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GREEDY AND LAZY REPRESENTATIONS IN NEGATIVE BASE SYSTEMS

Tomáš Hejda, Zuzana Masáková and Edita Pelantová

We consider positional numeration systems with negative real base $-\beta$, where $\beta > 1$, and study the extremal representations in these systems, called here the greedy and lazy representations. We give algorithms for determination of minimal and maximal $(-\beta)$-representation with respect to the alternate order. We also show that both extremal representations can be obtained as representations in the positive base β^2 with a non-integer alphabet. This enables us to characterize digit sequences admissible as greedy and lazy $(-\beta)$-representation. Such a characterization allows us to study the set of uniquely representable numbers. In the case that β is the golden ratio and the Tribonacci constant, we give the characterization of digit sequences admissible as greedy and lazy $(-\beta)$-representation using a set of forbidden strings.

Keywords: numeration systems, lazy representation, greedy representation, negative base, unique representation

Classification: 11A63, 37B10

1. INTRODUCTION

A positional number system is given by a real base α with $|\alpha| > 1$ and by a finite set of digits $\mathcal{A} \subset \mathbb{R}$, usually called alphabet. If $x \in \mathbb{R}$ can be expressed in the form $\sum_{i \leqslant k} x_i \alpha^i$ with coefficients $x_i \in \mathcal{A}$, we say that x has an α-representation in $\mathcal{A}$. The most important assumption on the choice of the alphabet $\mathcal{A}$ is that any positive real number x has at least one α-representation. This assumption implies that the cardinality of the alphabet $\#\mathcal{A}$ is at least $|\alpha|$.

It is well known that if the base α is a positive integer and the alphabet is $\mathcal{A} = \{0, 1, 2, \ldots, \alpha - 1\}$ then any positive real x has an α-representation and almost all positive reals (up to a countable number of exceptions) have unique representation. For example, in the decimal numeration system

$$\tfrac{1}{2} = 0.5000000\ldots = 0.4999999\ldots, \qquad \text{whereas} \qquad \tfrac{1}{3} = 0.333333\ldots.$$

If the base α is not an integer and the alphabet $\mathcal{A}$ is rich enough to represent all positive reals, then almost all $x \geqslant 0$ have infinitely many representations and one can choose among them "the nicest" one from some point of view.[1]

[1] Note that although most people prefer writing $\frac{1}{2} = 0.5$, the shopkeepers consider the representation $0.4999\ldots$ nicer than $0.5000\ldots$ for $x = \frac{1}{2}$.

The most studied numeration system with a positive base $\alpha \notin \mathbb{Z}$ uses the alphabet $\mathcal{A} = \{0, 1, 2, \ldots, \lfloor \alpha \rfloor\}$. The set of all α-representations of x understood as infinite words in the alphabet $\mathcal{A}$ is lexicographically ordered. The lexicographically greatest α-representation is considered to be the "nicest". Since this α-representation is computed by the so called greedy algorithm, it is referred to as the greedy representation. On the other hand, the lexicographically smallest β-representation is called lazy. One might also be interested in other representations in positive base systems, for an extensive overview we refer to [6].

The study of greedy representations for non-integer bases $\alpha > 1$ was initiated by Rényi in 1957 [9]. An interest in lazy representations for bases $\alpha \in (1, 2)$ with alphabet $\{0, 1\}$ was started in 1990 by works of Erdős, Joó and Komornik [4]. A systematic study of lazy representations for all bases $\alpha > 1$ can be found in the work of Dajani and Kraaikamp [2] from 2002.

Recently, Ito and Sadahiro [5] introduced a numeration system with a negative base $\alpha = -\beta < -1$ and with the alphabet $\mathcal{A} = \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$. They gave an algorithm for computing a $(-\beta)$-representation of $x \in \left[\frac{-\beta}{1+\beta}, \frac{1}{1+\beta}\right)$ and showed that the natural order on $\mathbb{R}$ corresponds to the alternate order on such $(-\beta)$-representations. Using a negative base, we can represent positive and negative numbers without an additional bit for the signum $\pm$. A family of transformations producing negative base representations of numbers for $1 < \beta < 2$ is studied in [1]. Among others, it is shown that although none of them gives the maximal representation in the alternate order, it is produced by a random algorithm, see Theorem 4.2. in [1]. In its proof, one can find out that the greedy representation is obtained by periodic application of two transformations.

In this article, we focus on negative bases $-\beta$, $\beta > 1$ in general, and deduce analogous result without introducing random $(-\beta)$-expansions. Our main result states that both extremal representations can be obtained using the positive base β^2 and a non-integer alphabet $\mathcal{B}$ by application of a transformation of the form $T(x) = \beta^2 x - D(x)$, where $D(x) \in \mathcal{B}$ (Theorems 4.1 and 4.2). Note that representations using a non-integer alphabet were considered by Pedicini [8]. This enables us to exploit results of [6] for giving necessary and sufficient conditions for identifying sequences admissible as greedy and lazy $(-\beta)$-expansions (Theorem 5.3). For $\beta = \phi$, the golden ratio, and $\beta = \mu$, the Tribonacci constant, we describe such sequences in terms of forbidden strings. Finally, we illustrate how the characterization of admissible greedy and lazy representations can be applied for the study of uniquely representable numbers (Section 7).

2. HOW TO OBTAIN α-REPRESENTATIONS OF REAL NUMBERS

In this chapter we recall a method for finding an α-representation of a given number with general real base α, $|\alpha| > 1$. It is clear that if we are able to find a representation for all numbers x from some bounded interval $J \subset \mathbb{R}$ containing 0, then we are also able to find an α-representation for any $x \in \bigcup_{k \in \mathbb{Z}} \alpha^k J$, i. e., for any real number (if 0 is an interior point of J or the base α is negative) or for all positive reals or all negative reals (if 0 is a boundary point of J and the base α is positive). Our definition below is a restriction of the very general numeration system considered by Thurston [11].

Definition 2.1. Given a base $\alpha \in \mathbb{R}$, $|\alpha| > 1$, a finite alphabet $\mathcal{A} \subset \mathbb{R}$ and a bounded interval $J \ni 0$. Let $D : J \mapsto \mathcal{A}$ be a mapping such that the transformation defined by $T(x) = \alpha x - D(x)$ maps J to J. The corresponding α-representation of x is a mapping $d = d_{\alpha,J,D} : J \mapsto \mathcal{A}^{\mathbb{N}}$,

$$d_{\alpha,J,D}(x) = x_1x_2x_3x_4\ldots, \quad \text{where } x_k = D\big(T^{k-1}(x)\big).$$

Let us comment the previous definition. By the definition of T, for any $x \in J$ we have $x = \frac{D(x)}{\alpha} + \frac{T(x)}{\alpha}$. As the value $T(x)$ is required to belong to J as well, we may use this formula recursively and obtain the mentioned α-representation of x in the alphabet $\mathcal{A}$:

$$x = \frac{D(x)}{\alpha} + \frac{D\big(T(x)\big)}{\alpha^2} + \frac{D\big(T^2(x)\big)}{\alpha^3} + \frac{D\big(T^3(x)\big)}{\alpha^4} + \cdots \tag{1}$$

Clearly, at any step of this recursion we have

$$x = \frac{x_1}{\alpha} + \frac{x_2}{\alpha^2} + \cdots + \frac{x_n}{\alpha^n} + \frac{T^n(x)}{\alpha^n}. \tag{2}$$

Example 2.2. In [9], Rényi defined for $\alpha = \beta > 1$ the mappings $T : [0,1) \mapsto [0,1)$ and $D : [0,1) \mapsto \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ by

$$D(x) = \lfloor \beta x \rfloor \quad \text{and} \quad T(x) = \beta x - D(x).$$

The β-representation of $x \in [0,1)$ corresponding to this choice of D and T is usually called the greedy β-expansion of x.

More general case of number systems with positive base is studied in [6], where the authors admit non-integer digits and, among other, give condition for a digit string to be admissible as a number expansion. We refer to [6] for an extensive list of useful literature.

Example 2.3. In [5], Ito and Sadahiro considered negative bases $\alpha = -\beta$, for any $\beta > 1$. On the interval $J = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ the mappings T and D are defined as follows:

$$D(x) = \left\lfloor -\beta x + \tfrac{\beta}{\beta+1} \right\rfloor \quad \text{and} \quad T(x) = -\beta x - D(x).$$

The corresponding alphabet, i. e., the range of D, is $\mathcal{A} = \big\{0, 1, \ldots, \lfloor\beta\rfloor\big\}$. Let us mention that for $\beta \notin \mathbb{N}$ the Ito–Sadahiro alphabet and the Rényi alphabet coincide. If that $\beta \in \mathbb{N}$, the digit $\lfloor\beta\rfloor$ occurs only in the infinite suffix $\big(\lfloor\beta\rfloor\big)^\omega$ and every number in J can also be represented in base $-\beta$ with the smaller alphabet $\mathcal{A}\setminus\big\{\lfloor\beta\rfloor\big\} = \{0, 1, \ldots, \beta-1\}$. However, such a representation cannot be obtained simply using the floor or the ceil function.

The Rényi β-expansion and the Ito–Sadahiro $(-\beta)$-expansion are "order-preserving", provided that we choose a suitable order on the set of representations which they produce.

