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UPPERS TO ZERO IN $R[x]$ AND ALMOST PRINCIPAL IDEALS

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Abstract. Let R be an integral domain with quotient field K and $f(x)$ a polynomial of positive degree in $K[x]$. In this paper we develop a method for studying almost principal uppers to zero ideals. More precisely, we prove that uppers to zero divisorial ideals of the form $I = f(x)K[x] \cap R[x]$ are almost principal in the following two cases:

- ▷ J, the ideal generated by the leading coefficients of I, satisfies $J^{-1} = R$.
- ▷ I^{-1} as the $R[x]$-submodule of $K(x)$ is of finite type.

Furthermore we prove that for $I = f(x)K[x] \cap R[x]$ we have:

- ▷ $I^{-1} \cap K[x] = (I :_{K(x)} I)$.
- ▷ If there exists $p/q \in I^{-1} - K[x]$, then $(q, f) \neq 1$ in $K[x]$. If in addition q is irreducible and I is almost principal, then $I' = q(x)K[x] \cap R[x]$ is an almost principal upper to zero.

Finally we show that a Schreier domain R is a greatest common divisor domain if and only if every upper to zero in $R[x]$ contains a primitive polynomial.

Keywords: almost principal ideal, divisorial ideal, greatest common divisor domain, Schreier domain, uppers to zero

MSC 2010: 13A05, 13B25, 13A15, 13F15

1. Introduction

Throughout this paper let R be a domain and let K denote its quotient field. Our terminology and notation come from [5], [9]. We first recall some preliminaries for concepts used in the paper. Recall that R is *Schreier* if R is integrally closed and for all $x, y, z \in R \setminus \{0\}$, $x \mid yz$ implies that $x = rs$ where $r \mid y$ and $s \mid z$. R is a Greatest Common Divisor domain (GCD domain) if any two nonzero elements in R have a greatest common divisor. A *pre-Schreier* domain R is a Schreier domain

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without the assumption that R is integrally closed. Schreier domains were introduced by Cohn in [4] where he showed that a GCD domain is a Schreier domain and also if R is a Schreier domain, so is $R[x]$. Examples of GCD domains include unique factorization domains. It is also well known that any GCD domain is integrally closed.

One of the motivations of this paper is to obtain certain conditions under which the category of Schreier domains and GCD domains coincide. To this end, in Section 1 we study the elements of such domains and characterize them via uppers to zero. Recall that a nonzero ideal I of $R[x]$ is an upper to zero if $I \cap R = 0$ and I is prime. Euclid's lemma on integral domains, Lemma 2.1, and the proof of Proposition 2.3 show that R is a GCD domain if and only if any upper to zero in $R[x]$ is principal. In fact, in Proposition 2.3 we prove that if R is a Schreier domain and if each upper to zero in $R[x]$ contains a primitive polynomial, then R is a GCD domain.

A fractional ideal of R is an R-submodule I of K such that there exists a nonzero $r \in R$ such that $rI \subseteq R$. Call a fractional ideal I *invertible* if $II^{-1} = R$ where $I^{-1} = (R :_K I) = \{x \in K \colon xI \subseteq R\}$. Furthermore, let $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup A_v$, where the union is taken over all finitely generated subideals A of I. Finally, let $F(D)$ denote the set of fractional ideals of D. The operation on $F(D)$ defined by $A \rightarrow (A^{-1})^{-1} = A_v$ where A ranges over $F(D)$ is called the v-operation. A fractional ideal $A \in F(D)$ is called a v-ideal if $A = A_v$ and a v-ideal A is said to be a v-ideal of finite type if $A = B_v$ where B is a finitely generated fractional ideal. Most of the facts about the v- and t-operations which we shall use can be found in [5, Section 32].

