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ON THE MEAN VALUE OF DEDEKIND SUM WEIGHTED BY THE QUADRATIC GAUSS SUM

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Abstract. Various properties of classical Dedekind sums $S(h,q)$ have been investigated by many authors. For example, Wenpeng Zhang, On the mean values of Dedekind sums, J. Théor. Nombres Bordx, 8 (1996), 429–442, studied the asymptotic behavior of the mean value of Dedekind sums, and H. Rademacher and E. Grosswald, Dedekind Sums, The Carus Mathematical Monographs No. 16, The Mathematical Association of America, Washington, D.C., 1972, studied the related properties. In this paper, we use the algebraic method to study the computational problem of one kind of mean value involving the classical Dedekind sum and the quadratic Gauss sum, and give several exact computational formulae for it.

Keywords: Dedekind sum, quadratic Gauss sum, mean value, identity

MSC 2010: 11L40, 11F20

1. Introduction

Let q be a natural number and h an integer prime to q. The classical Dedekind sum

$$S(h,q)=\sum_{a=1}^{q}\Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{ah}{q}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer};\\ 0, & \text{if } x \text{ is an integer},\end{cases}$$

describes the behaviour of the logarithm of the eta-function (see [6], [7]) under modular transformations. Several authors have studied the arithmetical properties of

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$S(h, q)$, and obtained many interesting results, see for example [2], [3], [4], [5], [8], [9], [10].

In this paper, we consider the computational problem of the mean value

$$\sum_{c=1}^{q}{}' G(c^2-1, \chi_0; q) S(c^2, q), \tag{1.1}$$

where $G(c, \chi; q)$ is the quadratic Gauss sum defined by $G(c, \chi; q) = \sum_{a=1}^{q} \chi(a) e(ca^2/q)$, $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$, $\sum_{c}'$ denotes the summation over all a such that $(a, q) = 1$, χ_0 denotes the principal character $\bmod\, q$.

About the mean value (1.1), it seems that none has studied it yet, at least we have not seen any related results before. This sum is interesting, because it has close relations with the class number h_p of the quadratic field $\mathbf{Q}(\sqrt{-p})$. In this paper, we use the algebraic method to study the computational problem of (1.1), and give several exact identities for it. That is, we shall prove the following three results:

Theorem 1. *Let p be an odd prime with $p \equiv 3 \bmod 4$, then for $q = p^\alpha$ with integer $\alpha \geqslant 2$, we have the identity*

$$\sum_{c=1}^{q}{}' G(c^2-1, \chi_0; q) S(c^2, q) = \frac{1}{6} \varphi^2(q) \Big(1 + \frac{1}{p}\Big).$$

Theorem 2. *For any prime $p > 3$ with $p \equiv 3 \bmod 4$, we have the identity*

$$h_p^2 = \frac{(p-1)(p-2)}{6} - \sum_{c=1}^{p-1} G(c^2-1, \chi_0; p) S(c^2, p).$$

Theorem 3. *Let p be an odd prime with $p \equiv 3 \bmod 4$, then for $q = p^\alpha$ with integer $\alpha \geqslant 2$ and real number $k \geqslant 0$, we have the identity*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |G(1, \chi^2; q)|^{2k} \cdot |L(1, \chi)|^2 = \frac{4^k \cdot \pi^2}{24} \cdot \varphi^3(q) \cdot q^{k-2} \cdot \Big(1 + \frac{1}{p}\Big),$$

where $\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ denotes the summation over all odd characters $\chi \bmod q$.

Some remarks. Theorem 2 is very interesting. In fact it gives a new class number formula for h_p with $p \equiv 3 \bmod 4$.

For the general odd square-full number $q \geqslant 3$, whether there exist computational formulae for $\sum_{c=1}^{q}{}' G(c^2-1,\chi_0;q)S(c^2,q)$ and $\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |G(1,\chi^2;q)|^{2k}\cdot|L(1,\chi)|^2$ are two open problems.

2. Several lemmas

In this section we give several lemmas which are necessary in the proof of our theorems.