Definition 2.4. Let $\mathcal{A} \subset \mathbb{R}$ be a finite alphabet ordered by the natural order "$<$" in $\mathbb{R}$. Let $x_1x_2x_3\ldots$ and $y_1y_2y_3\ldots$ be two different strings from $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^n$ for $n \in \mathbb{N}$, $n \geqslant 1$. Denote $k = \min\{i \mid x_i \neq y_i\}$. We write

- $x_1x_2x_3\ldots \prec_{\mathrm{lex}} y_1y_2y_3\ldots$ if $x_k < y_k$ and say that $x_1x_2x_3\ldots$ is smaller than $y_1y_2y_3\ldots$ in the lexicographical order;
- $x_1x_2x_3\ldots \prec_{\mathrm{alt}} y_1y_2y_3\ldots$ if $(-1)^k x_k < (-1)^k y_k$ and say that $x_1x_2x_3\ldots$ is smaller than $y_1y_2y_3\ldots$ in the alternate order.

Let us stress that we always compare only strings of the same length, finite or infinite.

Proposition 2.5. Let α, $\mathcal{A}$, J and D be as in Definition 2.1. Let numbers $x, y \in J$ and let $d_{\alpha,J,D}(x) = x_1x_2x_3\ldots$ and $d_{\alpha,J,D}(y) = y_1y_2y_3\ldots$ be their α-representations.

- If $\alpha > 1$ and $D(x)$ is non-decreasing then

$$x < y \quad \Longleftrightarrow \quad x_1x_2x_3\ldots \prec_{\mathrm{lex}} y_1y_2y_3\ldots.$$

- If $\alpha < -1$ and $D(x)$ is non-increasing then

$$x < y \quad \Longleftrightarrow \quad x_1x_2x_3\ldots \prec_{\mathrm{alt}} y_1y_2y_3\ldots.$$

P r o o f. We prove the statement for a negative base $\alpha < -1$. Let us denote $k = \min\{i \mid x_i \neq y_i\}$. As $x_\ell = y_\ell$ for all $\ell = 1, 2, \ldots, k-1$ we have according to (2) that

$$x < y \quad \Longleftrightarrow \quad \frac{T^{k-1}(x)}{\alpha^{k-1}} < \frac{T^{k-1}(y)}{\alpha^{k-1}}. \tag{3}$$

Since D is non-increasing and $x_k = D\big(T^{k-1}(x)\big) \neq y_k = D\big(T^{k-1}(y)\big)$, we have

$$x_k > y_k \quad \Longleftrightarrow \quad \frac{T^{k-1}(x)}{\alpha^{k-1}} < \frac{T^{k-1}(y)}{\alpha^{k-1}}. \tag{4}$$

First we discuss the case when k is even. Since $\alpha < 0$, combining (3) and (4), we obtain

$$x < y \quad \Longleftrightarrow \quad x_k < y_k \quad \Longleftrightarrow \quad x_1x_2x_3\ldots \prec_{\mathrm{alt}} y_1y_2y_3\ldots.$$

If k is odd, then we obtain

$$x < y \quad \Longleftrightarrow \quad x_k > y_k \quad \Longleftrightarrow \quad x_1x_2x_3\ldots \prec_{\mathrm{alt}} y_1y_2y_3\ldots.$$

The proof of the statement for a positive base is analogous. □

The above proposition allows one to give a condition of admissibility for a digit string to be a representation of a number according to the algorithm presented in Definition 2.1. Such an admissibility condition was given in [7] for the Rényi number system from Example 2.2, in [6] for more general positive base systems and in [5] for the number system from Example 2.3. Let us stress that the representations and consequently admissibility conditions depend on the given transformation T in the scheme J, D, T.

From now on, we focus on a different question. We do not fix any transformation, but among all representations of a number x with a given alphabet, we search for an extremal one.

Given a base α and an alphabet $\mathcal{A} \subset \mathbb{R}$ we put

$$J_{\alpha,\mathcal{A}} = \Big\{ \sum_{i=1}^{\infty} a_i \alpha^{-i} \,\Big|\, a_i \in \mathcal{A} \Big\},$$

the set of numbers representable with negative powers of α and the alphabet $\mathcal{A}$, and for any $x \in J_{\alpha,\mathcal{A}}$, we denote the set of its α-representations in $\mathcal{A}$ by

$$R_{\alpha,\mathcal{A}}(x) = \Big\{ x_1 x_2 x_3 \ldots \,\Big|\, x = \sum_{i=1}^{\infty} x_i \alpha^{-i} \text{ and } x_i \in \mathcal{A} \Big\}.$$

Proposition 2.5 suggests how to choose a suitable ordering on the set $R_{\alpha,\mathcal{A}}(x)$.

Definition 2.6. Let α be a real base with $|\alpha| > 1$ and let $\mathcal{A} \subset \mathbb{R}$ be a finite alphabet.

- Let $\alpha < -1$ and $x \in J_{\alpha,\mathcal{A}}$. Then the maximal and minimal elements of $R_{\alpha,\mathcal{A}}(x)$ with respect to the alternate order are called the greedy and lazy α-representations of x in the alphabet $\mathcal{A}$, respectively.
- Let $\alpha > 1$ and $x \in J_{\alpha,\mathcal{A}}$. Then the maximal and minimal elements of $R_{\alpha,\mathcal{A}}(x)$ with respect to the lexicographical order are called the greedy and lazy α-representations of x in the alphabet $\mathcal{A}$, respectively.

In order to justify that the maximal and minimal elements in $R_{\alpha,\mathcal{A}}(x)$ exist, realize that $R_{\alpha,\mathcal{A}}(x)$ is a compact subspace of $\mathcal{A}^{\mathbb{N}}$. For, it is a pre-image of a closed set $\{x\} \subset \mathbb{R}$ under the map $x_1 x_2 x_3 \ldots \mapsto \frac{x_1}{\alpha^1} + \frac{x_2}{\alpha^2} + \frac{x_3}{\alpha^3} + \cdots$ which is continuous with respect to the Cantor metric on $\mathcal{A}^{\mathbb{N}}$.

For positive bases $\alpha = \beta > 1$, the greedy β-representation in the alphabet $\mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ corresponds to the β-representation produced by the Rényi greedy algorithm described in Example 2.2. The method for obtaining the lazy representation in the same alphabet was described in [2].

Let us demonstrate how to obtain these extremal representations for the case of the golden mean $\phi = \frac{1+\sqrt{5}}{2}$. Since $\lceil \phi \rceil = 2$, we consider the alphabet $\mathcal{A} = \{0, 1\}$. As $\phi^2 = \phi + 1$, we have $\sum_{i=1}^{\infty} \phi^{-i} = \phi$ and so clearly,

$$J_{\phi,\mathcal{A}} = \left\{ \sum_{i=1}^{\infty} a_i \phi^{-i} \,\middle|\, a_i \in \{0,1\} \right\} = [0, \phi].$$

If the first digit in a ϕ-representation of an $x \in J_{\phi,\mathcal{A}}$ is $x_1 = 0$ then necessarily $x \in \frac{1}{\phi} J_{\phi,\mathcal{A}} = [0, 1]$. If the first digit in a ϕ-representation of x is $x_1 = 1$ then necessarily $x \in \frac{1}{\phi} + \frac{1}{\phi} J_{\phi,\mathcal{A}} = \big[\frac{1}{\phi}, \phi\big]$. It means that

- for $x \in [0, \frac{1}{\phi})$, all representations in $R_{\phi,\mathcal{A}}(x)$ have the first digit $x_1 = 0$;
- for $x \in (1, \phi]$, all representations in $R_{\phi,\mathcal{A}}(x)$ have the first digit $x_1 = 1$;
- for $x \in [\frac{1}{\phi}, 1]$, both 0 and 1 appear as x_1.

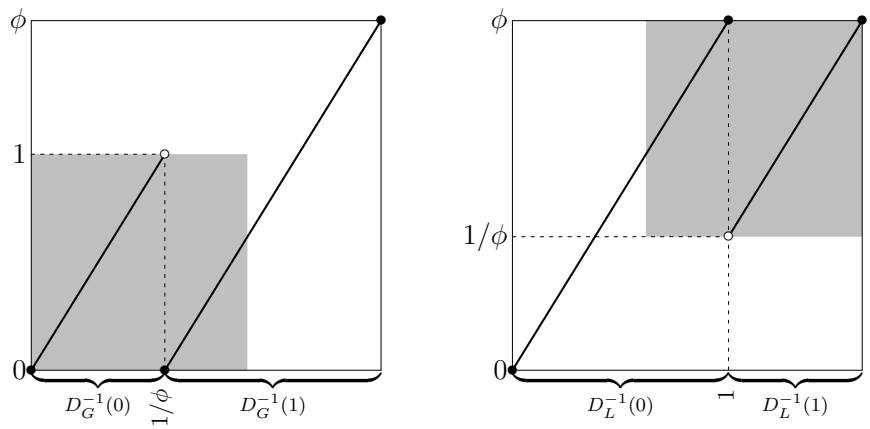

Fig. 1. The greedy and lazy transformations T_G and T_L for $\beta = +\phi$ with pre-images of 0 and 1 under D_G and D_L, respectively.

As we now consider the lexicographical order, the greedy representation of an $x \in \left[\frac{1}{\phi}, 1\right]$ has the first digit $x_1 = 1$ and the lazy representation must have $x_1 = 0$. Thus the corresponding digit assigning mappings are

$$D_G(x) = \begin{cases} 0 & \text{for } x \in \left[0, \frac{1}{\phi}\right), \\ 1 & \text{for } x \in \left[\frac{1}{\phi}, \phi\right], \end{cases} \quad \text{and} \quad D_L(x) = \begin{cases} 0 & \text{for } x \in [0, 1], \\ 1 & \text{for } x \in (1, \phi], \end{cases}$$

or equivalently

$$D_G(x) = \left\lfloor x + \tfrac{1}{\phi^2} \right\rfloor, \qquad D_L(x) = \lceil x \rceil - 1 \qquad \text{for any } x \in (0, \phi)$$

and

$$D_G(0) = D_L(0) = 0, \qquad D_G(\phi) = D_L(\phi) = 1.$$

The graphs of the corresponding transformations T_G and T_L are depicted in Figure 1.

