t - \int_{E_\delta} N|u_n(t)|^{q^+\alpha}\,\mathrm{d}t \\ &\leqslant (C_9|\bar{u}_n|^\alpha + C_{10})^{q^+} + \int_{[0,T]\setminus E_\delta} \gamma(t)\,\mathrm{d}t - 2^{-q^+\alpha}|\bar{u}_n|^{q^+\alpha}\delta N \end{aligned} \tag{3.13}$$

for large n. Hence, we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q^+\alpha}\varphi(u_n) \leqslant C_7^{q^+} - 2^{-q^+\alpha}\delta N,$$

and by the arbitrariness of $N > 0$, we have

$$\limsup_{n\to\infty} |\bar{u}_n|^{-q^+\alpha}\varphi(u_n) = -\infty,$$

which contradicts the boundedness of $\varphi(u_n)$. Hence $(|\bar{u}_n|)$ is bounded, and $\|u_n\|$ is bounded by (2.3) and (3.7).

The sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \text{ weakly in } W_T^{1,p(t)} \quad \text{and} \quad u_n \to u \text{ strongly in } C([0,T],\mathbb{R}^N) \tag{3.14}$$

and $\|u_n\|_\infty \leqslant C_{12}$ is bounded by Lemma 2.6, where C_{12} is a positive constant.

We conclude that

$$\left|\int_0^T (\nabla F(t,u_n(t)), u_n(t)-u(t))\,\mathrm{d}t\right| \leqslant a_0 T\|u_n-u\|_\infty \int_0^T b(t)\,\mathrm{d}t \to 0, \tag{3.15}$$

by (3.14) and assumption (A), where $a_0 = \max\limits_{0\leqslant s\leqslant C_{12}} a(s)$.

By Lemma 2.12 we have

$$\begin{aligned}\langle \varphi'(u_n), u_n-u\rangle = \int_0^T &[(|\dot u_n(t)|^{p(t)-2}\dot u_n(t), \dot u_n(t)-\dot u(t)) \\ &+ (\nabla F(t,u_n(t)), u_n(t)-u(t))]\,\mathrm{d}t,\end{aligned} \tag{3.16}$$

and $\langle \varphi'(u_n), u_n-u\rangle \to 0$ by the assumption of $\varphi'(u_n)\to 0$ and the boundness of $\{\|u_n\|\}$.

Then it follows from (2.5), (3.15), and (3.16) that

$$\begin{aligned}&\langle J'(u_n), u_n-u\rangle \\ &\quad= \int_0^T (|\dot u_n(t)|^{p(t)-2}\dot u_n(t), \dot u_n(t)-\dot u(t))\,\mathrm{d}t \\ &\quad= \langle \varphi'(u_n), u_n-u\rangle - \int_0^T (\nabla F(t,u_n(t)), u_n(t)-u(t))\,\mathrm{d}t \to 0.\end{aligned} \tag{3.17}$$

Moreover, since $J'(u)$ is a bounded linear function, we get $\langle J'(u), u_n-u\rangle \to 0$, which combined with (3.17) implies that

$$\lim_{n\to\infty} \langle J'(u_n)-J'(u), u_n-u\rangle = 0. \tag{3.18}$$

It follows from Lemma 2.14 and (3.18) that $\{u_n\}$ admits a convergent subsequence.

We now prove that φ satisfies the other conditions of the Saddle Point Theorem. Let $\widetilde{W}_T^{1,p(t)}$ be the subspace of $W_T^{1,p(t)}$ given by

$$\widetilde{W}_T^{1,p(t)} = \left\{u \in W_T^{1,p(t)} \colon \int_0^T u(t)\,\mathrm{d}t = 0\right\},$$

then we have

$$\varphi(u) \to \infty \tag{3.19}$$

as $\|u\| \to \infty$ in $\widetilde{W}_T^{1,p(t)}$. In fact, it follows from Lemma 2.7 that

$$\begin{aligned}
&\left|\int_0^T [H(t,u(t)) - H(t,0)]\,\mathrm{d}t\right| \\
&\qquad = \left|\int_0^T\int_0^1 (\nabla H(t,su(t)), u(t))\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\qquad \leqslant \int_0^T\int_0^1 f(t)|su(t)|^{\alpha}|u(t)|\,\mathrm{d}t + \int_0^T\int_0^1 g(t)|u(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad \leqslant \|u\|_\infty^{\alpha+1}\int_0^T f(t)\,\mathrm{d}t + \|u\|_\infty\int_0^T g(t)\,\mathrm{d}t \\
&\qquad \leqslant C_{13}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{(\alpha+1)/p^-} + C_{14}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} + C_{15}
\end{aligned}$$

for all $u \in \widetilde{W}_T^{1,p(t)}$ and some positive constants C_{13}, C_{14}, and C_{15}.