Lemma 1. *Let $q>2$ be an integer, then for any integer a with $(a,q)=1$ we have the identity*

$$S(a,q)=\frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2.$$

Proof. See Lemma 2 of [10]. □

Lemma 2. *Let $q>2$ be an odd square-full number, then we have the identity*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{\pi^2}{12}\frac{\varphi^3(q)}{q^2}\prod_{p\mid q}\Big(1+\frac{1}{p}\Big),$$

where $\sideset{}{^*}\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=-1}}$ *denotes the summation over all odd primitive characters $\chi \bmod q$.*

Proof. From the definition of Dedekind sums, Lemma 1 and the Möbius inversion formula (see Theorem 2.9 of [1]) we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2=\frac{\varphi(q)}{q^2}\pi^2\sum_{d\mid q}\mu(d)\frac{q}{d}S\Big(a,\frac{q}{d}\Big)=\pi^2\frac{\varphi(q)}{q}\sum_{d\mid q}\frac{\mu(d)}{d}S\Big(a,\frac{q}{d}\Big). \tag{2.1}$$

If $a=1$, then it is easy to see that

$$S(1,q)=\sum_{k=1}^{q-1}\Big(\frac{k}{q}-\frac{1}{2}\Big)^2=\frac{1}{12}\Big(q-3+\frac{2}{q}\Big).$$

So from this formula and (2.1) we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\varphi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d}\Big(\frac{q}{d}-3+\frac{2d}{q}\Big) \tag{2.2}$$

$$= \frac{\pi^2}{12}\varphi(q)\sum_{d|q}\frac{\mu(d)}{d^2} - \frac{\pi^2}{4}\frac{\varphi(q)}{q}\sum_{d|q}\frac{\mu(d)}{d} + \frac{\pi^2}{6}\frac{\varphi(q)}{q^2}\sum_{d|q}\mu(d)$$

$$= \frac{\pi^2}{12}\frac{\varphi^2(q)}{q}\Big[\prod_{p|q}\Big(1+\frac{1}{p}\Big)-\frac{3}{q}\Big].$$

Note that q is a square-full number, $\mu(q)$ and $\varphi(q)$ are two multiplicative functions,

$$\sum_{d|q}\mu(d)\frac{\varphi^2(q/d)}{q^2/d^2}=0 \quad \text{and} \quad \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \sum_{d|q}\sideset{}{^*}\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}} |L(1,\chi\chi_0)|^2.$$

From the Möbius inversion formula and (2.2) we immediately deduce

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}} |L(1,\chi\chi_0)|^2$$

$$= \sum_{d|q}\mu(d)\sum_{\substack{\chi \bmod q/d\\ \chi(-1)=-1}} |L(1,\chi)|^2$$

$$= \sum_{d|q}\mu(d)\Big\{\frac{\pi^2}{12}\frac{\varphi^2(q/d)}{q/d}\Big[\prod_{p|\frac{q}{d}}\Big(1+\frac{1}{p}\Big)-\frac{3}{q/d}\Big]\Big\}$$

$$= \frac{\pi^2}{12}\frac{\varphi^3(q)}{q^2}\prod_{p|q}\Big(1+\frac{1}{p}\Big),$$

where χ_0 denotes the principal character $\bmod\, q$. This proves Lemma 2. □

Lemma 3. *Let p be an odd prime, $\alpha \geqslant 2$ an integer. Then for any integer n with $(p,n)=1$, we have the identity*

$$\sideset{}{'}\sum_{b=1}^{p^\alpha} e\Big(\frac{nb^2}{p^\alpha}\Big)=0.$$

P r o o f. First, from the properties of the trigonometric sums we know that for any positive integer $h \geqslant 2$ and integer n with $(n,h)=1$, we have the identity

$$\sum_{u=0}^{h-1} e\Big(\frac{un}{h}\Big)=0.$$

Applying this identity and the properties of the reduced residue system mod p^α we have the identity

$$\sum_{b=1}^{p^\alpha}{}' e\Big(\frac{nb^2}{p^\alpha}\Big) = \sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}}{}' e\Big(\frac{n(up^{\alpha-1}+v)^2}{p^\alpha}\Big) = \sum_{u=0}^{p-1}\sum_{v=1}^{p^{\alpha-1}}{}' e\Big(\frac{2nuvp^{\alpha-1}+nv^2}{p^\alpha}\Big)$$
$$= \sum_{v=1}^{p^{\alpha-1}}{}' e\Big(\frac{nv^2}{p^\alpha}\Big)\sum_{u=0}^{p-1} e\Big(\frac{2nuv}{p}\Big) = 0.$$

This proves Lemma 3. □

Lemma 4. *Let p be an odd prime with $p \equiv 3 \bmod 4$, let α and n be two integers with $(n,p)=1$ and $\alpha \geqslant 2$. Then for any even character $\chi \bmod p^\alpha$, we have*

$$|G(n,\chi;p^\alpha)|^2 = 2\varphi(p^\alpha) + 2p^{\alpha-1}\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}+1)\Big[\Big(\frac{2rn}{p}\Big)C(1,p)-1\Big],$$

where $C(n,q) = \sum_{a=1}^{q} e(na^2/q)$ is the classical quadratic Gauss sum.