The transformations T_G and T_L map $[0, \phi] \mapsto [0, \phi]$. The points 0 and ϕ are fixed points of both transformations. However, we see that

- for any $x \in [0, \phi)$ there exists $n \geqslant 1$ such that $T_G^n(x) \in J_G = [0, 1)$ and $T_G(J_G) = J_G$;
- for any $x \in (0, \phi]$ there exists $n \geqslant 1$ such that $T_L^n(x) \in J_L = (\frac{1}{\phi}, \phi]$ and $T_L(J_L) = J_L$.

Moreover the intervals $[0, 1)$ and $(\frac{1}{\phi}, \phi]$ are the smallest intervals with the above described property. Such interval is called the attractor of a mapping in [2]. Let us point out that in both cases there exists an isolated fixed point $\phi = T_G(\phi)$ and $0 = T_L(0)$ outside the attractors. In fact, to find the greedy β-representation, Rényi used $J_G = [0, 1)$ as the domain of D and T in the scheme J, D, T. Even when the scheme J, D, T for finding the lazy representation is described, the restriction of the mappings D and T to the 'attractor' interval of T is preferred.

3. EXTREMAL REPRESENTATIONS IN NEGATIVE BASE SYSTEMS

Let us now fix a base $\alpha = -\beta$ for some $\beta > 1$, $\beta \notin \mathbb{N}$, and an alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. (We exclude integer bases because of the phenomena explained in Example 2.3.) Using the same arguments as in [8] it can be shown that the set I of numbers representable in this system is an interval, namely

$$I = \Big[\frac{-\beta\lfloor\beta\rfloor}{\beta^2-1}, \frac{\lfloor\beta\rfloor}{\beta^2-1}\Big] =: [l, r]. \tag{5}$$

Though the interval I (i. e., its boundary points) depends on the base β, we omit the symbol β it in the notation for simplicity. We denote by I_a the set of numbers which have a $(-\beta)$-representation starting with the digit $a \in \mathcal{A}$. Then

$$I_a = \frac{a}{-\beta} + \frac{1}{-\beta} I = \Big[l + \frac{\lfloor\beta\rfloor - a}{\beta}, r - \frac{a}{\beta}\Big] = \Big[\frac{a}{-\beta} + \frac{r}{-\beta}, \frac{a}{-\beta} + \frac{l}{-\beta}\Big]$$

and I can be written as a (not necessarily disjoint) union of intervals $I = \bigcup_{a\in\mathcal{A}} I_a$. Obviously, we have $-\beta x - a \in I$ for every $x \in I_a$.

Note that intervals I_a overlap, but only two at a time. If $\lfloor\beta\rfloor = 1$, then it is clear, as they are only two. If $\lfloor\beta\rfloor \geqslant 2$, then the right endpoint of I_{a+1} is smaller than the left endpoint of I_{a-1}, namely

$$\frac{a+1}{-\beta} + \frac{l}{-\beta} = \frac{a+1}{-\beta} + \frac{\lfloor\beta\rfloor}{\beta^2-1} < \frac{a-1}{-\beta} + \frac{1}{-\beta}\frac{\lfloor\beta\rfloor}{\beta^2-1} = \frac{a-1}{-\beta} + \frac{r}{-\beta},$$

which is equivalent to $2\beta - \lfloor\beta\rfloor - 2 > 0$.

Our aim is to provide an algorithm for finding extremal digit strings (with respect to the alternate order) representing a given x in the base $(-\beta)$. Assigning of the first digit of such an extremal representation must prefer — among all possible digits — the minimal or the maximal one in the alternate order.

We define

$$D_m(x) = \begin{cases} \lfloor\beta\rfloor & \text{for } x \in I_{\lfloor\beta\rfloor}, \\ a & \text{for } x \in I_a \setminus I_{a+1},\ a \in \mathcal{A},\ a \neq \lfloor\beta\rfloor, \end{cases}$$

and

$$D_v(x) = \begin{cases} 0 & \text{for } x \in I_0, \\ a & \text{for } x \in I_a \setminus I_{a-1},\ a \in \mathcal{A},\ a \neq 0, \end{cases}$$

and the corresponding transformations

$$T_m(x) = -\beta x - D_m(x) \qquad \text{and} \qquad T_v(x) = -\beta x - D_v(x).$$

It is readily seen that $D_m(x)$ prefers small digits (in the alternate order) and $D_v(x)$ prefers great digits, i. e.,

$$D_m(x) = a \quad \Longrightarrow \quad -\beta x - b \notin I \quad \text{for any } b \prec_{\text{alt}} a,\ b \in \mathcal{A}, \tag{6}$$

and

$$D_v(x) = a \quad \Longrightarrow \quad -\beta x - b \notin I \quad \text{for any } b \succ_{\text{alt}} a,\ b \in \mathcal{A}. \tag{7}$$

Let us stress that in accordance with Definition 2.4, alternate order on single digits is inverse to the usual one.

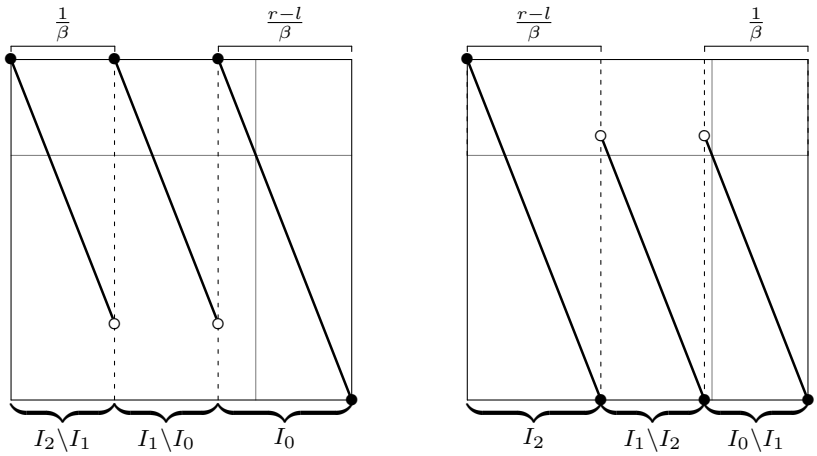

Fig. 2. Transformations T_v and T_m which prefer small and large digits in the alternate order, respectively (for a base $-\beta \in (-3,-2)$). The fractions $\frac{1}{\beta}$ and $\frac{r-l}{\beta}$ in the upper part of the figure represent two different lengths of intervals indicated in the lower part of the figure.

Proposition 3.1. Let $x \in I$.

- Denote $\varepsilon_0 = x$ and for all $i \geqslant 0$ put

  $$z_{2i+1} = D_v(\varepsilon_{2i}),\ \varepsilon_{2i+1} = T_v(\varepsilon_{2i}) \quad \text{and} \quad z_{2i+2} = D_m(\varepsilon_{2i+1}),\ \varepsilon_{2i+2} = T_m(\varepsilon_{2i+1}).$$

  Then $z_1 z_2 z_3 \ldots$ is the greedy $(-\beta)$-representation of x.

- Denote $\eta_0 = x$ and for all $i \geqslant 0$ put

  $$y_{2i+1} = D_m(\eta_{2i}),\ \eta_{2i+1} = T_m(\eta_{2i}) \quad \text{and} \quad y_{2i+2} = D_v(\eta_{2i+1}),\ \eta_{2i+2} = T_v(\eta_{2i+1}).$$

  Then $y_1 y_2 y_3 \ldots$ is the lazy $(-\beta)$-representation of x.

P r o o f. Suppose for contradiction that there exists a representation $x_1 x_2 x_3 \ldots \in R_{-\beta,\mathcal{A}}(x)$ such that $z_1 z_2 z_3 \ldots \prec_{\text{alt}} x_1 x_2 x_3 \ldots$. Let us denote $k = \min\{i \mid z_i \neq x_i\}$.

If $k = 2\ell$ is even, then the inequality $z_1 z_2 z_3 \ldots \prec_{\text{alt}} x_1 x_2 x_3 \ldots$ implies $z_{2\ell} < x_{2\ell}$ and thus $z_{2\ell} = a$ and $x_{2\ell} = b$ for some $a, b \in \mathcal{A}$, $a < b$. Since

$$\begin{aligned}\varepsilon_{2\ell-1} &= (-\beta)^{2\ell-1}\Bigg(x - \sum_{i=1}^{2\ell-1} z_i(-\beta)^{-i}\Bigg) = (-\beta)^{2\ell-1}\Bigg(x - \sum_{i=1}^{2\ell-1} x_i(-\beta)^{-i}\Bigg)\\ &= (-\beta)^{2\ell-1}\Bigg(x_{2\ell}(-\beta)^{-2\ell} + \sum_{i=2\ell+1}^{\infty} x_i(-\beta)^{-i}\Bigg) = -\frac{1}{\beta}\Bigg(b + \sum_{i=2\ell+1}^{\infty} x_i(-\beta)^{2\ell-i}\Bigg),\end{aligned}$$

we have $-\beta\varepsilon_{2\ell-1} - b \in I = [l, r]$. On the other hand, according to (6), $z_{2\ell} = D_m(\varepsilon_{2\ell-1}) = a$ implies that $-\beta\varepsilon_{2\ell-1} - b \notin I$, which is a contradiction. The case k odd is analogous.

