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THE CAUCHY PROBLEM FOR THE HOMOGENEOUS TIME-DEPENDENT OSEEN SYSTEM IN $\mathbb{R}^3$: SPATIAL DECAY OF THE VELOCITY

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Abstract. We consider the homogeneous time-dependent Oseen system in the whole space $\mathbb{R}^3$. The initial data is assumed to behave as $O(|x|^{-1-\varepsilon})$, and its gradient as $O(|x|^{-3/2-\varepsilon})$, when $|x|$ tends to infinity, where ε is a fixed positive number. Then we show that the velocity u decays according to the equation $|u(x,t)| = O(|x|^{-1})$, and its spatial gradient $\nabla_x u$ decreases with the rate $|x|^{-3/2}$, for $|x|$ tending to infinity, uniformly with respect to the time variable t. Since these decay rates are optimal even in the stationary case, they should also be the best possible in the evolutionary case considered in this article. We also treat the case $\varepsilon = 0$. Then the preceding decay rates of u remain valid, but they are no longer uniform with respect to t.

Keywords: Cauchy problem, time-dependent Oseen system, spatial decay, wake

MSC 2010: 35Q30, 65N30, 76D05

1. Introduction

Consider the homogeneous time-dependent Oseen system

$$\partial_t u(x,t) - \Delta_x u(x,t) + \tau \partial_1 u(x,t) + \nabla_x \pi(x,t) = 0, \quad \operatorname{div}_x u(x,t) = 0 \tag{1.1}$$

for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, under the initial condition

$$u(x,0) = a(x) \quad \text{for } x \in \mathbb{R}^3. \tag{1.2}$$

Here the initial data $a\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the parameter $\tau \in (0,\infty)$ are given, whereas the velocity $u\colon \mathbb{R}^3 \times [0,\infty) \mapsto \mathbb{R}^3$ and the pressure $\pi\colon \mathbb{R}^3 \times (0,\infty) \mapsto \mathbb{R}$ are unknown.

We are interested in the question as to what are the minimal assumptions on a such that the velocity u decays in the following way:

$$(1.3)\quad |\partial_x^\alpha u(x,t)| = O([|x|\nu_\tau(x)]^{-1-|\alpha|/2}) \quad \text{for } |x| \to \infty, \text{ uniformly in } t \in (0,\infty),$$

where α is a multiindex in $\mathbb{N}_0^3$ with $|\alpha| \leqslant 1$, and where the function ν_τ is defined by

$$(1.4)\qquad \nu_\kappa(z) := 1 + \kappa(|z| - z_1) \quad \text{for } \kappa \in (0,\infty),\ z \in \mathbb{R}^3.$$

Here and in the following, we use the notation $|\beta|$ for the length $\beta_1+\beta_2+\beta_3$ of $\beta \in \mathbb{N}_0^3$. The condition $|\alpha| \leqslant 1$ in (1.3) means that we consider the asymptotic behaviour of $u(\alpha = 0)$ and of the gradient of u with respect to $x(|\alpha| = 1)$. The term $\nu_\tau(x)$ may be seen as a mathematical manifestation of the wake extending behind a rigid body moving with constant velocity through a viscous incompressible fluid. We further note that the decay rates of u and $\nabla_x u$ given by (1.3) are the best possible, in the sense that no better rates could be detected in general in the stationary case ([1], [18, Chapter IX], [16], [24], [4]). Returning to the question concerning the minimal conditions on a such that (1.3) holds, an ideal answer would, of course, consist in requiring that

$$|\partial^\alpha a(y)| = O([|y|\nu_\tau(y)]^{-1-|\alpha|/2}) \quad \text{for } |y| \to \infty,$$

with α as in (1.3). However, although we did obtain a decay rate as in (1.3) under this condition, we could not show this rate to be uniform in $t \in (0,\infty)$. In order to get a decay uniform in t, we had to assume there is an $\varepsilon > 0$ with

$$|\partial^\alpha a(y)| = O([|y|\nu_\tau(y)]^{-1-|\alpha|/2-\varepsilon}) \quad \text{for } |y| \to \infty.$$