Proof. Since χ is an even character mod p^α, so $\chi(-1)=1$. Then from the definition of $G(n,\chi;p^\alpha)$ we have

$$|G(n,\chi;p^\alpha)|^2 = \sum_{a=1}^{p^\alpha}{}'\sum_{b=1}^{p^\alpha}{}' \chi(a)\overline{\chi}(b) e\Big(\frac{n(a^2-b^2)}{p^\alpha}\Big) \tag{2.3}$$
$$= \sum_{a=1}^{p^\alpha}{}' \chi(a)\sum_{b=1}^{p^\alpha}{}' e\Big(\frac{nb^2(a^2-1)}{p^\alpha}\Big).$$

Let $(a^2-1,p^\alpha)=p^m$. If $m \leqslant \alpha-2$, then note that $(n(a^2-1)/p^m, p)=1$, and from Lemma 3 we have

$$\sum_{u=1}^{p^\alpha}{}' e\Big(\frac{nu^2(a^2-1)}{p^\alpha}\Big) = p^m\sum_{u=1}^{p^{\alpha-m}}{}' e\Big(\frac{nu^2(a^2-1)/p^m}{p^{\alpha-m}}\Big) = 0. \tag{2.4}$$

If $m=\alpha$, then

$$\sum_{u=1}^{p^\alpha}{}' e\Big(\frac{nu^2(a^2-1)}{p^\alpha}\Big) = \varphi(p^\alpha). \tag{2.5}$$

If $m=\alpha-1$, then $a=rp^{\alpha-1}\pm 1$, $1\leqslant r\leqslant p-1$. Note that for any prime p with $p\nmid n$, by Theorem 7.5.4 of [10] we have

$$C(n,p)=\Big(\frac{n}{p}\Big)C(1,p), \tag{2.6}$$

where $(\frac{x}{p})$ is the Legendre symbol.

Then from (2.6) we get

$$\begin{aligned}\sideset{}{'}\sum_{u=1}^{p^\alpha} e\Big(\frac{nu^2(a^2-1)}{p^\alpha}\Big)&=p^{\alpha-1}\sideset{}{'}\sum_{u=1}^{p} e\Big(\frac{nu^2(a^2-1)/p^{\alpha-1}}{p}\Big)\\&=p^{\alpha-1}\Big[\Big(\frac{\pm 2rn}{p}\Big)C(1,p)-1\Big].\end{aligned} \tag{2.7}$$

Note that for any even character $\chi \bmod p^\alpha$, we have

$$\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}+1)\Big[\Big(\frac{2rn}{p}\Big)C(1,p)-1\Big]=\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}-1)\Big[\Big(\frac{-2rn}{p}\Big)C(1,p)-1\Big].$$

So from (2.3)–(2.7) we get

$$|G(n,\chi;p^\alpha)|^2=2\varphi(p^\alpha)+2p^{\alpha-1}\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}+1)\Big[\Big(\frac{2rn}{p}\Big)C(1,p)-1\Big].$$

This proves Lemma 4. $\square$

Lemma 5. *Let p be an odd prime, let α and n be two integers with $(n,p)=1$ and $\alpha\geqslant 2$. Then for any even primitive character $\chi \bmod p^\alpha$, we have the identity*

$$|G(n,\chi;p^\alpha)|^2=2p^\alpha+2p^\alpha\Big(\frac{-2n}{p}\Big)\frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})},$$

where $\chi_2(a)=(\frac{a}{p})$ is the Legendre symbol, and $\tau(\chi)=\sum_{a=1}^{p^\alpha}\chi(a)e(a/p^\alpha)$ denotes the classical Gauss sum.

