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Abstract. For compact hypersurfaces with constant mean curvature in the unit sphere, we give a comparison theorem between eigenvalues of the stability operator and that of the Hodge Laplacian on 1-forms. Furthermore, we also establish a comparison theorem between eigenvalues of the stability operator and that of the rough Laplacian.

1 Introduction

Let M be an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $\mathbb{S}^{n+1}(1)$. We let h_{ij} denote the components of the second fundamental form, S stand for the norm square of the second fundamental form, H be the mean curvature of M, respectively. A Schrödinger operator

$$J = -\Delta - (S + n)\,,$$

where Δ denotes the Laplace-Beltrami operator, is called a Jacobi operator. Since the spectral behavior is directly related to the instability of both minimal hypersurfaces and hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}(1)$ (for example, see [2], [10]), many mathematicians studied the first and the second eigenvalues of such Jacobi operator. The first eigenvalue of J on hypersurfaces in $\mathbb{S}^{n+1}(1)$ was studied by Simons [10] and Wu [11]. Ei Soufi and Ilias [6] studied the second eigenvalue of the Jacobi operator above. In 1993, Alencar, do Carmo and Colares [1] studied the stability of hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}(1)$. Similarly to the case of both minimal hypersurfaces and hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}(1)$, we have a notion of Jacobi operator corresponding to compact hypersurfaces with constant scalar curvsture. For the first eigenvalue and the second eigenvalue of such Jacobi operator, the readers who are interested in it see [4], [8].

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Recently, Savo [9] considered compact minimal hypersurfaces of the unit sphere and proved a comparison theorem between the spectrum of the stability operator J and that of the Hodge Laplacian on 1-forms. In this paper, we consider hypersurfaces of the unit sphere with constant mean curvature. Now we state our result as follows:

Theorem 1. *Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature H. We denote the norm square of the second fundamental form by S. Then*

$$\lambda^J_\alpha \leq -2(n-1) + \lambda^{\Delta_1}_{m(\alpha)} + n|H| \max_M \sqrt{S}\,, \tag{1}$$

where λ^J_α is the α-th eigenvalue of J, $\lambda^{\Delta_1}_{m(\alpha)}$ is the $m(\alpha)$-th eigenvalue of the Hodge Laplacian Δ_1 with respect to 1-form. Here $m(\alpha) = \binom{n+2}{2}(\alpha-1)+1$.

In particular, Savo [9] has proved that for compact minimal hypersurfaces of the unit sphere, it holds that

$$\lambda^J_\alpha \leq -2(n-1) + \lambda^{\Delta_1}_{m(\alpha)}\,. \tag{2}$$

Hence, the Theorem 1 above extends Theorem 1 in [9]. On the other hand, for eigenvalues of the stability operator J and the rough Laplacian, we have the following result:

Theorem 2. *Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature. We have*

$$\lambda^J_\alpha \leq -(n-1) + \lambda^{D^*D}_{m(\alpha)}\,, \tag{3}$$

where λ^J_α is the α-th eigenvalue of J, $\lambda^{D^*D}_{m(\alpha)}$ is the $m(\alpha)$-th eigenvalue of the rough Laplacian D^*D with respect to 1-form. Here $m(\alpha) = \binom{n+2}{2}(\alpha-1)+1$.

2 Proof of Theorems

Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature. We adopt the following index convention:

$$1 \leq i,j,k,l \leq n\,, \quad 1 \leq A,B \leq n+2\,.$$

Choosing a local orthonormal frame $\{e_1,\ldots,e_n,e_{n+1}\}$ and the dual coframe $\{\omega_1,\ldots,\omega_n,\omega_{n+1}\}$ such that when restricted on M, $\{e_1,\ldots,e_n\}$ forms a local orthonormal frame on M. Hence, $\omega_{n+1}=0$ on M and the following structure equations (see [5]):

$$\begin{aligned} dx &= \omega_i e_i\,,\\ de_i &= \omega_{ij}e_j + h_{ij}\omega_j e_{n+1} - \omega_i x\,,\\ de_{n+1} &= -h_{ij}\omega_j e_i\,, \end{aligned}$$