Houston [7] mentioned that being upper to zero is equivalent to $I = f(x)K[x] \cap R[x]$ for some irreducible polynomial $f(x)$ of $K[x]$. An ideal I of $R[x]$ is called almost principal if there exists a polynomial $f(x) \in I$ of positive degree and a nonzero $s \in R$ such that $sI \subseteq (f(x))R[x]$. Whenever I is an upper to zero this is equivalent to the existence of some $s \in R$ for which $sI \subseteq f(x)R[x]$. Furthermore, I is said to be divisorial if $I_v = (I^{-1})^{-1} = I$.

In the second section of this paper we develop a method for studying almost principal ideals as it comes in the sequel. Note that each almost principal ideal of the form $I = f(x)K[x] \cap R[x]$ is divisorial. (If $\exists s \in R$, $sI \subseteq f(x)R[x]$, then $I_v \subseteq f(x)K[x]$. Since $I_v \subseteq R[x]$, we obtain $I_v \subseteq f(x)K[x] \cap R[x] = I$. In addition, the definition of operations implies the converse statement and thus the result follows.)

As was asked by Houston and studied further by several authors, specially by Hamann, Houston, Johnson [6], the almost principal property is related to the following question:

Question. When is $f(x)K[x] \cap R[x]$ divisorial?

In [6, Proposition 1.15] the authors obtained a number of conditions in terms of I^{-1} and $[I : I]$ equivalent to the almost principal property which are represented here:

(1) I is almost principal.
(2) $I^{-1}K[x] = (1/f)K[x]$.
(3) $I^{-1} \not\subseteq K[x]$.
(4) I^{-1} is not a ring.
(5) $I^{-1} \neq [I : I]$.
(6) There exists $g(x) \in R[x] - I$ with $g(x)I \subseteq f(x)R[x]$.

Furthermore, the work of Houston and Zafrullah [8] also concerns divisorial and almost principal ideals. Although they mentioned no knowledge for divisorial ideals being almost principal in general, in Proposition 1.9 they proved the following:

Let $I = f(x)K[x] \cap R[x]$ be an upper to zero, and let J denote the ideal generated by the constant terms of the elements of I. If $J^{-1} = R$ and $I^{-1} \neq R[x]$, then I is almost principal. Thus if $J^{-1} = R$ and I is divisorial, then it is almost principal. (Note that $I^{-1} \neq R[x]$, otherwise $I_v = (I^{-1})^{-1} = R[x]$ which contradicts $I_v = I$. Hence I is almost principal.)

This result particularly motivated us to study the implication "divisorial ideals of the form $I = f(x)K[x] \cap R[x]$ are almost principal". In fact, we prove this in the following two cases:

(1) The ideal generated by the leading coefficients of I, say J, satisfies $J^{-1} = R$; see Proposition 3.3.
(2) I^{-1} as the $R[x]$-submodule of $K(x)$ is of finite type (finitely generated for example); see Proposition 3.5.

Furthermore, in Theorem 3.1 we prove that if there exists $p/q \in I^{-1} - K[x]$ whenever $I = f(x)K[x] \cap R[x]$ is an ideal of $R[x]$, then $(q, f) \neq 1$ in $K[x]$. This will be applied to deduce that almost principal uppers to zero play an important role in studying almost principal ideals. More precisely, in Proposition 3.2 we prove the following:

Let $I = f(x)K[x] \cap R[x]$ be an almost principal ideal, and suppose that there exists $p/q \in I^{-1} - K[x]$ for which q is irreducible. Then $J = q(x)K[x] \cap R[x]$ is an almost principal upper to zero.

In GCD domains any multiple of gcd of any two elements could be excluded, [9, Theorem 49]. Generally, two elements x, y in a domain R are called *v-coprime* if $xR \cap yR = xyR$; see [12]. Further, $x, y \in R \backslash \{0\}$ are said to be *coprime* if x and y have no nonunit common factor in R. Note that x, y being v-coprime implies x, y coprime but not conversely. In a pre-Schreier domain two coprime elements are v-coprime; see [4, Proposition 3.3]. In addition it is useful to note that $\gcd(a, b) = 1 \iff \forall x \in R$ $((x \mid a, b) \implies x \mid 1)$ and so the negation of a, b are coprime would be $\exists x \in R$ $((x \mid a, b), x \nmid 1)$. Moreover, if a and b are v-coprime, in any domain, and $a \mid by$ then $a \mid y$; see, e.g., [12]. This observation improves Euclid's lemma on integral domains:

Lemma 2.1. *Let R be an integral domain. Suppose that $uR \cap aR = uaR$ and u divides ab for some elements a, b, u in R. Then $u \mid b$.*

Tang [10, Theorem I], has provided a proof of the fact that an integral domain R is a GCD domain if and only if any prime ideal of $R[x]$ lying over zero is principal. If R is noetherian, since a noetherian GCD domain is Unique Factorization Domain (UFD), $R[x]$ (being noetherian and GCD) is UFD and a height one prime ideal in a UFD is principal. Now we provide a simple proof of this result without the noetherian assumption (that is, the same assumptions as in [10, Theorem I]). As a matter of fact we offer an alternative proof, using the fact that if R is Schreier then so is $R[X]$.

Theorem 2.2. *Let R be an integral domain. Then R is a GCD domain if and only if any upper to zero is principal.*

Proof. ($\Longleftarrow$): Recall that a Generalized GCD domain (GGCD domain) is a domain in which the intersection of any two invertible ideals is again an invertible ideal, cf. [1]. From [2, Theorem 15] it is known that an integral domain is a GGCD if and only if every upper to zero is invertible. Thus R is a GGCD domain. Finally, since it is also obvious that a GGCD domain is a GCD domain if and only if every invertible ideal is principal, the result follows.

($\Longrightarrow$): We use the facts that (i) a GCD domain is Schreier, (ii) if R is Schreier then so is $R[x]$ and (iii) in a Schreier domain every irreducible element is prime. Now let P be an upper to zero, i.e. P is a prime ideal in $R[x]$ such that $P \cap R = (0)$. Let $f \in P \setminus \{0\}$ be of the least degree. Then $\deg(f) \geqslant 1$. Since R is a GCD domain we can write $f = df_1$ where d is a GCD of the coefficients of f and f_1 is a primitive polynomial. Since $P \cap R = (0)$ and since P is a prime, $f_1 \in P$. Now being a primitive polynomial of the least degree, f_1 is irreducible, and in a Schreier domain an irrededucible element is a prime. So $f_1 R$ is a prime ideal contained in P. But being an upper to zero, P is of height one and so $P = f_1 R$. □

As we mentioned earlier every GCD domain is Schreier, but in general the converse is not necessarily true; see [11] for several examples of Schreier domains that are not GCD domains. We use the notion of a primitive polynomial to accomplish that. Recall that a polynomial $f(x)$ is primitive if the coefficients of f have no nonunit common factor. Stated in ring-theoretic terms, a polynomial is primitive if the ideal generated by the coefficients of $f(x)$, that is, $A_f = (a_0, a_1, \ldots, a_n)$ is not contained in any proper principal ideal. Products of primitive polynomials are primitive over a Schreier domain. It is easy to see that over any domain R the factors of a primitive polynomial in $R[x]$ are again primitive. For more details see [3]. Note that in a Schreier domain every irreducible element is prime. The following result deals more seriously with such phenomena.

Proposition 2.3. *A Schreier domain R is a GCD domain if and only if every upper to zero in $R[x]$ contains a primitive polynomial.*

Proof. Let P be an upper to zero in $R[x]$ and let f be a primitive polynomial in P. We can assume f to be of the least degree. But then f is irreducible and hence prime because $R[x]$ is Schreier. Since $f \in P$ and since P is of height one we have $P = fR[x]$. So every upper to zero in $R[x]$ is principal. To show that R is a GCD domain, let $a, b \in R \setminus \{0\}$. Then by [5, Corollary 34.9],

$$(a + bx)K[x] \cap R[x] = (a + bx)(A_{(a+bx)})^{-1}R[x]$$

which is principal. As a matter of fact $(a + bx)K[x] \cap R[x]$ being principal forces $(a + bx)(a, b)^{-1}R[x]$ principal which forces $(a, b)^{-1}$ principal and which means that $\gcd(a, b)$ exists. As a conclusion $(A_{(a+bx)})^{-1}$ is invertible and hence principal because in a Schreier domain every invertible ideal is principal, cf. [2, Theorem 1]. But then $aR \cap bR$ is principal for each pair $a, b \in R \setminus \{0\}$. The converse is obvious. □