By (1.7), Lemma 2.2, and (3.3) we have

$$\begin{aligned}
\int_0^T [G(u(t)) - G(0)]\,\mathrm{d}t &= \int_0^T\int_0^1 (\nabla G(su(t)) - \nabla G(0), u(t))\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^T\int_0^1 \frac{1}{s}(\nabla G(su(t)) - \nabla G(0), su(t))\,\mathrm{d}s\,\mathrm{d}t \\
&\geqslant \int_0^T\int_0^1 \frac{1}{s}(-rs^{\beta}|u(t)|^{\beta})\,\mathrm{d}s\,\mathrm{d}t \\
&\geqslant \frac{-rT^{p^-}}{\beta}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big) - C_{16}
\end{aligned}$$

for all $u \in \widetilde{W}_T^{1,p(t)}$ and some positive constant C_{16}. Hence, we have

$$\begin{aligned}
\varphi(u) - \int_0^T F(t,0)\,\mathrm{d}t &= \int_0^T \frac{1}{p(t)}|\dot u(t)|^{p(t)}\,\mathrm{d}t + \int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t \\
&= \int_0^T \frac{1}{p(t)}|\dot u(t)|^{p(t)} + \int_0^T [G(u(t)) - G(0)]\,\mathrm{d}t \\
&\quad + \int_0^T [H(t,u(t)) - H(t,0)]\,\mathrm{d}t \\
&\geqslant \Big(\frac{1}{p^+} - \frac{rT^{p^-}}{\beta}\Big)\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t - C_{13}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{(\alpha+1)/p^-} \\
&\quad - C_{14}\Big(\int_0^T |\dot u(t)|^{p(t)}\,\mathrm{d}t\Big)^{1/p^-} - C_{15} - C_{16}
\end{aligned}$$

for all $u \in \widetilde{W}_T^{1,p(t)}$, which implies (3.19) by Proposition 2.1.

Moreover, we have

$$\begin{aligned}\varphi(x) = \int_0^T F(t,x)\,\mathrm{d}t &\leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,\mathrm{d}t - \int_{E_\delta} N|x|^{q^+\alpha}\,\mathrm{d}t \\ &\leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,\mathrm{d}t - NM^{q^+\alpha}\operatorname{meas} E_\delta \\ &\leqslant \int_{[0,T]\setminus E_\delta} \gamma(t)\,\mathrm{d}t - N\operatorname{meas} E_\delta\end{aligned}$$

for all $|x| \geqslant M$ by (3.11), which implies that

$$\varphi(x) \to -\infty \tag{3.20}$$

as $|x| \to \infty$ in $\mathbb{R}^N$ by the arbitrariness of N.

We have proved that the fuctional φ satisfies all the conditions of the Saddle Point Theorem, so we know that φ has at least one critical point by the Saddle Point Theorem, which is a periodic solution of system (1.1). The proof is completed. □

Now we prove Theorem 3.4.

P r o o f of Theorem 3.4. Let $\gamma = \log_{2\lambda}(2\mu)$. Then $0 \leqslant \gamma < p^-$. For $|x| > 1$ there exists a positive integer n such that

$$n - 1 < \log_{2\lambda}|x| \leqslant n.$$

Then one has $|x|^\gamma > (2\lambda)^{(n-1)\gamma} = (2\mu)^{n-1}$ and $|x| \leqslant (2\lambda)^n$. Hence we have

$$F_1(t,x) \leqslant 2\mu F_1\Big(t, \frac{x}{2\lambda}\Big) \leqslant \ldots \leqslant (2\mu)^n F_1\Big(t, \frac{x}{(2\lambda)^n}\Big) \leqslant 2\mu|x|^\gamma a_0 b(t)$$

for a.e. $t \in [0,T]$ and all $|x| > 1$ by (i) in Theorem 3.4 and assumption (A), where $a_0 = \max\limits_{0\leqslant s\leqslant 1} a(s)$. Moreover, one obtains

$$F_1(t,x) \leqslant (2\mu|x|^\gamma + 1)a_0 b(t) \tag{3.21}$$

for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$, where $0 \leqslant \gamma < p^-$.