Let us state our results in more detail. To this end, we have to introduce some notation. Put $e_1 := (1,0,0)$ and $A^c := \mathbb{R}^3 \setminus A$ for $A \subset \mathbb{R}^3$. By $\mathfrak{H}$, we denote the usual heat kernel in $\mathbb{R}^3$, that is,

$$(1.5)\qquad \mathfrak{H}(z,t) := (4\pi t)^{-3/2} \mathrm{e}^{-|z|^2/(4t)} \quad \text{for } z \in \mathbb{R}^3,\ t \in (0,\infty).$$

We define the volume potential $\mathfrak{I}^{(\kappa)}(c)\colon\ \mathbb{R}^3 \times (0,\infty) \mapsto \mathbb{R}^3$ by

$$(1.6)\qquad \mathfrak{I}^{(\kappa)}(c)(x,t) := \int_{\mathbb{R}^3} \mathfrak{H}(x - y - \kappa t e_1, t) c(y)\,\mathrm{d}y \quad \text{for } x \in \mathbb{R}^3,\ \kappa, t \in (0,\infty),$$

and for suitable functions $c\colon\ \mathbb{R}^3 \mapsto \mathbb{R}^3$.

Our main result may now be stated as follows:

Theorem 1.1. *Fix the parameters* $\tau, S_0, \sigma_0, \delta_0, \delta_1 \in (0,\infty)$, $\kappa_0 \in [0,1]$, $S \in (S_0,\infty)$. *Let* $a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ *with*

$$|a(y)| \leqslant \delta_0[|y|, \nu_\tau(y)]^{-1-\kappa_0} \quad \text{for } y \in B^c_{S_0}. \tag{1.7}$$

Then the function $\mathfrak{I}^{(\tau)}(a)$ *is well defined and belongs to* $C^\infty(\mathbb{R}^3 \times (0,\infty))^3$.

If $\kappa_0 > 0$, *we have*

$$|\mathfrak{I}^{(\tau)}(a)(x,t)| \leqslant C_1(\delta_0 + \|a|B_{S_0}\|_1)[|x|\nu_\tau(x)]^{-1} \quad \text{for } x \in B^c_S,\ t \in (0,\infty), \tag{1.8}$$

where the constant $C_1 > 0$ *depends on* S_0, S, κ_0 *and* τ, *with the dependence on* τ *being such that* C_1 *is an increasing function of* τ.

Otherwise, not excluding the case $\kappa_0 = 0$, *we have*

$$|\mathfrak{I}^{(\tau)}(a)(x,t)| \leqslant C_2(\delta_0 + \|a|B_{S_0}\|_1)(1+t)^{\sigma_0}[|x|\nu_\tau(x)]^{-1} \tag{1.9}$$

for $x \in B^c_S$, $t \in (0,\infty)$, *with the constant* $C_2 > 0$ *depending on* S_0, S, σ_0 *and* τ, *and again being an increasing function of* τ.

Now additionally suppose that $a|\overline{B_{S_0}}^c \in W^{1,1}_{\mathrm{loc}}(\overline{B_{S_0}}^c)^3$ *and*

$$|\partial^\beta a(y)| \leqslant \delta_1[|y|\nu_\tau(y)]^{-1-|\beta|-\kappa_0} \quad \text{for } y \in \overline{B_{S_0}}^c,\ \beta \in \mathbb{N}^3_0 \text{ with } |\beta| \leqslant 1. \tag{1.10}$$

Then, if $\kappa_0 > 0$, *there is a constant* C_3 *of the same type as the constant* C_1 *in* (1.8) *such that*

$$|\partial^\beta_x \mathfrak{I}^{(\tau)}(a)(x,t)| \leqslant C_3(\delta_1 + \|a|B_{S_0}\|_1)[|x|\nu_\tau(x)]^{-1-|\beta|/2} \tag{1.11}$$

for $x \in B^c_S$, $t \in (0,\infty)$, $\beta \in \mathbb{N}^3_0$ *with* $|\beta| \leqslant 1$. *If we do not exclude the case* $\kappa_0 = 0$, *we obtain*

$$|\partial^\beta_x \mathfrak{I}^{(\tau)}(a)(x,t)| \leqslant C_4(\delta_1 + \|a|B_{S_0}\|_1)(1+t)^{\sigma_0}[|x|\nu_\tau(x)]^{-1-|\beta|/2} \tag{1.12}$$

for x, t, β *as in* (1.11), *with* C_4 *being the same kind of constant as* C_2 *in* (1.9).