Very similarly one proves the statement for the lazy representation. □

Note that for $\beta \in (1,2)$, the description of greedy $(-\beta)$-representations of Proposition 3.1 can be deduced from the proof of Theorem 4.2 in [1].

4. REPRESENTATIONS IN BASE β^2 WITH NON-INTEGER ALPHABETS

The algorithm for obtaining extremal $(-\beta)$-representations of a number x, described in Proposition 3.1, does not fit in the scheme of the Definition 2.1 for negative base $\alpha = -\beta$. In particular, there is no transformation $T(x) = -\beta x - D(x)$ which generates for every x the greedy (or lazy) $(-\beta)$-representation. This fact complicates the description of digit strings occurring as greedy or lazy representations. Nevertheless, we manage to overcome this handicap.

Defining $T_G := T_m T_v$ and $T_L := T_v T_m$ we obtain transformations $I \to I$ which produce the greedy and lazy $(-\beta)$-representations. The price to be paid is that the digit assigning functions D_G and D_L are not integer-valued. For, we have

$$T_G(x) = T_m\big(T_v(x)\big) = \beta^2 x + \beta D_v(x) - D_m\big(-\beta x - D_v(x)\big),$$

i. e.,

$$T_G(x) = \beta^2 x - D_G(x), \tag{8}$$

where

$$D_G(x) = -\beta D_v(x) + D_m\big(-\beta x - D_v(x)\big) \in \mathcal{B},$$

with the alphabet $\mathcal{B} = \{-b\beta + a \mid a, b \in \mathcal{A}\}$, which has $(\#\mathcal{A})^2$ distinct elements, as $\beta \notin \mathbb{N}$.

Let us describe the mappings T_G and D_G (resp. T_L and D_L) explicitly. As T_G is a composition of two piecewise linear functions, T_G is also of this kind. Its points of discontinuity are described as follows,

$$\mathcal{D}is_G = \mathcal{D}is_v \cup T_v^{-1}\big(\mathcal{D}is_m\big), \tag{9}$$

where $\mathcal{D}is_v$ and $\mathcal{D}is_m$ are points of discontinuity of T_v and T_m respectively. Clearly, $\#\mathcal{D}is_v = \#\mathcal{D}is_m = \#\mathcal{A} - 1$. We can easily see that

$$\mathcal{D}is_m = \Big\{ \frac{a}{-\beta} + \frac{l}{-\beta} \,\Big|\, a \in \mathcal{A},\ a \neq 0 \Big\} \tag{10}$$

and

$$\mathcal{D}is_v = \Big\{ \frac{a}{-\beta} + \frac{r}{-\beta} \,\Big|\, a \in \mathcal{A},\ a \neq \lfloor \beta \rfloor \Big\}. \tag{11}$$

As every discontinuity point $\frac{a}{-\beta} + \frac{l}{-\beta}$ of $\mathcal{D}is_m$ lies in $(r-1, r]$ and for every letter $a \in \mathcal{A}$ one has $(r-1, r] \subset T_v(I_a)$, the set $T_v^{-1}(\mathcal{D}is_m)$ has $\#\mathcal{A} \times (\#\mathcal{A} - 1)$ points. Thus the union in (9) has $(\#\mathcal{A})^2 - 1$ elements. The interval I therefore splits into $(\#\mathcal{A})^2$ subintervals on which the transformation T_G is continuous. Explicitly, for every $a \in \mathcal{A}$, $a \neq 0$, we have

$$T_v^{-1}\Big(\frac{a}{-\beta} + \frac{l}{-\beta}\Big) = \Big\{ \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{l}{(-\beta)^2} \,\Big|\, b \in \mathcal{A} \Big\}.$$

As $-\beta r = l$, every discontinuity point $\frac{b}{-\beta} + \frac{r}{-\beta}$ of $\mathcal{D}is_v$ can be written in the form $\frac{b}{-\beta} + \frac{l}{(-\beta)^2}$, and thus

$$\mathcal{D}is_G = \Big\{ \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{l}{(-\beta)^2} \,\Big|\, a, b \in \mathcal{A},\ (b,a) \neq (\lfloor\beta\rfloor, 0) \Big\}.$$

In order to obtain an explicit description of the mapping T_G, let us define

$$\gamma_{ba} \coloneqq l + \frac{\lfloor\beta\rfloor - b}{\beta} = \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{l}{(-\beta)^2}, \quad \text{for any } a, b \in \mathcal{A}. \tag{12}$$

Note that if $(b,a) = (\lfloor\beta\rfloor, 0)$, then $\gamma_{ba} = l$. The set of words of length two over the alphabet $\mathcal{A}$ can be ordered by the alternate order, namely

$$\lfloor\beta\rfloor 0 \prec_{\text{alt}} \lfloor\beta\rfloor 1 \prec_{\text{alt}} \cdots \prec_{\text{alt}} \lfloor\beta\rfloor\lfloor\beta\rfloor \prec_{\text{alt}} (\lfloor\beta\rfloor - 1)0 \prec_{\text{alt}} \cdots \\ \cdots \prec_{\text{alt}} 1\lfloor\beta\rfloor \prec_{\text{alt}} 00 \prec_{\text{alt}} \cdots \prec_{\text{alt}} 0\lfloor\beta\rfloor.$$

Taking into account that $\beta \notin \mathbb{N}$, one can easily see that

$$ab \prec_{\text{alt}} cd \quad \Longleftrightarrow \quad \gamma_{ab} = \frac{a}{-\beta} + \frac{b+l}{(-\beta)^2} < \frac{c}{-\beta} + \frac{d+l}{(-\beta)^2} = \gamma_{cd}.$$

In particular, this means that the assignment $ab \leftrightarrow -a\beta + b$, for $a, b \in \mathcal{A}$, is a bijection. On the set $\mathcal{A}^2$ we can define the predecessor and successor functions by

$$P(ab) \coloneqq \max\{cd \mid cd \in \mathcal{A}^2,\ cd \prec_{\text{alt}} ab\},$$
$$S(ab) \coloneqq \min\{cd \mid cd \in \mathcal{A}^2,\ cd \succ_{\text{alt}} ab\},$$

where maximum and minimum is taken again with respect to the alternate order. Of course, the predecessor is not defined for the minimal element $\min\mathcal{A}^2 = \lfloor\beta\rfloor 0$ and the successor is not defined for the maximal element $\max\mathcal{A}^2 = 0\lfloor\beta\rfloor$.

Now we are ready to provide an explicit description of the digit assigning map $D_G : I \to \mathcal{B} = \{-b\beta + a \mid a, b \in \mathcal{A}\}$, namely

$$D_G(x) = \begin{cases} -b\beta + a, & \text{if } x \in \big[\gamma_{ba}, \gamma_{S(ba)}\big) \text{ for some } ba \neq 0\lfloor\beta\rfloor, \\ \lfloor\beta\rfloor, & \text{if } x \in \big[\gamma_{ba}, r\big] \text{ for } ba = 0\lfloor\beta\rfloor. \end{cases} \tag{13}$$

Define a morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ by

$$\psi(-b\beta + a) = ba.$$

Note that ψ is well defined as $\beta \notin \mathbb{N}$. The action of the morphism ψ can be naturally extended to infinite words by setting $\psi(x_1x_2x_3\ldots) = \psi(x_1)\psi(x_2)\psi(x_3)\ldots$. All the above considerations can be summarized in the following theorem.

Theorem 4.1. Let $\beta > 1$, $\beta \notin \mathbb{N}$, $\mathcal{A} = \big\{0, 1, \ldots, \lfloor\beta\rfloor\big\}$. For an $x \in I$, denote by $d_G(x) = d_{\beta^2,I,D_G}(x)$ the β^2-representation of x obtained in accordance with Definition 2.1 by the transformation $T_G : I \to I$, defined by (8). Then

- $d_G(x)$ is the greedy β^2-representation of x in the alphabet $\mathcal{B}$;
- $\psi\big(d_G(x)\big)$ is the greedy $(-\beta)$-representation of x in the alphabet $\mathcal{A}$.

For the sake of completeness, let us provide an explicit description of the lazy transformation $T_L = T_v T_m : I \to I$ and the corresponding digit assigning map $D_L : I \to \mathcal{B}$, such that

$$T_L(x) = \beta^2 x - D_L(x), \tag{14}$$

for $x \in I$. Defining

$$\delta_{ba} \coloneqq \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{r}{(-\beta)^2}, \quad \text{for any } a, b \in \mathcal{A}.$$

we can write the set of discontinuity points of T_L as

$$\mathcal{D}is_L = \big\{\delta_{ba} \,\big|\, a, b \in \mathcal{A},\ ba \neq 0\lfloor\beta\rfloor\big\}.$$

Note that for $ba = 0\lfloor\beta\rfloor$ one has $\delta_{ba} = r$. Again,

$$ab \prec_{\text{alt}} cd \quad \iff \quad \delta_{ab} < \delta_{cd}.$$

We now have the digit assigning function $D_L : I \to \mathcal{B} = \{-b\beta + a \mid a, b \in \mathcal{A}\}$, given by

$$D_L(x) = \begin{cases} -\lfloor\beta\rfloor\beta, & \text{if } x \in \big[l, \delta_{ba}\big] \text{ for } ba = \lfloor\beta\rfloor 0, \\ -b\beta + a, & \text{if } x \in \big(\delta_{P(ba)}, \delta_{ba}\big] \text{ for } ba \neq \lfloor\beta\rfloor 0. \end{cases} \tag{15}$$

Theorem 4.2. Let $\beta > 1$, $\beta \notin \mathbb{N}$, $\mathcal{A} = \big\{0, 1, \ldots, \lfloor\beta\rfloor\big\}$. For an $x \in I$, denote by $d_L(x) = d_{\beta^2, I, D_L}(x)$ the β^2-representation of x obtained in accordance with Definition 2.1 by the transformation $T_L : I \to I$, defined by (14). Then

- $d_L(x)$ is the lazy β^2-representation of x in the alphabet $\mathcal{B}$;
- $\psi\big(d_L(x)\big)$ is the lazy $(-\beta)$-representation of x in the alphabet $\mathcal{A}$.