P r o o f. First, for any primitive character $\chi \bmod p^\alpha$ we have

$$\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}+1)=-1. \tag{2.8}$$

In fact, from the properties of the classical Gauss sum we have

$$\sum_{r=1}^{p-1} \chi(rp^{\alpha-1}+1) = \frac{1}{\tau(\overline{\chi})} \sum_{r=1}^{p-1} \sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{a(rp^{\alpha-1}+1)}{p^\alpha}\Big)$$
$$= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{a}{p^\alpha}\Big) \sum_{r=1}^{p-1} e\Big(\frac{ar}{p}\Big) = -\frac{\tau(\overline{\chi})}{\tau(\overline{\chi})} = -1.$$

So formula (2.8) is correct.

On the other hand, from the properties of the classical Gauss sum we also have

$$\sum_{r=1}^{p-1} \chi\Big(rp^{\alpha-1}+1\Big)\Big(\frac{r}{p}\Big) = \frac{1}{\tau(\overline{\chi})} \sum_{r=1}^{p-1} \Big(\frac{r}{p}\Big) \sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{a(rp^{\alpha-1}+1)}{p^\alpha}\Big) \tag{2.9}$$
$$= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p^\alpha} \overline{\chi}(a) e\Big(\frac{a}{p^\alpha}\Big) \sum_{r=1}^{p-1} \Big(\frac{r}{p}\Big) e\Big(\frac{ar}{p}\Big)$$
$$= \frac{C(1,p)}{\tau(\overline{\chi})} \sum_{a=1}^{p^\alpha} \overline{\chi}(a) \Big(\frac{a}{p}\Big) e\Big(\frac{a}{p^\alpha}\Big) = \frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})} C(1,p),$$

where we have used the identity $C(1,p) = \sum_{a=1}^{p-1} (\frac{a}{p}) e(\frac{a}{p})$.

From the properties of the classical Gauss sum we know that $C^2(1,p) = (\frac{-1}{p})p$; applying (2.8), (2.9) and Lemma 4 we immediately deduce that

$$|G(n,\chi;p^\alpha)|^2 = 2\varphi(p^\alpha) + 2p^{\alpha-1} + 2p^{\alpha-1} \sum_{r=1}^{p-1} \chi(rp^{\alpha-1}+1) \Big(\frac{2rn}{p}\Big) C(1,p)$$
$$= 2p^\alpha + 2p^{\alpha-1} \Big(\frac{2n}{p}\Big) \frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})} C^2(1,p)$$
$$= 2p^\alpha + 2p^\alpha \Big(\frac{-2n}{p}\Big) \frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})}.$$

This proves Lemma 5. □

3. Proof of the theorems

In this section, we use the lemmas from Section 2 to complete the proof of our theorems. First we prove Theorem 1. From Lemma 4 we know that for any non-primitive even character $\chi \bmod p^\alpha$, we have

$$|G(n,\chi;p^\alpha)|^2 = 0. \tag{3.1}$$

In fact, if $\chi = \chi_0$, the principal character $\bmod\, p^\alpha$, then from Lemma 3 we know that (3.1) is correct. If χ is a non-primitive even character $\bmod\, p^\alpha$ and $\chi \neq \chi_0$, then χ must be an even character $\bmod\, p^{\alpha-1}$. So from Lemma 4 we have

$$\begin{aligned}
|G(n,\chi;p^\alpha)|^2 &= 2\varphi(p^\alpha) + 2p^{\alpha-1}\sum_{r=1}^{p-1}\chi(rp^{\alpha-1}+1)\Big[\Big(\frac{2rn}{p}\Big)G(1;p)-1\Big] \\
&= 2\varphi(p^\alpha) + 2p^{\alpha-1}\sum_{r=1}^{p-1}\Big[\Big(\frac{2rn}{p}\Big)G(1;p)-1\Big] \\
&= 2p^{\alpha-1}\sum_{r=1}^{p-1}\Big(\frac{2rn}{p}\Big)C(1,p) = 0.
\end{aligned}$$

So (3.1) is also correct.