It follows from (i) in Theorem 3.5 and from Lemma 2.7 that

$$
\begin{aligned}
(3.22)\quad \left|\int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,\mathrm{d}t\right| &= \left|\int_0^T\int_0^1 (\nabla F_2(t,\bar{u}+s\tilde{u}(t)),\tilde{u}(t))\,\mathrm{d}s\,\mathrm{d}t\right| \\
&\leqslant \int_0^T\int_0^1 f(t)|\bar{u}+s\tilde{u}(t)|^{\alpha}|\tilde{u}(t)|\,\mathrm{d}t + \int_0^T\int_0^1 g(t)|\tilde{u}(t)|\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant 2^{p^- -1}(|\bar{u}|^{\alpha} + \|\tilde{u}\|_\infty^{\alpha})\|\tilde{u}\|_\infty \int_0^T f(t)\,\mathrm{d}t + \|\tilde{u}\|_\infty \int_0^T g(t)\,\mathrm{d}t \\
&= \left(\left(\frac{1}{2p^+}\right)^{1/p^-}\frac{\|\tilde{u}\|_\infty}{4C_0}\right)\left((2^{p^-+1})(2p^+)^{1/p^-}C_0\int_0^T f(t)\,\mathrm{d}t\right)|\bar{u}|^{\alpha} \\
&\quad + 2^{p^- -1}\|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)\,\mathrm{d}t + \|\tilde{u}\|_\infty\int_0^T g(t)\,\mathrm{d}t \\
&\leqslant \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t + D_0|\bar{u}|^{q^+\alpha} + D_1\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{(\alpha+1)/p^-} \\
&\quad + D_2\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{1/p^-} + D_3
\end{aligned}
$$

for all $u \in W_T^{1,p(t)}$ with the same C_0 as in Lemma 2.7, and some positive constants D_0, D_1, D_2, D_3. Hence, we have

$$
\begin{aligned}
\varphi(u) &\geqslant \int_0^T \frac{1}{p(t)}|\dot{u}(t)|^{p(t)}\,\mathrm{d}t + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t - \int_0^T F_1(t,-\tilde{u}(t))\,\mathrm{d}t \\
&\quad + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]\,\mathrm{d}t \\
&\geqslant \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t - D_0|\bar{u}|^{q^+\alpha} - D_1\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{(\alpha+1)/p^-} \\
&\quad - D_2\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{1/p^-} - (2\mu\|\tilde{u}(t)\|_\infty^{\gamma} + 1)\int_0^T a_0 b(t)\,\mathrm{d}t \\
&\quad + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t - D_4 \\
&\geqslant \frac{1}{2p^+}\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t - D_1\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{(\alpha+1)/p^-} - D_2\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{1/p^-} \\
&\quad + |\bar{u}|^{q^+\alpha}\left\{\frac{1}{|\bar{u}|^{q^+\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})\,\mathrm{d}t + \int_0^T F_2(t,\bar{u})\,\mathrm{d}t\right] - D_0\right\} \\
&\quad - D_5\left(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\right)^{\gamma/p^-} - D_6
\end{aligned}
$$

for all $u \in W_T^{1,p(t)}$ and some positive constants D_5, D_6, which implies that

$$\varphi(u) \to \infty$$

as $\|u\| \to \infty$ by Proposition 2.1. By Theorem 1.1 and Corollary 1.1 in [12], we complete the proof. $\square$

Next we prove Theorem 3.5.

Proof of Theorem 3.5. Let (u_k) be a minimizing sequence of φ. It follows from (i) of Theorem 3.5 and from Lemma 2.7 that

$$\begin{aligned}
\varphi(u_k) \geqslant{} & \frac{1}{p^+}\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t + \int_0^T (h(t), u_k(t))\,\mathrm{d}t + \int_0^T \gamma_1(t)\,\mathrm{d}t \\
& + \int_0^T F_2(t, \bar{u}_k)\,\mathrm{d}t + \int_0^T\int_0^1 (\nabla F_2(t, \bar{u}_k + s\tilde{u}_k(t), \tilde{u}_k(t))\,\mathrm{d}s\,\mathrm{d}t \\
\geqslant{} & \frac{1}{p^+}\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t + \|\tilde{u}_k\|_\infty \int_0^T h(t)\,\mathrm{d}t - \|\tilde{u}_k\|_\infty \int_0^T g(t)\,\mathrm{d}t + D_7 \\
\geqslant{} & \frac{1}{p^+}\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t - D_8\Bigl(\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/p^-} - D_9
\end{aligned}$$