In [4], the relation between the lazy and the greedy representations of numbers in a numeration system with a positive base $\beta \in (1, 2)$ and the alphabet $\{0, 1\}$ was established. Namely, if $x = x_1 x_2 x_3 \ldots$ is the greedy β-representation of $x \in J_{\beta,\mathcal{A}} = \big[0, \frac{1}{\beta - 1}\big]$, then $(1-x_1)(1-x_2)(1-x_3)\ldots$ is the lazy β-representation of $y = \frac{1}{\beta-1} - x$. Generalization of this relation for numeration systems with a base $\beta > 1$ and the alphabet $\mathcal{A} = \big\{0, 1, \ldots, \lfloor\beta\rfloor\big\}$ was presented in [2]. Comparing the formulas for D_G and D_L, one can deduce a similar symmetry between greedy and lazy $(-\beta)$-representations.

Proposition 4.3. Let $z_1 z_2 z_3 z_4 \ldots$ be the greedy $(-\beta)$-representation of a number $z \in I$ and let $y_1 y_2 y_3 y_4 \ldots$ be the lazy $(-\beta)$-representation of a number $y \in I$. Then

$$y_i + z_i = \lfloor\beta\rfloor \quad \text{for every } i \geqslant 1 \qquad \iff \qquad y + z = -\frac{\lfloor\beta\rfloor}{\beta + 1}.$$

From now on, we shall concentrate on the properties of the greedy transformation T_G only.

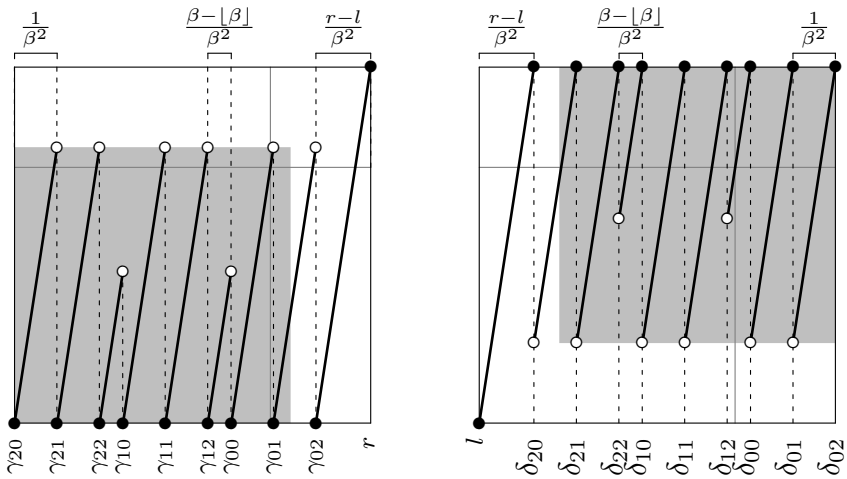

Fig. 3. The greedy and lazy transformations T_G and T_L. This figure corresponds to a base $-\beta \in (-3,-2)$.

5. ADMISSIBILITY

The transformation T_G has the following property:

For every $x \in [l, l+1)$ one has $T_G(x) \in [l, l+1)$. Moreover, for every $x \in I \setminus [l, l+1)$, $x \neq r$, there exists an exponent $k \in \mathbb{N}$ such that $T_G^k(x) \in [l, l+1)$.

The above fact implies that, in general, some digits from the alphabet $\mathcal{B}$ do not appear infinitely many times in the greedy β^2-representation of any number $x \in I$.

Lemma 5.1. Let d_G be as in Theorem 4.1. Denote by $\mathcal{A}_G$ the minimal alphabet, such that the β^2-representation $d_G(x)$ of every $x \in [l, l+1)$ has digits in $\mathcal{A}_G$. Then

$$\mathcal{A}_G = \big\{-b\beta + a \mid a, b \in \mathcal{A},\ b \geqslant 1\big\} \cup \big\{a \,\big|\, a \in \mathcal{A},\ a < \beta\{\beta\}\big\},$$

where $\{\beta\} = \beta - \lfloor\beta\rfloor$ is the fractional part of β. Moreover, for every $x \in I \setminus \{r\}$, the β^2-representation $d_G(x)$ has a suffix over $\mathcal{A}_G$.

P r o o f . The fact that a given $x \in I$ satisfies $T_G^k(x) \in [l, l+1)$ for sufficiently large k means that the greedy representation of x eventually uses only such digits $-b\beta + a$ that the corresponding interval $[\gamma_{ba}, \gamma_{S(ba)})$ has a non-empty intersection with $[l, l+1)$. Such digits satisfy

$$l + 1 > \gamma_{ba} = \frac{b}{-\beta} + \frac{a}{(-\beta)^2} + \frac{l}{(-\beta)^2}.$$

This is equivalent to

$$\beta\big(\beta - \lfloor\beta\rfloor\big) > -b\beta + a. \tag{16}$$

The left-hand side of the above inequality is positive. The right-hand side is negative whenever $b \geqslant 1$. If $b = 0$ then the condition for a is written as $a < \beta\{\beta\}$. □

Remark 5.2. From the above lemma it follows that the maximal digit $\max\mathcal{A}_G$ satisfies

$$0 \leqslant \max\mathcal{A}_G = -0\cdot\beta + \lceil\{\beta\}\beta\rceil - 1.$$

The condition $\mathcal{A}_G = \mathcal{B}$ is equivalent to the fact that the maximal digit $\max\mathcal{B} = -0\cdot\beta + \lfloor\beta\rfloor$ of the alphabet $\mathcal{B}$ also satisfies (16). This gives the condition

$$\beta^2 - \lfloor\beta\rfloor\beta - \lfloor\beta\rfloor > 0,$$

which happens if and only if β is strictly greater than the larger root of $x^2 - \lfloor\beta\rfloor x - \lfloor\beta\rfloor$. Therefore the condition $\mathcal{A}_G = \mathcal{B}$ is satisfied precisely by the numbers

$$\beta \in \bigcup_{m\in\mathbb{N}} \Big(\frac{m+\sqrt{m^2+4m}}{2}, m+1\Big).$$

As a consequence of Lemma 5.1, for the description of admissible digit strings in the greedy representations, it is reasonable to consider the transformation T_G restricted to the interval $[l, l+1)$. Denote for any $X = -b\beta + a \in \mathcal{A}_G$

$$l_X = \gamma_{ba} \quad \text{and} \quad r_X = \begin{cases} \gamma_{S(ba)} & \text{if } X \neq \max\mathcal{A}_G, \\ l+1 & \text{if } X = \max\mathcal{A}_G, \end{cases} \tag{17}$$

where $\gamma_{ba} = -b\beta + a$, see (12). Then we have a disjoint union

$$[l, l+1) = \bigcup_{X\in\mathcal{A}_G} [l_X, r_X)$$

and the digit function D_G restricted to $[l, l+1)$ can be written as

$$D_G(x) = X \quad \text{if } x \in [l_X, r_X).$$

The mapping $D_G : [l, l+1) \to \mathcal{A}_G$, and the corresponding $T_G : [l, l+1) \to [l, l+1)$, $d_G : [l, l+1) \to \mathcal{A}_G^{\mathbb{N}}$ are right continuous.

In order to formulate the result about admissible greedy representations which is derived using a result of [6], let us define the left continuous mapping $D_G^* : (l, l+1] \to \mathcal{A}_G$ by

$$D_G^*(x) = X \quad \text{if } x \in (l_X, r_X].$$

The left continuous mappings $T_G^* : (l, l+1] \to (l, l+1]$, $d_G^* : (l, l+1] \to \mathcal{A}_G^{\mathbb{N}}$ are defined accordingly. The mappings T and T^* are illustrated in Figure 4. Note that for every $x \in (l, l+1]$, one has

$$D_G^*(x) = \lim_{\varepsilon\to 0+} D_G(x-\varepsilon), \quad T_G^*(x) = \lim_{\varepsilon\to 0+} T_G(x-\varepsilon), \quad d_G^*(x) = \lim_{\varepsilon\to 0+} d_G(x-\varepsilon).$$

Theorem 5.3. Let $X_1X_2X_3\ldots \in \mathcal{A}_G^{\mathbb{N}}$. Then there exists an $x \in [l, l+1)$ such that $d_G(x) = X_1X_2X_3\ldots$ if and only if for every $k \geqslant 1$

$$X_{k+1}X_{k+2}X_{k+3}\ldots \prec \begin{cases} d_G^*\big(T_G^*(l+1)\big) & \text{if } X_k = \max\mathcal{A}_G, \\ d_G^*\big(l + \{\beta\}\big) & \text{if } X_k = -b\beta + \lfloor\beta\rfloor,\ X_k \neq \max\mathcal{A}_G. \end{cases} \tag{18}$$

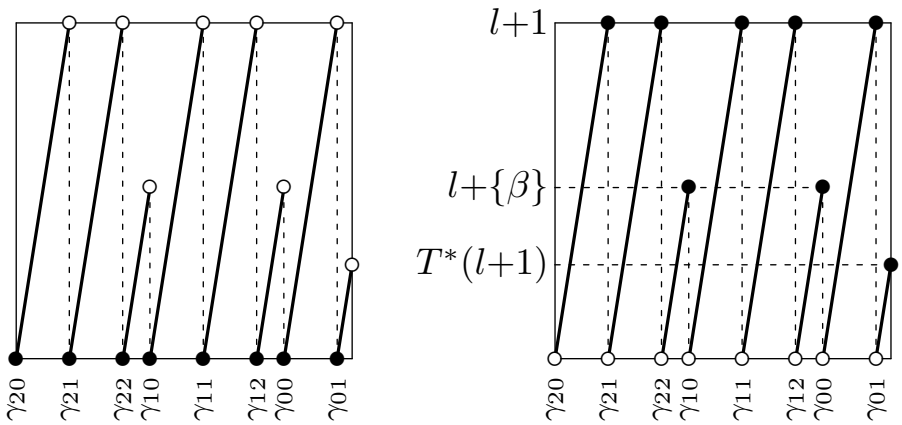

Fig. 4. Right and left continuous mappings T and T^* defined on $[l, l+1)$ and $(l, l+1]$, respectively. This figure corresponds to a base $-\beta \in (-3, -2)$.