where h_{ij} denote the components of the second fundamental form of x, in which we used the summation convention on repeated indices. We will take this convention in the later part without any confusion. The Gauss equations (see [5], [7]) are

$$\begin{aligned} R_{ijkl} &= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}) \,, \\ R_{ij} &= R_{ikjk} = (n-1)\delta_{ij} + nHh_{ij} - h_{ik}h_{jk} \,, \\ R &= n(n-1) + n^2H^2 - S \,, \end{aligned} \tag{4}$$

where R stands for the scalar curvature and $S = \sum_{ij} h_{ij}^2$ is the norm square of the second fundamental form, $H = \frac{1}{n}h_{ii}$ is the mean curvature of x. The Codazzi equations are given by

$$h_{ijk} = h_{ikj} \,, \quad \text{for } i,j,k = 1,\ldots,n \,.$$

Let f be a smooth function on M. The first and the second covariant derivatives of f are defined by

$$df = f_i\omega_i \,,$$

$$f_{ij}\omega_j = df_i + f_j\omega_{ji} \,.$$

Let a be a fixed vector in $\mathbb{R}^{n+2}$. Define

$$f^a = \langle a, x\rangle, \quad g^a = \langle a, e_{n+1}\rangle \,.$$

Then we have the following lemma:

Lemma 1. *(see [3]) Under the conceptions above, we have*

$$\begin{aligned} f_i^a &= \langle a, e_i\rangle \,, \quad g_i^a = -h_{ij}f_j^a, \\ f_{ij}^a &= h_{ij}g^a - f^a\delta_{ij} \,, \\ g_{ij}^a &= h_{ij}f^a - h_{ik}h_{jk}g^a - h_{ijk}f_k^a \,. \end{aligned}$$

Define the Hodge Laplacian Δ_p by

$$\Delta_p = d\delta + \delta d \colon A^p(M) \to A^p(M)$$

where $\delta = (-1)^{n(p+1)} * d* \colon A^p(M) \to A^{p-1}(M)$. For any $\psi \in A^p(M)$, one has

$$\Delta_p\,\psi = D^*D(\psi) - \mathrm{Ric}(\psi) \,,$$

where D^*D denotes the rough Laplacian which is given by

$$D^*D(\psi) = \sum_i (D_{e_i}D_{e_i} - D_{D_{e_i}e_i})\psi \,.$$

In particular, when $\xi = \xi_i\omega_i \in A^1(M)$, $D^*D(\xi) = \xi_{j,ii}\omega_j$, where the second covariant derivatives of ξ is defined by

$$\xi_{i,jk}\omega_k = d\xi_{i,j} + \xi_{k,j}\omega_{ki} + \xi_{i,k}\omega_{kj} \,.$$

In particular, for $f \in C^\infty(M)$, we have $\Delta_0 f = f_{ii} = \Delta f$. By (4), one gets

$$\mathrm{Ric}(\xi) = \xi_i R_{ij} e_j = (n-1)\xi + nH h_{ij}\xi_i e_j - h_{ik}h_{jk}\xi_i e_j$$

and hence,

$$D^*D(\xi) = \Delta_1\xi + (n-1)\xi + nH h_{ij}\xi_i e_j - h_{ik}h_{jk}\xi_i e_j. \tag{5}$$

Lemma 2. *Let a be a fixed vector in $\mathbb{R}^{n+2}$ and $a^\top$ denote the orthogonal projection onto M. Then*

$$\Delta_1 a^\top = -nH h_{ij} f_j^a e_i - n f_i^a e_i\,, \tag{6}$$

$$D^*D(a^\top) = -f_i^a e_i - h_{ik}h_{jk} f_i^a e_j\,. \tag{7}$$

Proof. By a direct calculation, one has from Lemma 1

$$\begin{aligned}\Delta_1 a^\top &= \Delta_1(\langle a^\top, e_i\rangle\omega_i) = \Delta_1(df^a) = d(\Delta f^a)\\ &= nH dg^a - n df^a = -nH h_{ij} f_j^a e_i - n f_i^a e_i\,.\end{aligned}$$