Remark 2.4. As the proof of Proposition 2.3 shows, it is enough to assume that $R[x]$ is pre-Schreier. But notice that if $R[x]$ is pre-Schreier then actually R and hence $R[x]$ is Schreier; see [2, Corollary 10]. So Proposition 2.3 will not be improved if we change the hypothesis "R is Schreier" with "$R[x]$ is pre-Schreier".

Let $I = f(x)K[x] \cap R[x]$ be an ideal of $R[x]$ which is not an upper to zero in general. As we set in the Introduction, let $I^{-1} = (R[x] :_{K(x)} I)$ and $I_v = (I^{-1})^{-1}$. Throughout this section by $a \mid b$ we mean that a divides b and by $\gcd(a, b)$ we mean their greatest common divisor in $K[x]$ which is a Principal Ideal Domain (PID) and it makes sense to speak about them.

Note that if I is an upper to zero and almost principal, then $\exists p(x)/q(x) \in I^{-1} - K[x]$. Note that if $\exists s \in R$ with $sI \subseteq f(x)R[x]$, simply take $s/f \in I^{-1} - K[x]$. In the first theorem of this section we explore the converse statement. In fact this result is a helpful trick to prove the result that was dubbed obvious in [6, Lemma 1.14].

Theorem 3.1. *Let R be an integral domain with quotient field K, $f(x)$ a polynomial of positive degree in $R[x]$, and $I = f(x)K[x] \cap R[x]$. Then we have:*

(i) $I^{-1} \cap K[x] = (I_{K(x)} I)$.

(ii) *If there exists $p/q \in I^{-1} - K[x]$, then $(q, f) \neq 1$ in $K[x]$.*

Proof. (i) The inclusion ($\subseteq$) is easy. Take any $g \in I^{-1} \cap K[x]$. For any $h \in I, h(x) = f(x)l(x) \in R[x]$ for some $l(x) \in K[x]$. So $h(x)g(x) = f(x)l(x)g(x) \in R[x]$, and so it is in $I = fK[x] \cap R[x]$. For the inclusion ($\supseteq$), take any $t \in (I :_{K(x)} I)$ with $t \in I^{-1}$. We will prove that $t \in K[x]$. Write down $t = p(x)/q(x)$, where $p(x), q(x)$ are relatively prime in $K[x]$. Suppose that $\deg(q(x)) > 0$, and take any non zero element $f(x)l(x) \in I$. Then

$$f(x)l(x)\frac{p(x)}{q(x)} \in I, \tag{2.1.1}$$

and so $f(x)l(x)(p(x)/q(x)) = f(x)r(x)$ for some $r(x) \in K[x]$. We obtain that $l(x)p(x) = q(x)r(x)$, i.e., $q(x) \mid l(x)$ and so $\exists s(x) \in K[x]$, $l(x) = s(x)q(x)$. Now (2.1.1) gives us $f(x)s(x)p(x) \in I$. Starting the story for $s(x)p(x)$ instead of $l(x)$ we deduce that $q(x) \mid s(x)p(x)$ in $K[x]$. Since $(p(x), q(x)) = 1$, $q(x) \mid s(x)$ and so $q(x)^2 \mid l(x)$. Iterating this procedure, we get that $l(x) = 0$, which is a contradiction. Hence $t \in K[x]$.