P r o o f . Applying Theorem 2.5 from [6] to the greedy transformation T_G, we obtain a necessary and sufficient condition for a string $X_1X_2X_3\ldots$ over the alphabet $\mathcal{A}_G$ to be admissible, namely

$$d(l_{X_k}) \preceq X_kX_{k+1}X_{k+2}\ldots \prec d^*(r_{X_k}) \qquad \text{for all } k \geqslant 1. \tag{19}$$

Here and in what follows, for simplicity of notation we omit in the proof the subscripts in $d_G = d$, $T_G = T$, $d^*_G = d^*$, $T^*_G = T^*$.

Since $d(l_X) = Xd\big(T(l_X)\big)$ and $d^*(r_X) = Xd^*\big(T^*(r_X)\big)$, we can rewrite conditions (19) as

$$d\big(T(l_{X_k})\big) \preceq X_{k+1}X_{k+2}\ldots \prec d^*\big(T^*(r_{X_k})\big) \qquad \text{for all } k \geqslant 1. \tag{20}$$

Let us study the left-hand side inequality. From the description of $T = T_G$, it is clear that for every letter $X \in \mathcal{A}_G$, we have $T(l_X) = l$ (see also Figure 4) and thus, the string on the left-hand side of (20) reads

$$d\big(T(l_{X_k})\big) = d(l) = (\min\mathcal{A}_G)^\omega.$$

Consequently, the left-hand side condition is not limiting.

In order to study the right-hand side condition from (20), realize that if $X \neq \max\mathcal{A}_G$, we have

$$T^*(r_X) = \begin{cases} l+1 & \text{for } X \neq -b\beta + \lfloor\beta\rfloor, \\ l+\{\beta\} & \text{for } X = -b\beta + \lfloor\beta\rfloor, \end{cases}$$

see Figure 4. As $r_X = l+1$ for $X = \max\mathcal{A}_G$, the right-hand side condition can be written as

$$X_{k+1}X_{k+2}\ldots \prec \begin{cases} d^*\big(T^*(l+1)\big) & \text{for } X_k = \max\mathcal{A}_G, \\ d^*(l+1) & \text{for } X_k \neq -b\beta + \lfloor\beta\rfloor,\ X_k \neq \max\mathcal{A}_G, \\ d^*\big(l+\{\beta\}\big) & \text{for } X_k = -b\beta + \lfloor\beta\rfloor,\ X_k \neq \max\mathcal{A}_G. \end{cases} \tag{21}$$

We have $d^*(l+1) = \max\mathcal{A}_G\, d^*\big(T^*(l+1)\big)$. Therefore if $X_{k+1} < \max\mathcal{A}_G$, then the condition

$$X_{k+1}X_{k+2}\ldots \prec d^*(l+1) = \max\mathcal{A}_G\, d^*\big(T^*(l+1)\big)$$

is satisfied. If $X_{k+1} = \max\mathcal{A}_G$, then the condition is equivalent to

$$X_{k+2}\ldots \prec d^*\big(T^*(l+1)\big),$$

which is ensured by verifying the first of (21) for the shifted index $k+1$. This completes the proof. □

Remark 5.4. Using Proposition 4.3 one can derive an analogous necessary and sufficient condition for admissible lazy β^2-representations $X_1X_2X_3\ldots$ of numbers in $x \in (r-1, r]$ over the alphabet $\mathcal{A}_L = \lfloor\beta\rfloor - \mathcal{A}_G$.

6. NEGATIVE GOLDEN RATIO

Let us illustrate the previous results and their implications on the example of the negative base $-\beta$ where $\beta = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. Real numbers representable in base $-\phi$ over the alphabet $\mathcal{A} = \{0, 1\}$ form the interval $J_{-\phi,\mathcal{A}} = I = \big[-1, \frac{1}{\phi}\big] = [l, r]$.

The greedy and lazy $(-\phi)$-representation can be obtained from the greedy and lazy ϕ^2-representation over the alphabet $\mathcal{B} = \{-\phi, -\phi+1, 0, 1\}$, applying the morphism $\psi : \mathcal{B}^* \to \mathcal{A}^*$ given by

$$\psi(-\phi) = 10, \quad \psi(-\phi+1) = 11, \quad \psi(0) = 00, \quad \psi(1) = 01.$$

The greedy and lazy ϕ^2-representations are generated by the transformation

$$T_G(x) = \phi^2 x - D_G(x), \quad T_L(x) = \phi^2 x - D_L(x), \quad x \in \big[-1, \tfrac{1}{\phi}\big],$$

where the digit assigning maps D_G and D_L are

$$D_G(x) = \begin{cases} -\phi & \text{for } x \in \big[-1, -\frac{1}{\phi}\big), \\ -\phi+1 & \text{for } x \in \big[-\frac{1}{\phi}, -\frac{1}{\phi^2}\big), \\ 0 & \text{for } x \in \big[-\frac{1}{\phi^2}, 0\big), \\ 1 & \text{for } x \in \big[0, \frac{1}{\phi}\big], \end{cases} \quad D_L(x) = \begin{cases} -\phi & \text{for } x \in \big[-1, -\frac{1}{\phi^2}\big], \\ -\phi+1 & \text{for } x \in \big(-\frac{1}{\phi^2}, 0\big], \\ 0 & \text{for } x \in \big(0, \frac{1}{\phi^3}\big], \\ 1 & \text{for } x \in \big(\frac{1}{\phi^3}, \frac{1}{\phi}\big]. \end{cases}$$

The graph of the transformations T_G, T_L together with their restriction to the intervals $[l, l+1) = [-1, 0)$, $(r-1, r] = (-\frac{1}{\phi^2}, \frac{1}{\phi}]$ are drawn in Figure 5.

Remark 6.1. From the graphs of T_G and T_L we see that

- the greedy ϕ^2-representation of an $x \in \big(-1, \frac{1}{\phi}\big)$ eventually uses only digits from the alphabet $\mathcal{A}_G = \{-\phi, -\phi+1, 0\}$, while its lazy ϕ^2-representation eventually uses only digits $\mathcal{A}_L = \{-\phi+1, 0, 1\}$; in particular,
  - if $x \in \big[0, \frac{1}{\phi}\big)$, i. e., outside $[l, l+1)$, then its greedy $(-\phi)$-representation has the form $(01)^M z_1 z_2 z_3\ldots$, where M is the minimal positive integer such that $x' = x - \sum_{k=1}^{M} \frac{1}{\phi^{2k}} < 0$ and $z_1 z_2 z_3\ldots$ is the greedy $(-\phi)$-representation for the number x';

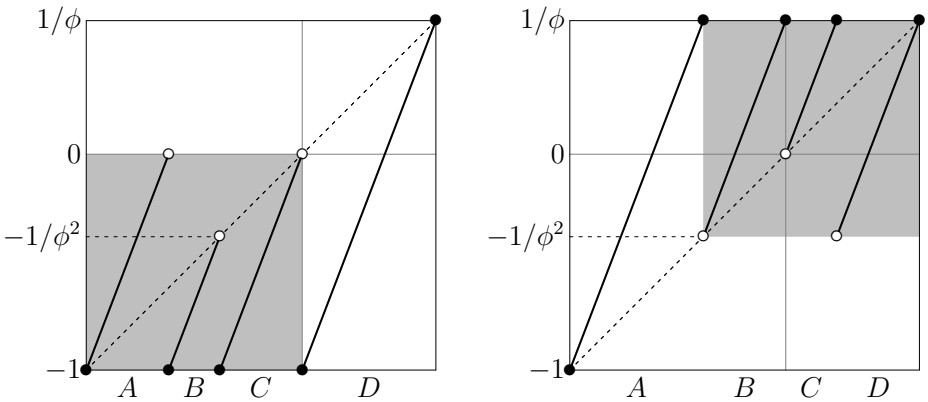

Fig. 5. Greedy (left) and lazy (right) transformations T_G and T_L for the base $-\phi$.

- if $x \in \left(-1, -\frac{1}{\phi^2}\right]$, i. e., outside $(r-1, r]$, then its lazy $(-\phi)$-representation has the form $(10)^M y_1 y_2 y_3 \ldots$, where M is the minimal positive integer such that $x'' = x + \phi \sum_{k=1}^{M} \frac{1}{\phi^{2k}} > -\frac{1}{\phi^2}$ and $y_1 y_2 y_3 \ldots$ is the lazy $(-\phi)$-representation of x'';

- the greedy and lazy $(-\phi)$-representations of $\frac{1}{\phi}$ are both equal to $(01)^\omega$;
- the greedy and lazy $(-\phi)$-representations of -1 are both equal to $(10)^\omega$;
- the greedy $(-\phi)$-representation of 0 is $01(10)^\omega$, while its lazy $(-\phi)$-representation is $11(01)^\omega = 1(10)^\omega$.