Hence (6) is proved. Substituting ξ in (5) by $a^\top$ and using (6), we obtain (7). □

Lemma 3. *Let ξ be a vector field on M and a, b be two independent fixed vectors in $\mathbb{R}^{n+2}$. Then we have*

$$\begin{aligned}\Delta\big(\langle a, e_{n+1}\rangle\langle b^\top, \xi\rangle\big) = &\big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nH h_{ij}\xi_i - S\xi_j\big) f_j^b g^a\\ &- 2h_{ij}h_{ik}\xi_k f_j^a g^b - 2h_{ij}\xi_{k,i} f_j^a f_k^b - 2\xi_{ii} f^b g^a + 2h_{ij}\xi_i f_j^a f^b\\ &+ nH\xi_j f^a f_j^b + 2h_{ij}\xi_{i,j} g^b g^a.\end{aligned}$$

Proof. Given a point $p \in M$, let $\{e_i\}_{i=1}^n$ be an orthonormal frame which is geodesic at p. Then $\Delta f = e_i e_i(f)$ and we have from (5), Lemma 1 and Lemma 2,

$$\begin{aligned}\Delta\langle b^\top, \xi\rangle &= \langle D^*D(b^\top), \xi\rangle + 2f_{ij}^b\xi_{i,j} + \langle b^\top, D^*D(\xi)\rangle\\ &= -\xi_j f_j^b - h_{ik}h_{jk}\xi_i f_j^b + 2(h_{ij}g^b - f^b\delta_{ij})\xi_{i,j}\\ &\quad + \langle e_j, \Delta_1\xi\rangle f_j^b + (n-1)\xi_j f_j^b + nH h_{ij}\xi_i f_j^b - h_{ik}h_{jk}\xi_i f_j^b\\ &= \big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nH h_{ij}\xi_i\big) f_j^b\\ &\quad + 2h_{ij}\xi_{i,j} g^b - 2\xi_{i,i} f^b\,,\end{aligned}$$

$$\begin{aligned}\big\langle\nabla\langle a, e_{n+1}\rangle, \nabla\langle b^\top, \xi\rangle\big\rangle &= g_i^a\big(\langle D_{e_i} b^\top, \xi\rangle + \langle b^\top, D_{e_i}\xi\rangle\big)\\ &= g_i^a(f_{ij}^b\xi_j + f_j^b\xi_{j,i})\\ &= -h_{ij}h_{ik}\xi_k f_j^a g^b + h_{ij}\xi_i f_j^a f^b - h_{ij}\xi_{k,i} f_j^a f_k^b\,.\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta\big(\langle a, e_{n+1}\rangle\langle b^\top,\xi\rangle\big) &= \langle a, e_{n+1}\rangle\Delta\langle b^\top,\xi\rangle + \langle b^\top,\xi\rangle\Delta\langle a, e_{n+1}\rangle \\
&\quad + 2\big\langle\nabla\langle a, e_{n+1}\rangle, \nabla\langle b^\top,\xi\rangle\big\rangle \\
&= g^a\Big(\big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i\big)f_j^b \\
&\quad + 2h_{ij}\xi_{i,j}g^b - 2\xi_{i,i}f^b\Big) + (nHf^a - Sg^a)\xi_j f_j^b \\
&\quad + 2(-h_{ij}h_{ik}\xi_k f_j^a g^b + h_{ij}\xi_i f_j^a f^b - h_{ij}\xi_{k,i}f_j^a f_k^b) \\
&= \big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i - S\xi_j\big)f_j^b g^a \\
&\quad - 2h_{ij}h_{ik}\xi_k f_j^a g^b - 2h_{ij}\xi_{k,i}f_j^a f_k^b - 2\xi_{i,i}f^b g^a + 2h_{ij}\xi_i f_j^a f^b \\
&\quad + nH\xi_j f^a f_j^b + 2h_{ij}\xi_{i,j}g^b g^a.
\end{aligned}$$

We conclude the proof of Lemma 3. $\square$

Now we are in a position to prove Theorem 1.