For $q = p^\alpha$ with $\alpha \geqslant 2$ and $\chi \bmod q$, note that the Gauss sum $\tau(\chi) = \sum\limits_{a=1}^{q}\chi(a)e(\frac{a}{q}) = 0$, if χ is not a primitive character $\bmod\, q$; if χ is a primitive character $\bmod\, q$, then we have $|\tau(\chi)|^2 = q$. So from Lemma 1 we have

$$\begin{aligned}
(3.2)\qquad & \sideset{}{'}\sum_{c=1}^{q} G(c^2-1,\chi_0;q)S(c^2,q) \\
&= \frac{1}{\pi^2 q}\sum_{\substack{d|q\,\frac{d^2}{\varphi(d)}\sum\limits_{\chi \bmod d} \\ \chi(-1)=-1}} \sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{a=1}^{q}\chi(c^2)e\Big(\frac{a^2(c^2-1)}{q}\Big)|L(1,\chi)|^2 \\
&= \frac{1}{\pi^2 q}\sum_{\substack{d|q\,\frac{d^2}{\varphi(d)}\sum\limits_{\chi \bmod d} \\ \chi(-1)=-1}} \sideset{}{'}\sum_{a=1}^{q}e\Big(\frac{-a^2}{q}\Big)\sideset{}{'}\sum_{c=1}^{q}\chi(c^2)e\Big(\frac{a^2c^2}{q}\Big)|L(1,\chi)|^2 \\
&= \frac{1}{\pi^2 q}\sum_{\substack{d|q\,\frac{d^2}{\varphi(d)}\sum\limits_{\chi \bmod d} \\ \chi(-1)=-1}} \sideset{}{'}\sum_{a=1}^{q}\overline{\chi}(a^2)e\Big(\frac{-a^2}{q}\Big)\sideset{}{'}\sum_{c=1}^{q}\chi((ac)^2)\; e\Big(\frac{(ac)^2}{q}\Big)|L(1,\chi)|^2 \\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\Big|\sideset{}{'}\sum_{c=1}^{q}\chi(c^2)\; e\Big(\frac{c^2}{q}\Big)\Big|^2\cdot|L(1,\chi)|^2 \\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}|G(1,\chi^2\chi_0;q)|^2\cdot|L(1,\chi)|^2.
\end{aligned}$$

It is clear that if χ is an odd primitive character $\bmod\, p^\alpha$ with $\alpha \geqslant 2$ and $p \equiv 3 \bmod 4$, then χ^2 is an even primitive character $\bmod\, p^\alpha$. So from (3.1), (3.2) and Lemma 5 we get

$$\sum_{c=1}^{q}{}' G(c^2-1,\chi_0;q)S(c^2,q) \tag{3.3}$$
$$= \frac{1}{\pi^2}\frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |G(1,\chi^2;q)|^2 \cdot |L(1,\chi)|^2$$
$$= \frac{2}{\pi^2}\frac{q^2}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \Big(1+\Big(\frac{-2}{p}\Big)\frac{\tau(\overline{\chi}^2\chi_2)}{\tau(\overline{\chi}^2)}\Big)\cdot |L(1,\chi)|^2.$$

From Lemma 5 we also know that $(\frac{-2}{p})\tau(\overline{\chi}^2\chi_2)/\tau(\overline{\chi}^2)$ is a real number, so noting that $\overline{\tau(\chi)} = \overline{\chi}(-1)\tau(\overline{\chi})$, from (3.3) we have

$$\sum_{c=1}^{q}{}' G(c^2-1,\chi_0;q)S(c^2,q) \tag{3.4}$$
$$= \frac{2}{\pi^2}\frac{q^2}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \Big(1+\Big(\frac{-2}{p}\Big)\frac{\overline{\tau(\overline{\chi}^2\chi_2)}}{\overline{\tau(\overline{\chi}^2)}}\Big)\cdot |L(1,\overline{\chi})|^2$$
$$= \frac{2}{\pi^2}\frac{q^2}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \Big(1+\Big(\frac{2}{p}\Big)\frac{\tau(\chi^2\chi_2)}{\tau(\chi^2)}\Big)\cdot |L(1,\overline{\chi})|^2$$
$$= \frac{2}{\pi^2}\frac{q^2}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \Big(1-\Big(\frac{-2}{p}\Big)\frac{\tau(\overline{\chi}^2\chi_2)}{\tau(\overline{\chi}^2)}\Big)\cdot |L(1,\chi)|^2,$$

where we have used the identity $(\frac{-1}{p}) = -1$.