(ii) First of all note that we can assume that $(p, q) = 1$. Now suppose the contrary and assume that $(q, f) = 1$. Since $p(x)/q(x) \in I^{-1}$, for any $l(x) \in K[x]$ for which $f(x)l(x) \in I$ we have $f(x)l(x)(p(x)/q(x)) \in R[x]$. Thus $f(x)l(x)p(x) = q(x)r(x) \in R[x]$ for some $r(x) \in R[x]$. Hence $q(x) \mid f(x)l(x)p(x)$, i.e., $q(x) \mid l(x)$. Therefore, $\exists s(x) \in K[x]$ s.t. $l(x) = s(x)q(x)$. This gives us $f(x)s(x)p(x) = r(x) \in R[x]$ and so $q(x) \mid s(x)$ or $q(x)^2 \mid l(x)$. Arguing as in (i) finally we end up with $l(x) = 0$. $\square$

With Theorem 3.1, the following proposition is merely a corollary.

Proposition 3.2. *Let $I = f(x)K[x] \cap R[x]$ be an almost principal ideal and suppose that there exists $p/q \in I^{-1} - K[x]$ for which q is irreducible. Then $J = q(x)K[x] \cap R[x]$ is an almost principal upper to zero.*

Proof. By Theorem 3.1(ii), $(q, f) \neq 1$ and since q is irreducible we have $f(x) = g(x)q(x)$ for some $g(x) \in K[x]$. Hence $\exists 0 \neq s \in R$ such that $sg(x) \in R[x]$. Now let $t(x) = sf(x)$ and $g_1(x) = sg(x)$, then [6, Lemma 1.5] implies that J is almost principal. Note that since $I = t(x)K[x] \cap R[x]$ is almost principal, we claim that $I_1 = g_1(x)K[X] \cap R[x]$ and $J = q(x)K[x] \cap R[x]$ are both almost principal. Thus J is an almost principal upper to zero. $\square$

Now we give affirmative answers for the implication "divisorial ideals are almost principal" in two new situations.

Proposition 3.3. *Let $I = f(x)K[x] \cap R[x]$ be an upper to zero, and let J denote the ideal generated by the leading coefficients of I. If $J^{-1} = R$ and I is not almost principal, then $I^{-1} = R[x]$. As a corollary, I is not divisorial.*

Proof. Suppose the contrary and assume that $I^{-1} \neq R[x]$. Then there exists an element $\psi \in I^{-1} - R[x]$. Let $\psi = a_n x^n + \ldots + a_0$ be of the minimum degree with such property. For any $g \in I$ which is of the form $g = b_m x^m + \ldots + b_0$, we have $\psi \cdot g \in R[x]$ and so $a_n b_m \in R$. We claim that $b_m \cdot \psi \in R[x]$. Suppose the result fails to hold, and for an $1 \leqslant i < n$, $a_i b_m$ is not in R. Put $\theta = b_m \psi - b_m a_n x^n$. Note that $\theta \in I^{-1} - R[x]$, $\deg \theta < \deg \psi$, which is a contradiction. As g was arbitrary we get that $a_i \in J^{-1} = R$ and so $\psi \in R[x]$. Hence $I^{-1} = R[x]$. The corollary now is proved since $I_v = (I^{-1})^{-1} = R[x] \neq I$. $\square$

Corollary 3.4. *Let $I = f(x)K[x] \cap R[x]$ be an upper to zero, and let J denote the ideal generated by the leading coefficients of I. Then I divisorial implies that I is almost principal or J^{-1} is not equal to R.*

Note that I^{-1} is a v-ideal and a v-ideal A is said to be of finite type if there is a finitely generated ideal B contained in A with $A_v = B_v$. Hence in the following result we can replace "of finite type" with "finitely generated" harmlessly.

Proposition 3.5. *Let $I = f(x)K[x] \cap R[x]$ be an upper to zero, and assume that I is not almost principal. If I^{-1} as the $R[x]$-submodule of $K(x)$ is of finite type, then I is not divisorial.*

Proof. Proceeding contrapositively, suppose that I is divisorial. Since I^{-1} is finitely generated, there exists a nonzero element c of R such that cg is in $R[x]$ for all g in I^{-1}. It yields that $c \in (I^{-1})^{-1}$. Since I is divisorial, we claim that $c \in I$. But this is a contradiction since I is an upper and so $I \cap R = 0$. $\square$

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