Let us now apply Theorem 5.3 to the case $\beta = \phi$. Denote for simplicity the digits of the alphabet $\mathcal{B} = \{-\phi, -\phi+1, 0, 1\}$ by

$$A = -\phi \quad < \quad B = -\phi + 1 \quad < \quad C = 0 \quad < \quad D = 1.$$

With this notation, we have $\mathcal{A}_G = \{A, B, C\}$ and $\mathcal{A}_L = \{B, C, D\}$.

Proposition 6.2. A string $X_1 X_2 X_3 \ldots$ over the alphabet $\mathcal{A}_G = \{A, B, C\}$ is the greedy ϕ^2-representation of a number $x \in [-1, 0)$ if and only if it does not contain a factor from the set $\{BC, B^\omega, C^\omega\}$.

A string $X_1 X_2 X_3 \ldots$ over the alphabet $\mathcal{A}_L = \{B, C, D\}$ is the lazy ϕ^2-representation of a number $x \in (\frac{1}{\phi^2}, \frac{1}{\phi}]$ if and only if it does not contain a factor from the set $\{CB, B^\omega, C^\omega\}$.

P r o o f. Since both $l + 1 = 0$ and $l + \{\phi\} = -\frac{1}{\phi^2}$ are fixed points of the transformation T_G^*, we obtain

$$d_G^*\big(T_G^*(l+1)\big) = 0^\omega = C^\omega, \quad d_G^*\big(l + \{\phi\}\big) = (-\phi + 1)^\omega = B^\omega,$$

and from Theorem 5.3, a string $X_1X_2X_3\ldots$ over the alphabet $\mathcal{A}_G$ is the greedy ϕ^2-representation of a number $x \in [-1, 0)$ if and only if for all $k \geqslant 1$,

$$X_{k+1}X_{X+2}\ldots \prec \begin{cases} C^\omega & \text{if } X_k = C, \\ B^\omega & \text{if } X_k = B. \end{cases}$$

It is not difficult to see that such condition is satisfied exactly by the strings in $\{A, B, C\}^*$ which do not contain a factor from the set $\{BC, B^\omega, C^\omega\}$.

The statement about lazy ϕ^2-representations is derived by symmetry $A \leftrightarrow D$, $B \leftrightarrow C$, (cf. Proposition 4.3). □

It follows from the facts above that only digits $B = -\phi + 1$ and $C = 0$ are common to both greedy and lazy ϕ^2-representation restricted to the intervals $[l, l + 1)$, $(r - 1, r]$, respectively. However, due to the forbidden strings BC in greedy and CB in lazy ϕ^2-representation, no string over the alphabet $\{B, C\}$ is admissible in both cases. Combining with Remark 6.1, we have the following result.

Corollary 6.3. The points $x = -1$ and $x = \frac{1}{\phi}$ are the only points in $[-1, \frac{1}{\phi}]$ which have a unique $(-\phi)$-representation over the alphabet $\{0, 1\}$.

Proposition 6.2 provides a combinatorial criterion for admissibility of representations in base ϕ^2 in the non-integer alphabet $\mathcal{A}_G$. One can also rewrite the admissibility of a digit string in base $-\phi$ using forbidden strings in the original alphabet $\{0, 1\}$.

Proposition 6.4. A digit string $x_1x_2x_3\ldots$ over the alphabet $\{0, 1\}$ is a greedy $(-\phi)$-representation of some $x \in [-1, 0)$ if and only if

(i) it does not start with the prefix $1^{2k}0$, nor $0^{2k-1}1$, $k \geqslant 1$;

(ii) it does not end with the suffix 0^ω nor 1^ω;

(iii) it does not contain the factor $10^{2k}1$, nor $01^{2k}0$, $k \geqslant 1$.

Proof. ($\Rightarrow$): Realize that grouping the digits in the greedy $(-\phi)$-representation of x to pairs, we obtain a string over the alphabet $\{(10), (11), (00), (01)\}$. This is justified by the fact that the greedy $(-\phi)$-representation of x over $\{0, 1\}$ is obtained from the greedy ϕ^2-representation of x over $\{A, B, C, D\}$ applying $\psi(A) = 10$, $\psi(B) = 11$, $\psi(C) = 00$, $\psi(D) = 01$.

From Proposition 6.2, one can derive that the forbidden strings over the alphabet $\{(10), (11), (00), (01)\}$ are (01), (11)(00), $(11)^\omega$, $(00)^\omega$. It is thus obvious that the greedy representation over the alphabet $\{0, 1\}$ cannot have a suffix 0^ω, 1^ω, i. e. (ii) holds. In order to prove (i), realize that $0^{2k-1}1 = (00)^{k-1}(01)$ contains the forbidden digit (01). Similarly, $1^{2k}0 = (11)^k0$ is a prefix of either $(11)^k(01)$ or $(11)^k(00)$, where the former contains the forbidden digit (01) and the latter the forbidden string (11)(00). As for (iii), note that the factor $10^{2k}1$ is read in the alphabet $\{(10), (11), (00), (01)\}$ either as $(10)(00)^{k-1}(01)$ or as a part of $(*1)(00)^k(1*)$. In both cases it necessarily contains either (01) or (11)(00). The string $01^{2k}0$ is excluded in a similar way.

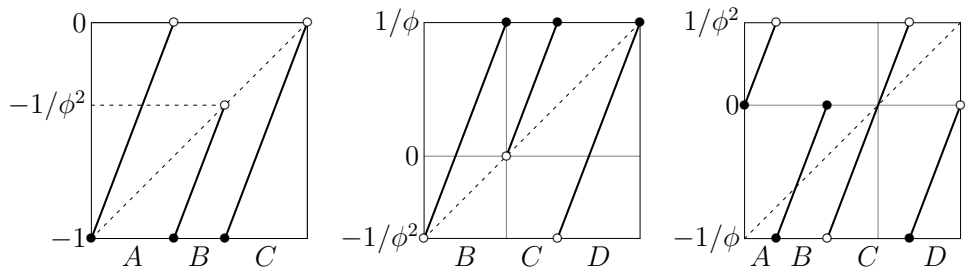

Fig. 6. Transformations T_G (left), T_L (middle) and T_{IS}^2 (right) in the base ϕ^2 that correspond to the greedy, lazy and Ito–Sadahiro representations in the base $-\phi$.

($\Leftarrow$): We must show that a string over $\{0,1\}$ satisfying conditions (i) – (iii) does not contain forbidden strings of pairs (01), (11)(00), $(11)^\omega$, $(00)^\omega$. Note that the brackets are always around pairs $x_{2i+1}x_{2i+2}$. Conditions (i) – (iii) are equivalent to the fact that the string $x_1x_2x_3\ldots$ is of the form

$$0^{2k_1}1^{2k_2+1}0^{2k_3+1}1^{2k_4+1}\ldots, \qquad \text{with } k_i \geqslant 0. \tag{22}$$

Thus the blocks of 1's always start at odd-indexed positions and blocks of 0's (except the prefix) start at even-indexed positions. This implies the impossibility of occurrence of (01), (11)(00). The impossibility of $(11)^\omega$, $(00)^\omega$ is obvious. □

Let us now compare the greedy and lazy $(-\phi)$-representations with those obtained by the Ito–Sadahiro algorithm, presented in Example 2.3. We denote by D_{IS} and T_{IS} the corresponding transformations which now read $D_{IS}, T_{IS} : \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right) \mapsto \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$ defined

$$D_{IS}(x) = \left\lfloor -\phi x + \tfrac{1}{\phi} \right\rfloor \quad \text{and} \quad T_{IS}(x) = -\phi x - \left\lfloor -\phi x + \tfrac{1}{\phi} \right\rfloor.$$

As follows from the admissibility condition in [5], a digit sequence $x_1x_2x_3\ldots$ over the alphabet $\{0,1\}$ is the Ito–Sadahiro $(-\phi)$-representation of some $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$ if and only if

$$10^\omega \preceq_{\text{alt}} x_ix_{i+1}x_{i+2}\ldots \prec_{\text{alt}} 010^\omega \quad \text{for all } i \geqslant 1. \tag{23}$$

One can easily derive that $x_1x_2x_3\ldots$ is an Ito–Sadahiro $(-\phi)$-representation if and only if it does not contain any string $10^{2k+1}1$ for $k \geqslant 0$ as a substring, and it does not end with 010^ω.

The example $x = -\frac{1}{2}$ shows that, in general, the Ito–Sadahiro algorithm does not assign to $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$ any extremal representation. We have the following $(-\phi)$-representations of $-\frac{1}{2}$:

Ito–Sadahiro	$x_1x_2x_3\ldots = (100)^\omega$	$= 100\,100\,100\ldots,$
lazy	$y_1y_2y_3\ldots = 1(001110)^\omega$	$= 100\,111\,000\ldots,$
greedy	$z_1z_2z_3\ldots = (111000)^\omega$	$= 111\,000\,111\ldots.$

Clearly, $y_1y_2y_3\ldots \prec_{\mathrm{alt}} x_1x_2x_3\ldots \prec_{\mathrm{alt}} z_1z_2z_3\ldots$. In fact, one can show that for no element of the interval $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$ the Ito–Sadahiro $(-\phi)$-representation is extremal with respect to the alternate order.

Proposition 6.5. The Ito–Sadahiro $(-\phi)$-representation is not extremal for any $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$.