Note that since the constants $C_1, \dots, C_4$ in Theorem 1.1 are increasing functions of τ, we may let τ tend to zero in (1.8), (1.9), (1.11), (1.12). Actually an estimate of $\mathfrak{I}^{(\tau)}(a)$ with $\tau = 0$ is implicitly given in the second part of the proof of Lemma 4.1; this estimate corresponds to the limit case $\tau = 0$ in (1.8) and (1.11), respectively, and holds even for $\kappa_0 = 0$.

Lemma 2.3 below states conditions on a such that the function $\mathfrak{I}^{(\tau)}(a)$ is the velocity part of a solution to (1.1), (1.2), with vanishing pressure. This type of result justifies the title of this paper. But we do not attempt to identify those functions that are the velocity part of a solution to (1.1), (1.2) *and* may be represented by the volume potential $\mathfrak{I}^{(\tau)}(a)$. It is for a different reason that we study the latter function: we are interested in the asymptotic behaviour of solutions to the time-dependent Navier-Stokes system with Oseen term,

$$\partial_t u - \Delta_x u + \tau \partial_1 u + \tau (u \cdot \nabla_x) u + \nabla_x p = f, \quad \mathrm{div}_x u = 0 \quad \text{in } \overline{\Omega}^c \times (0, T_0) \tag{1.13}$$

for some $T_0 \in (0, \infty]$, under Dirichlet boundary conditions on $\partial\Omega \times (0, T_0)$ and a decay condition at infinity,

$$u | \partial\Omega \times (0, T_0) = b, \quad u(x,t) \to 0 \ (|x| \to \infty) \quad \text{for } t \in (0, T_0), \tag{1.14}$$

and under an initial condition,

$$u(x, 0) = a(x) \quad \text{for } x \in \overline{\Omega}^c, \tag{1.15}$$

where $\Omega \subset \mathbb{R}^3$ is an open and bounded set, so that $\overline{\Omega}^c$ is an exterior domain. In [12], we considered the situation that the velocity part u of a solution to (1.13)–(1.15) may be represented in the form

$$u(x,t) = \mathfrak{R}^{(\tau)}(f - \tau(u \cdot \nabla_x)u)(x,t) + \mathfrak{I}^{(\tau)}(a)(x,t) + \mathfrak{V}^{(\tau)}(\Phi)(x,t) \tag{1.16}$$

for $x \in \overline{\Omega}^c$, $t \in (0, T_0)$ ([12, Theorem 3.1]), where $\mathfrak{R}^{(\tau)}(f - \tau(u \cdot \nabla_x)u)(x,t)$ is a convolution integral on $\overline{\Omega}^c \times (0, T_0)$ pertaining to a fundamental solution of (1.1) and the function $f - \tau(u \cdot \nabla_x)u$. The single layer potential $\mathfrak{V}^{(\tau)}(\Phi)(x,t)$ is also defined as a convolution integral, but the integral in question extends over $\partial\Omega \times (0, T_0)$, and the convolution involves a function Φ which solves an integral equation on $\partial\Omega \times (0, T_0)$. We discussed in [12] how formula (1.16) may be exploited in order to show that u decays as indicated in (1.3). The proof we presented in [12] for this result uses Theorem 1.1 in order to deal with the term $\mathfrak{I}^{(\tau)}(a)(x,t)$ in (1.16). The work at hand is motivated by this role of Theorem 1.1 in a theory on asymptotic behaviour of solutions to (1.13)–(1.15).