Proof. (of Theorem 1) Let $\{E_A\}_{A=1}^{n+2}$ be a fixed orthonormal basis of $\mathbb{R}^{n+2}$. Define

$$X_{AB}^\top = \langle E_A, e_{n+1}\rangle E_B^\top - \langle E_B, e_{n+1}\rangle E_A^\top$$

and

$$u_{AB} = \langle X_{AB}^\top, \xi\rangle = -u_{BA}.$$

Let

$$f^A = \langle E_A, x\rangle, \quad g^A = \langle E_A, e_{n+1}\rangle\,.$$

Then from Lemma 3, we have

$$\begin{aligned}
\Delta u_{AB} &= \Delta\big(\langle E_A, e_{n+1}\rangle\langle E_B^\top,\xi\rangle\big) - \Delta\big(\langle E_B, e_{n+1}\rangle\langle E_A^\top,\xi\rangle\big) \\
&= \big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i - S\xi_j\big)(f_j^B g^A - f_j^A g^B) \\
&\quad - 2h_{ij}h_{ik}\xi_k(f_j^A g^B - f_j^B g^A) - 2h_{ij}\xi_{k,i}(f_j^A f_k^B - f_j^B f_k^A) \\
&\quad - 2\xi_{ii}(f^B g^A - f^A g^B) + 2h_{ij}\xi_i(f_j^A f^B - f_j^B f^A) \\
&\quad + nH\xi_j(f^A f_j^B - f^B f_j^A) \\
&= (n-2-S)u_{AB} + v_{AB}\,,
\end{aligned}$$

where

$$\begin{aligned}
v_{AB} &= \big(\langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i\big)(f_j^B g^A - f_j^A g^B) \\
&\quad - 2h_{ij}h_{ik}\xi_k(f_j^A g^B - f_j^B g^A) - 2h_{ij}\xi_{k,i}(f_j^A f_k^B - f_j^B f_k^A) \\
&\quad - 2\xi_{i,i}(f^B g^A - f^A g^B) + 2h_{ij}\xi_i(f_j^A f^B - f_j^B f^A) \\
&\quad + nH\xi_j(f^A f_j^B - f^B f_j^A)\,.
\end{aligned}$$

Let λ_α^J be the α-th eigenvalue of J and φ_α be the orthonormal eigenfunction corresponding to λ_α^J, that is,

$$J\varphi_\alpha = \lambda_\alpha^J \varphi_\alpha, \quad \int\limits_M \varphi_\alpha \varphi_\beta = \delta_{\alpha\beta}. \tag{8}$$

Denote by $V_m^{\Delta_1}$ the direct sum of the first m eigenspaces of Δ_1 such that the following orthogonality relations

$$\int\limits_M \langle X_{AB}^\top, \xi\rangle \varphi_1 = \cdots = \int\limits_M \langle X_{AB}^\top, \xi\rangle \varphi_{\alpha-1} = 0 \tag{9}$$

hold for all A, B. Note that $X_{AB}^\top$ is skew symmetric. Hence, we know that (9) has $\binom{n+2}{2}(\alpha-1)$ homogenous linear equations in $\xi \in V_m^{\Delta_1}$. If we let

$$m(\alpha) := \binom{n+2}{2}(\alpha-1)+1\,,$$

then we can find a non-zero vector field $\xi \in V_{m(\alpha)}^{\Delta_1}$ such that the function u_{AB} is orthogonal to the first $\alpha-1$ eigenfunctions of J for all A, B. By the Rayleigh-Ritz principle, we have

$$\begin{aligned}
\lambda_\alpha^J \int\limits_M u_{AB}^2 &\le \int\limits_M u_{AB} J\, u_{AB} \\
&= -\int\limits_M u_{AB}\Delta\, u_{AB} - \int\limits_M (S+n)u_{AB}^2 \\
&= -\int\limits_M \Big(2(n-1)u_{AB}^2 + u_{AB}v_{AB}\Big).
\end{aligned} \tag{10}$$