Combining (3.3), (3.4) and Lemma 2 we immediately deduce the identity

$$\sum_{c=1}^{q}{}' G(c^2-1,\chi_0;q)S(c^2,q) = \frac{2}{\pi^2}\frac{q^2}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{1}{6}\varphi^2(q)\Big(1+\frac{1}{p}\Big).$$

This proves Theorem 1. □

Now we prove Theorem 2. For any prime $p \equiv 3 \bmod 4$, let χ_2 denote the Legendre symbol, then $\chi_2(-1) = -1$, so from Theorem 9.17 of [1] and Lemma 1 we have

$$\Big|\sum_{c=1}^{p-1}\chi_2(c^2)e\Big(\frac{c^2}{p}\Big)\Big|^2 = |\mathrm{i}\sqrt{p}-1|^2 = p+1, \tag{3.5}$$

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2}. \tag{3.6}$$

If $\chi(-1)=-1$ and $\chi\neq\chi_2$, noting that $\chi_2(-1)=-1$, we also have

$$\begin{aligned}
(3.7)\qquad \left|\sum_{c=1}^{p-1}\chi(c^2)e\Big(\frac{c^2}{p}\Big)\right|^2 &= \left|\sum_{c=1}^{p-1}\chi(c)\Big(1+\chi_2(c)\Big)e\Big(\frac{c}{p}\Big)\right|^2\\
&= |\tau(\chi)+\tau(\chi\chi_2)|^2 = p\cdot\left|1+\frac{\tau(\chi\chi_2)}{\tau(\chi)}\right|^2\\
&= p\cdot\Big(2+\frac{\tau(\chi\chi_2)}{\tau(\chi)}+\frac{\overline{\tau(\chi\chi_2)}}{\overline{\tau(\chi)}}\Big)\\
&= p\cdot\Big(2+\frac{\tau(\chi\chi_2)}{\tau(\chi)}-\frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})}\Big).
\end{aligned}$$

Combining (3.2), (3.5), (3.6), (3.7) and noting that $L(1,\chi_2)=\pi h_p/\sqrt{p}$, we have

$$\begin{aligned}
&\sum_{c=1}^{p-1}G(c^2-1,\chi_0;p)S(c^2,p)\\
&=\frac{1}{\pi^2}\frac{p^2}{p-1}\sum_{\substack{\chi\neq\chi_2\\ \chi(-1)=-1}}\Big(2+\frac{\tau(\chi\chi_2)}{\tau(\chi)}-\frac{\tau(\overline{\chi}\chi_2)}{\tau(\overline{\chi})}\Big)\cdot|L(1,\chi)|^2+\frac{1}{\pi^2}\frac{p(p+1)}{p-1}\cdot|L(1,\chi_2)|^2\\
&=\frac{1}{\pi^2}\frac{2p^2}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2-\frac{1}{\pi^2}\frac{p^2-p}{p-1}\cdot|L(1,\chi_2)|^2\\
&=\frac{(p-1)(p-2)}{6}-\frac{p}{\pi^2}\cdot|L(1,\chi_2)|^2=\frac{(p-1)(p-2)}{6}-h_p^2.
\end{aligned}$$

This proves Theorem 2. □

Finally, we prove Theorem 3. In fact, from the method of proof of Theorem 2 (see formulae (3.3) and (3.4)) we have

$$\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|G(1,\chi^2;q)|^{2k}\cdot|L(1,\chi)|^2 &= \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|G(1,\chi^2;q)|^{2k}\cdot|L(1,\chi)|^2\\
&= \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}(2q)^k\Big(1+\Big(\frac{-2}{p}\Big)\frac{\tau(\overline{\chi}^2\chi_2)}{\tau(\overline{\chi}^2)}\Big)^k\cdot|L(1,\chi)|^2\\
&= \frac{(2q)^k}{2}\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\Big[\Big(1-\Big(\frac{2}{p}\Big)\frac{\tau(\overline{\chi}^2\chi_2)}{\tau(\overline{\chi}^2)}\Big)^k+\Big(1+\Big(\frac{2}{p}\Big)\frac{\tau(\overline{\chi}^2\chi_2)}{\tau(\overline{\chi}^2)}\Big)^k\Big]\cdot|L(1,\chi)|^2\\
&= 2^{2k-1}\cdot q^k\cdot\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{4^k\cdot\pi^2}{24}\cdot\varphi^3(q)\cdot q^{k-2}\cdot\prod_{p|q}\Big(1+\frac{1}{p}\Big).
\end{aligned}$$

This completes the proof of Theorem 3. □

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