Let us indicate some further references related to the work at hand. Knightly [22] proved that solutions to (1.13)–(1.15) exhibit a wake, but he required various smallness conditions. Mizumachi [27], too, studied the asymptotic behaviour of solutions to (1.13)–(1.15), showing (1.3) for $\alpha = 0$, under assumptions that are more restrictive than those in [12]. In his proof, he also estimated the potential $\mathfrak{I}^{(\tau)}(a)$ ([27, p. 514–515]), establishing (1.8). But he did not consider the other estimates presented in Theorem 1.1, and his conditions on a are stronger than those in that latter theorem. In fact, he suppposed there is a stationary solution u_S of (1.13) such that $a(x) = u_s(x) + |x|^{-2}$ for $|x| \to \infty$ ([27, (1.11)]).

The representation formula (1.16), introduced in [12, Theorem 3.1], is a consequence of results from [5]–[11]. We refer to [10, Section 1] for an overview of this part of our theory. References [9] and [11] give a proof of (1.3) for the case that u is the velocity part of a solution to the Oseen system (1.1) in an exterior domain in $\mathbb{R}^3$ (instead of the whole space $\mathbb{R}^3$), under Dirichlet boundary conditions, with [9] requiring that a and f have compact support, whereas [11] handles a more general situation. Theorem 1.1 enters into the theory in [11].

Existence results for problem (1.13)–(1.15) were established by Heywood [19], [20], who used variational arguments, by Solonnikov [30] (solutions in Sobolev and Hölder spaces, as a consequence of an extensive linear theory), and by Miyakawa [26] and Shibata [28] (mild solutions). L^p-L^q-estimates for the Oseen system in $\overline{\Omega}^c$ were treated by Kobayashi and Shibata [23] (space dimension $n = 3$), [15] ($n \geqslant 3$), [14] (local L^p-L^q-estimates in the case $n \geqslant 3$), and by Bae, Jin [2] (weighted L^p-norms). The temporal decay of spatial L^p-norms of the velocity part of solutions to (1.13)–(1.15) was studied by Masuda [25], Heywood [20, p. 675], Shibata [28], Enomoto, Shibata [15] (case $n \geqslant 3$), and Bae, Roh [3]. Finally, as concerns the Cauchy problem (1.1), (1.2), Knightly [21, p. 507] and Takahashi [31] deduced results on pointwise spatial and temporal decay of solutions to (1.1), (1.2) by exploiting a theory on asymptotic behaviour of solutions to the instationary Stokes system.

2. Auxiliary results

Here and in the following, we write C for numerical constants, and $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ for constants depending on parameters $\gamma_1, \ldots, \gamma_n \in (0, \infty)$ for some $n \in \mathbb{N}$. For $z \in \mathbb{R}^3$, we put $z' := (z_2, z_3)$, so $z = (z_1, z')$ and $|z|^2 = z_1^2 + |z'|^2$. Recall the definition of the heat kernel $\mathfrak{H}$ in (1.5). The following estimate of $\mathfrak{H}$ was shown in [29].

Lemma 2.1. $\mathfrak{H} \in C^\infty(\mathbb{R}^3 \times (0, \infty))$ *and*

$$|\partial_t^l \partial_z^\beta \mathfrak{H}(z,t)| \leqslant \mathfrak{C}(l, |\beta|)(|z|^2 + t)^{-3/2-|\beta|/2-l} \mathrm{e}^{-|z|^2/(8t)} \tag{2.1}$$

for $z \in \mathbb{R}^3$, $t \in (0, \infty)$, $\beta \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$.

The ensuing result is well known and will be used frequently.

Lemma 2.2. $\int_{\mathbb{R}^3} \mathfrak{H}(z,t)\,\mathrm{d}z = 1$ *for* $t \in (0, \infty)$.