Proof. Let us consider an $x \in \left[-\frac{1}{\phi}, \frac{1}{\phi^2}\right)$ with the Ito–Sadahiro $(-\phi)$-representation $y_1y_2y_3\ldots$. Suppose that $y_1y_2y_3\ldots$ is also a greedy $(-\phi)$-representation of x. Then by Remark 6.1, it is of the form

$$y_1y_2y_3\ldots = (01)^M x_1x_2x_3\ldots, \quad \text{for some } M \geqslant 0,$$

where the string $x_1x_2x_3\ldots$ is the greedy representation of some $x \in [-1, 0)$. The string $x_1x_2x_3\ldots$ thus satisfies conditions of Proposition 6.4, and, consequently, is of the form (22). However, this is not compatible with the admissibility condition (23) for Ito–Sadahiro $(-\phi)$-representations.

Suppose now that the Ito–Sadahiro $(-\phi)$-representation $y_1y_2y_3\ldots$ is also lazy. The argumentation is similar, noting that the forbidden strings for the lazy $(-\phi)$-representation can be derived from Proposition 6.4 by replacing $0 \leftrightarrow 1$. □

7. UNIQUE $(-\beta)$-REPRESENTATIONS

In [1], it is shown that for $1 < \beta < 2$, the set of numbers with a unique $(-\beta)$-representation is of Lebesgue measure zero. The authors also show that for $\beta < \phi$, such numbers are only two. Let us show that although the measure is always zero, the set of numbers with unique $(-\beta)$-representation can be uncountable.

Example 7.1. Let $\beta = \mu$ be the Tribonacci constant, i. e., the real root $\mu \approx 1.83$ of $x^3 - x^2 - x - 1$. Since $\lfloor \mu \rfloor = 1$ and $\mu > \phi$, the greedy μ^2-representation of numbers in the interval $[l, l+1) = [-\frac{\mu}{\mu^2-1}, \frac{1}{\mu(\mu^2-1)})$ may use all possible digits $-\mu$, $-\mu+1$, 0, 1, see Lemma 5.1. We shall again use the notation

$$A = -\mu, \quad B = -\mu + 1, \quad C = 0, \quad D = 1.$$

Our aim is to derive the admissibility condition for digit strings $X_1X_2X_3\ldots$ over the alphabet $\mathcal{A}_G = \{A, B, C, D\}$ using Theorem 5.3.

With the notation (17), omitting subscripts of $T_G = T$, $d_G = d$, one can verify that

$$\begin{aligned} l + \{\mu\} &= r_C, \\ T^*\big(l + \{\mu\}\big) &= T^*(r_C) = l + 1 = r_D, \\ (T^*)^2\big(l + \{\mu\}\big) &= T^*(r_D) = r_B, \\ (T^*)^3\big(l + \{\mu\}\big) &= T^*(r_B) = l + \{\mu\}, \end{aligned}$$

see also Figure 7. The strings used in the admissibility condition from Theorem 5.3 are thus purely periodic and very strongly related, namely

$$d^*\big(l + \{\mu\}\big) = Cd^*(l+1) = CDd^*\big(T^*(l+1)\big) = CDBd^*\big(l + \{\mu\}\big) = (CDB)^\omega.$$

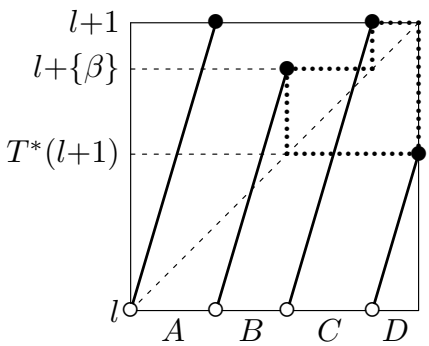

Fig. 7. The graph of the transformation T_G^* for the Tribonacci constant.

Therefore, a digit string $X_1X_2X_3\ldots$ over the alphabet $\mathcal{A}_G$ is an admissible greedy μ^2-representation if and only if for all $k \geqslant 1$,

$$X_{k+1}X_{k+2}X_{k+3}\ldots \prec \begin{cases}(CDB)^\omega & \text{if } X_k = B,\\ (BCD)^\omega & \text{if } X_k = D,\end{cases}$$

which is equivalent to the requirement that the string $X_1X_2X_3\ldots$ does not contain a factor from $\{BD, DC, DD, (BCD)^\omega\}$.

By symmetry $A \leftrightarrow D$, $B \leftrightarrow C$, we obtain that a digit string $X_1X_2X_3\ldots$ over the alphabet $\mathcal{A}_L = \mathcal{A}_G$ is an admissible lazy μ^2-representation if and only if it does not contain a factor from the set $\{CA, AB, AA, (CBA)^\omega\}$.

From this, we can see that any digit string over the alphabet $\{B, C\}$ is admissible both as the greedy and lazy μ^2-representation, thus numbers represented by such a string have only one representation in base $-\mu$. There are uncountably many such numbers.

Theorem 7.2. Let $\beta > 1 + \sqrt{3}$, $\beta \notin \mathbb{N}$. Then there exist uncountably many numbers in $J_{-\beta,\mathcal{A}}$ having a unique $(-\beta)$-representation over the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$.

P r o o f . We will show that there are uncountably many numbers in the interval $J_{-\beta,\mathcal{A}} = [l, r)$ for which the lazy and the greedy $(-\beta)$-representations coincide. Necessarily, such a number x (with the exception of $x = l$ and $x = r$) must lie in the intersection $[l, l+1) \cap (r-1, r] = (r-1, l+1)$. The alphabet for the β^2-representation of such an $x \in (r-1, l+1)$ is $\mathcal{A}_G \cap \mathcal{A}_L$.

Recall that the requirement (18) from Theorem 5.3 for the admissible greedy β^2-representations controls the string only after digits X_k of the form $X_k = -b\beta + \lfloor\beta\rfloor$ and $X_k = \max\mathcal{A}_G$. By symmetry (Proposition 4.3), the requirement on lazy β^2-representations controls the string only after digits X_k of the form $X_k = -b\beta + 0$ and $X_k = \min\mathcal{A}_L$. Realize that

$$\max\mathcal{A}_G = 0 \cdot \beta + a \quad \text{and} \quad \min\mathcal{A}_L = \lfloor\beta\rfloor \cdot \beta + \lfloor\beta\rfloor - a$$

for some $a \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$, see Remark 5.2. Therefore any digit string over the original alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ avoiding 0 and $\lfloor\beta\rfloor$ is allowed both as a greedy and lazy $(-\beta)$-representation. If $\beta > 3$, then $\mathcal{A} \setminus \{0, \lfloor\beta\rfloor\}$ has at least two elements and thus we have uncountably many numbers with unique $(-\beta)$-representation.

Let now β satisfy $1 + \sqrt{3} < \beta < 3$. The above argument is not sufficient, since $\mathcal{A} \setminus \{0, \lfloor\beta\rfloor\} = \{1\}$, so we need to refine our study of admissible greedy and lazy strings. As $1 + \sqrt{3}$ is a root of $x^2 - 2x - 2$, by Remark 5.2, the alphabets $\mathcal{A}_G$, $\mathcal{A}_L$ of greedy and lazy β^2-representations are equal to the full alphabet $\mathcal{B} = \{-b\beta + a \mid 0 \leqslant a, b \leqslant 2\}$.

Let us show that the string $d^*_G(l + \{\beta\})$ starts with a digit $X \in \mathcal{B}$, $X \geqslant 0$. For that, we have to verify that

$$l + \{\beta\} > l_0 = \frac{l}{(-\beta)^2},$$

where we use notation (17). This is equivalent to $\beta^2 - 2\beta - 2 > 0$, which is true precisely when $\beta > 1 + \sqrt{3}$.

From the condition (18) we derive that any string over the alphabet $\{X \in \mathcal{B} \mid X < 0\}$ is admissible as a greedy β^2-representation. By symmetry, any string over the alphabet $\{X \in \mathcal{B} \mid X > -2\beta + 2\}$ is admissible as a lazy β^2-representation. Altogether, every string over the alphabet

$$\{X \in \mathcal{B} \mid -2\beta + 2 < X < 0\} = \{-\beta, -\beta + 1, -\beta + 2\}$$

is admissible both as the greedy and as the lazy β^2-representation of some number $x \in (r - 1, l + 1)$, and therefore x has a unique $(-\beta)$-representation. Obviously, there are uncountably many such numbers. □

Remark 7.3. In fact, as pointed out by one of the referees, for $\beta > 3$ the proof tells more than stated in the theorem. The set U of all numbers with $(-\beta)$-representations over the alphabet $\mathcal{A} \setminus \{0, \lfloor\beta\rfloor\}$ forms a self-similar set containing $\lfloor\beta\rfloor - 1$ copies of itself, each time β times smaller than the original set. It implies that the Hausdorff dimension of the set U equals to $\log_\beta(\lfloor\beta\rfloor - 1)$. The set of numbers with a unique $(-\beta)$-representation contains U and therefore is of Hausdorff dimension at least $\log_\beta(\lfloor\beta\rfloor - 1)$.

8. CONCLUSIONS

Our main tool in this paper was to view the $(-\beta)$-representations in the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ as strings of pairs of digits in $\mathcal{A}$, which amounts, in fact, to considering the alphabet $\mathcal{B} = -\beta \cdot \mathcal{A} + \mathcal{A}$ and the base β^2. Such an approach puts forward the utility of studying number systems with positive base and a non-integer alphabet, as was already started by Pedicini [8] or Kalle and Steiner [6]. Obtaining new results for such systems — for example analogous to those of de Vries and Komornik [3] or Schmidt [10] would probably contribute also to the knowledge about negative base systems.

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