It follows from $u_{AB} = \xi_l(f_l^B g^A - f_l^A g^B)$ that

$$\sum_{A,B} u_{AB}^2 = \xi_l\xi_k \sum_{A,B}(f_l^B g^A - f_l^A g^B)(f_k^B g^A - f_k^A g^B) = 2|\xi|^2, \tag{11}$$

$$\begin{aligned}
\sum_{A,B} u_{AB}v_{AB} = \xi_l\Big\{&(\langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i) \\
&\times \sum_{A,B}(f_j^B g^A - f_j^A g^B)(f_l^B g^A - f_l^A g^B) \\
&- 2h_{ij}h_{ik}\xi_k \sum_{A,B}(f_j^A g^B - f_j^B g^A)(f_l^B g^A - f_l^A g^B) \\
&- 2h_{ij}\xi_{k,i} \sum_{A,B}(f_j^A f_k^B - f_j^B f_k^A)(f_l^B g^A - f_l^A g^B) \\
&- 2\xi_{i,i} \sum_{A,B}(f^B g^A - f^A g^B)(f_l^B g^A - f_l^A g^B)
\end{aligned} \tag{12}$$

$$
\begin{aligned}
&+ 2h_{ij}\xi_i \sum_{A,B}(f_j^A f^B - f_j^B f^A)(f_l^B g^A - f_l^A g^B) \\
&+ nH\xi_j \sum_{A,B}(f^A f_j^B - f^B f_j^A)(f_l^B g^A - f_l^A g^B)\Big\} \\
= {} & \xi_l\Big\{2(\langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i)\delta_{jl} + 4h_{ij}h_{ik}\xi_k\delta_{jl}\Big\} \\
= {} & 2\langle \xi, \Delta_1\xi\rangle + 2nHh_{ij}\xi_i\xi_j,
\end{aligned}
$$

where we used

$$\sum_{A,B}\langle E_A, X\rangle\langle Y, E_B\rangle = \langle X, Y\rangle$$

for any X, Y. Applying (11) and (12) to (10) yields

$$
\begin{aligned}
\lambda_\alpha^J \int_M |\xi|^2 \le & - \int_M \Big(2(n-1)|\xi|^2 + \langle \xi, \Delta_1\xi\rangle + nHh_{ij}\xi_i\xi_j\Big) \\
\le & - 2(n-1)\int_M |\xi|^2 + \lambda_{m(\alpha)}^{\Delta_1}\int_M |\xi|^2 + n|H|\max_M \sqrt{S}\int_M |\xi|^2
\end{aligned}
\tag{13}
$$

which shows that

$$\lambda_\alpha^J \le -2(n-1) + \lambda_{m(\alpha)}^{\Delta_1} + n|H|\max_M \sqrt{S}.$$

We complete the proof of Theorem 1. □

Proof. (of Theorem 2) From (5), we have

$$\big\langle \xi, D^*D(\xi)\big\rangle = \langle \xi, \Delta_1\xi\rangle + (n-1)|\xi|^2 + nHh_{ij}\xi_i\xi_j - h_{ik}h_{jk}\xi_i\xi_j. \tag{14}$$

Putting (14) into (13), one gets

$$
\begin{aligned}
\lambda_\alpha^J \int_M |\xi|^2 \le & - \int_M \Big(2(n-1)|\xi|^2 + \langle \xi, \Delta_1\xi\rangle + nHh_{ij}\xi_i\xi_j\Big) \\
= & - \int_M \Big((n-1)|\xi|^2 + \langle \xi, D^*D(\xi)\rangle + h_{ik}h_{jk}\xi_i\xi_j\Big) \\
\le & - \int_M \Big((n-1)|\xi|^2 + \langle \xi, D^*D(\xi)\rangle\Big) \\
\le & - (n-1)\int_M |\xi|^2 + \lambda_{m(\alpha)}^{D^*D}\int_M |\xi|^2,
\end{aligned}
$$

which gives

$$\lambda_\alpha^J \le -(n-1) + \lambda_{m(\alpha)}^{D^*D}.$$

Thus, the proof of Theorem 2 is completed. □

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