We remark that for our purposes, it would be enough to know that the integral in Lemma 2.2 is bounded independently of $t \in (0,\infty)$. Since the function a introduced in Section 1 may be written as $\chi_{B_{S_0}} a + \chi_{B^c_{S_0}} a$, with the first term of the sum belonging to $L^1(\mathbb{R}^3)^3$ and the second to $L^\infty(\mathbb{R}^3)^3$, the ensuing lemma implies in particular that the volume potential $\mathfrak{I}^{(\tau)}(a)$ introduced in (1.6) is well defined and belongs to $C^\infty(\mathbb{R}^3 \times (0,\infty))^3$.

Lemma 2.3. *Let $\kappa \in (0,\infty)$, $p \in [1,\infty]$, $c \in L^p(\mathbb{R}^3)^3$. Then*

$$\int_{\mathbb{R}^3} |\partial_x^\beta \mathfrak{H}(x - \kappa t e_1 - y, t) c(y)| \,\mathrm{d}y < \infty \quad \text{for } x \in \mathbb{R}^3,\ t \in (0,\infty),\ \beta \in \mathbb{N}_0^3. \tag{2.2}$$

Moreover, $\mathfrak{I}^{(\kappa)}(c) \in C^\infty(\mathbb{R}^3 \times (0,\infty))^3$ and

$$\partial_x^\beta \mathfrak{I}^{(\kappa)}(c)(x,t) = \int_{\mathbb{R}^3} \partial_x^\beta \mathfrak{H}(x - \kappa t e_1 - y, t) c(y) \,\mathrm{d}y \tag{2.3}$$

for x, t, β as in (2.2),

$$\partial_t \mathfrak{I}^{(\kappa)}(c)(x,t) - \Delta_x \mathfrak{I}^{(\kappa)}(c)(x,t) + \kappa \partial_1 \mathfrak{I}^{(\kappa)}(c)(x,t) = 0 \quad (x \in \mathbb{R}^3,\ t > 0). \tag{2.4}$$

If $p < \infty$ and if c belongs to the closure of the set $\{\varphi \in C_0^\infty(\mathbb{R}^3)^3 \colon \operatorname{div}\varphi = 0\}$ in $L^p(\mathbb{R}^3)^3$, then $\operatorname{div}_x \mathfrak{I}^{(\kappa)}(c)(x,t) = 0$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$. If $c \in C^0(\mathbb{R}^3)^3$, we further have

$$\mathfrak{I}^{(\kappa)}(c) \in C^0(\mathbb{R}^3 \times [0,\infty))^3 \quad \text{and} \quad \mathfrak{I}^{(\kappa)}(c)(x,0) = c(x) \quad \text{for } x \in \mathbb{R}^3. \tag{2.5}$$

Proof. Let $R, \delta, M \in (0,\infty)$ with $\delta < M$. For $z \in B_R, y \in B^c_{2R}$, we have $|z-y| \geqslant |y| - R \geqslant |y|/2$, so we find by Lemma 2.1 that

$$\begin{aligned} &|\partial_t^l \partial_z^\beta \mathfrak{H}(z-y,t) c(y)| \\ &\qquad \leqslant \mathfrak{C}(l,|\beta|) \delta^{-3/2-|\beta|/2-l} [\mathrm{e}^{-|y|^2/(32M)} \chi_{B^c_{2R}}(y) + \chi_{B_{2R}}(y)] |c(y)| \end{aligned} \tag{2.6}$$

($z \in B_R$, $y \in \mathbb{R}^3$, $t \in (\delta, M)$, $\beta \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$). The right-hand side of (2.6) is integrable with respect to $y \in \mathbb{R}^3$. Thus it follows by a standard application of Lebesgue's theorem and by the properties of the heat kernel that the statements of Lemma 2.3 up to and including (2.4) hold in $B_R \times (\delta, M)$, and thus in $\mathbb{R}^3 \times (0,\infty)$.