for all k and some constants D_8 and D_9, which implies that $(\|\tilde{u}_k\|_\infty)$ is bounded by Lemma 2.7. On the other hand, in a way similar to the proof of Theorem 3.4, we have

$$\Bigl|\int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})]\,\mathrm{d}t\Bigr| \leqslant D_{10}\Bigl(\Bigl(\int_0^T |\dot{u}(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/p^-} + 1\Bigr)$$

for all $u \in W_T^{1,p(t)}$ and some positive constant D_{10}, which implies that

$$\begin{aligned}
\varphi(u_k) \geqslant{} & \frac{1}{p^+}\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)\,\mathrm{d}t - \int_0^T F_1(t, -\tilde{u}_k(t))\,\mathrm{d}t \\
& + \int_0^T F_2(t, \bar{u}_k)\,\mathrm{d}t + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u}_k)]\,\mathrm{d}t \\
\geqslant{} & \frac{1}{p^+}\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t - a(\|\tilde{u}_k\|_\infty)\int_0^T b(t)\,\mathrm{d}t \\
& - D_{10}\Bigl(\Bigl(\int_0^T |\dot{u}_k(t)|^{p(t)}\,\mathrm{d}t\Bigr)^{1/p^-} + 1\Bigr) \\
& + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)\,\mathrm{d}t + \int_0^T F_2(t, \bar{u}_k)\,\mathrm{d}t
\end{aligned}$$

for all positive integers k. It follows from the boundedness of $(\tilde{u}_k)$ that $(\bar{u}_k)$ is bounded. Hence φ has a bounded minimizing sequence (u_k). Now Theorem 3.5 follows from Theorem 1.1 and Corollary 1.1 in [12]. $\square$

P r o o f of Theorem 3.6. In view of the proofs of Theorem 3.4 and 3.5, the conclusion of Theorem 3.6 holds. The proof is complete. □

4. Examples

In this section we give three examples to illustrate our results.

E x a m p l e 4.1. In system (1.1), let $p(t) = 4 + 2\cos\omega t$, and let

$$G(x) = -\frac{1}{14T^2}|x_1|^2 \quad \text{and} \quad H(t,x) = -|x|^{1+\alpha},$$

where ω denotes the positive constant $2\pi/T$, $0 < \alpha < 1$.

This shows that all conditions of Theorem 3.2 are satisfied, where

$$\beta = 2, \quad p^- = 2, \quad q^+ = 2.$$

By Theorem 3.2, system (1.1) has at least one periodic solution.

R e m a r k 4.1. Here F satisfies the conditions of our Theorem 3.2, but for $F(t,x)$ the results mentioned in [21] do not hold because $F(t,x)$ is neither generalized superquadratic nor generalized subquadratic in x.

E x a m p l e 4.2. In system (1.1), let $p(t) = 6 + \cos\omega t$, and let

$$F_1(t,x) = |x|^4 \quad \text{and} \quad F_2(t,x) = |\sin\omega t||x|^3,$$

where ω denotes the positive constant $2\pi/T$. Then $F_1(t,x)$ is $(1,8)$ subconvex, and

$$|\nabla F_2(t,x)| = 3|\sin\omega t||x|^2.$$

This shows that all conditions of Theorem 3.4 are satisfied, where

$$\alpha = 2, \quad p^- = 5, \quad q^+ = \frac{5}{4}.$$

By Theorem 3.4, system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

E x a m p l e 4.3. In system (1.1), let $p(t) = 6 + t$, and let

$$F_1(t,x) = (|x|^2 + \ln(1+|x|^2)) \quad \text{and} \quad F_2(t,x) = (h(t),x),$$

where $h \in L^1([0,T],\mathbb{R}^N)$ with $\int_0^T h(t) = 0$. Then $F_1(t,x)$ is $(1/2,1)$ subconvex and $F_2(t,x)$ satisfies the other conditions of Theorem 3.5, so system (1.1) has at least one solution which minimizes φ on $W_T^{1,p(t)}$.

References

[1] *M. S. Berger, M. Schechter*: On the solvability of semilinear gradient operator equations. Adv. Math. *25* (1977), 97–132.

[2] *X.-L. Fan, X. Fan*: A Knobloch-type result for $p(t)$-Laplacian systems. J. Math. Anal. Appl. *282* (2003), 453–464.

[3] *X.-L. Fan, D. Zhao*: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. *263* (2001), 424–446.

[4] *X.-L. Fan, Q.-H. Zhang*: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal., Theory Methods Appl. *52* (2003), 1843–1852.

[5] *G.-H. Fei*: On periodic solutions of superquadratic Hamiltonian systems. Electron. J. Differ. Equ., paper No. 8 (2002), 1–12.

[6] *J.-X. Feng, Z.-Q. Han*: Periodic solutions to differential systems with unbounded or periodic nonlinearities. J. Math. Anal. Appl. *323* (2006), 1264–1278.

[7] *Q. Jiang, C. L. Tang*: Periodic and subharmonic solutions of a class of subquadratic Hamiltonian systems. J. Math. Anal. Appl. *328* (2007), 380–389.

[8] *S.-J. Li, M. Willem*: Applications of local linking to critical point theory. J. Math. Anal. Appl. *189* (1995), 6–32.

[9] *J. Q. Liu*: A generalized saddle point theorem. J. Differ. Equations *82* (1989), 372–385.

[10] *S. Ma, Y. Zhang*: Existence of infinitely many periodic solutions for ordinary p-Laplacian systems. J. Math. Anal. Appl. *351* (2009), 469–479.

[11] *J. Mawhin*: Semi-coercive monotone variational problems. Bull. Cl. Sci., V. Sér., Acad. R. Belg. *73* (1987), 118–130.

[12] *J. Mawhin, M. Willem*: Critical point theory and Hamiltonian systems. Applied Mathematical Sciences, Vol. 74. Springer, New York, 1989.

[13] *P. H. Rabinowitz*: Periodic solutions of Hamiltonian systems. Comm. Pure Appl. Math. *31* (1978), 157–184.

[14] *P. H. Rabinowitz*: On subharmonic solutions of Hamiltonian systems. Commun. Pure Appl. Math. *33* (1980), 609–633.

[15] *C.-L. Tang*: Periodic solutions of non-autonomous second order systems with γ-quasisubadditive potential. J. Math. Anal. Appl. *189* (1995), 671–675.

[16] *C.-L. Tang*: Periodic solutions of non-autonomous second order systems with sublinear nonlinearity. Proc. Am. Math. Soc. *126* (1998), 3263–3270.

[17] *C.-L. Tang, X.-P. Wu*: Periodic solutions for second order systems with not uniformly coercive potential. J. Math. Anal. Appl. *259* (2001), 386–397.

[18] *C.-L. Tang, X.-P. Wu*: Notes on periodic solutions of subquadratic second order systems. J. Math. Anal. Appl. *285* (2003), 8–16.

[19] *C.-L. Tang, X.-P. Wu*: A note on periodic solutions of nonautonomous second order systems. Proc. Am. Math. Soc. *132* (2004), 1295–1303.

[20] *Z.-L. Tao, C.-L. Tang*: Periodic and subharmonic solutions of second order Hamiltonian systems. J. Math. Anal. Appl. *293* (2004), 435–445.

[21] *X.-J. Wang, R. Yuan*: Existence of periodic solutions for $p(t)$-Laplacian systems. Nonlinear Anal., Theory Methods Appl. *70* (2009), 866–880.

[22] *M. Willem*: Oscillations forcées de systèms hamiltoniens. Sémin. Anal. Non Linéaire. Univ. Besancon, 1981.

[23] *X.-P. Wu*: Periodic solutions for nonautonomous second-order systems with bounded nonlinearity. J. Math. Anal. Appl. *230* (1999), 135–141.

[24] *X.-P. Wu, C.-L. Tang*: Periodic solutions of a class of non-autonomous second-order systems. J. Math. Anal. Appl. *236* (1999), 227–235.

[25] *B. Xu, C.-L. Tang*: Some existence results on periodic solutions of ordinary p-Laplacian systems. J. Math. Anal. Appl. *333* (2007), 1228–1236.
[26] *Y.-W. Ye, C.-L. Tang*: Periodic solutions for some nonautonomous second order Hamiltonian systems. J. Math. Anal. Appl. *344* (2008), 462–471.
[27] *V. V. Zhikov*: Averaging of functionals in the calculus of variations and elasticity theory. Math. USSR, Izv. *29* (1987), 33–66.
[28] *V. V. Zhikov*: On passage to the limit in nonlinear variational problems. Suss. Acad. Sci, Sb., Math. *183* (1992), 47–84.

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