Let $p \in [1,\infty)$, and suppose that $c \in L^p(\mathbb{R}^3)^3$. Then we deduce from Lemma 2.1, 2.2 and Young's inequality for integrals that

$$\|\partial^\beta \mathfrak{I}^{(\kappa)}(c)(\ ,t)\|_p \leqslant \int_{\mathbb{R}^3} |\partial_z^\beta \mathfrak{H}(z - \kappa t e_1, t)| \,\mathrm{d}z \|c\|_p \leqslant \mathfrak{C} t^{-|\beta|/2} \|c\|_p$$

for $t \in (0,\infty)$, $\beta \in \mathbb{N}_0^3$ with $|\beta| \leqslant 1$. Therefore equation (2.3), an integration by parts and a density argument yield that $\operatorname{div}_x \mathfrak{I}^{(\kappa)}(c) = 0$, under the assumptions on c specified in the passage following equation (2.4). As for (2.5), we refer to [17, proof of Theorem 1.2.1]. $\square$

The following estimate of $|z-\kappa t e_1|^2+t$ constitutes the basic observation that allows us to detect a wake in time-dependent Oseen flows.

Lemma 2.4 ([5, Lemma 2])**.** *Let $\kappa \in (0,\infty)$. Then*

$$(|z-\kappa t e_1|^2+t)^{-1} \leqslant \mathfrak{C}(K)\max\{1,\kappa\}(\varrho_{K,\kappa}(z)+t)^{-1}$$

for $z\in\mathbb{R}^3,\ \kappa,t\in(0,\infty)$, where $\varrho_{K,\kappa}(z):=|z|^2$ for $z\in B_K$, and $\varrho_{K,\kappa}(z):=|z|\nu_\kappa(z)$ else.

Another useful observation is stated in

Lemma 2.5 ([13, Lemma 4.8])**.** *$\nu_\kappa(x-y)^{-1}\leqslant \mathfrak{C}\max\{1,\kappa\}(1+|y|)\nu_\kappa(x)^{-1}$ for $x,y\in\mathbb{R}^3,\ \kappa\in(0,\infty)$.*

By exploiting the exponential factor in the inequality in Lemma 2.1, we get

Lemma 2.6. *Let $\varepsilon,t\in(0,\infty),\ z\in\mathbb{R}^3,\ \beta\in\mathbb{N}_0^3$ with $|\beta|\leqslant 1$. Then*

$$|\partial_z^\beta \mathfrak{H}(z,t)|\leqslant \mathfrak{C}(\varepsilon)(1+t)^{\varepsilon/2}(1+|z|)^{-\varepsilon}(|z|^2+t)^{-3/2-|\beta|/2}.$$

P r o o f. If $|z|\leqslant 1$, Lemma 2.6 is an immediate consequence of Lemma 2.1. Suppose that $|z|\geqslant 1$. Then

$$\begin{aligned} \mathrm{e}^{-|z|^2/(8t)} &= (t/|z|^2)^{\varepsilon/2}(|z|^2/t)^{\varepsilon/2}\mathrm{e}^{-|z|^2/(8t)} \\ &\leqslant \mathfrak{C}(\varepsilon)(t/|z|^2)^{\varepsilon/2}\leqslant \mathfrak{C}(\varepsilon)t^{\varepsilon/2}(1+|z|)^{-\varepsilon}, \end{aligned}$$

where the last inequality is valid because $|z|\geqslant 1$. Now Lemma 2.6 follows from Lemma 2.1. □

To end this section, two technical lemmas that will be used frequently later on.

Lemma 2.7. *Let $z\in\mathbb{R}^3$. Then*

$$(|z|+z_1)(|z|-z_1)=|z'|^2,\quad (1+|z|)\nu_1(z)\geqslant(1+|z'|)^2/8. \tag{2.7}$$

P r o o f. The first equation is obvious. The second follows from the first by observing that in the case $|z'|\geqslant 1$, we have

$$(1+|z|)\nu_1(z)\geqslant |z|(|z|-z_1)\geqslant(|z|+z_1)(|z|-z_1)/2=|z'|^2/2\geqslant(1+|z'|)^2/8.$$

If $|z'|\leqslant 1$, the second inequality in (2.7) is immediate. □

Lemma 2.8. *Let $\varepsilon \in (0,1]$ and $z \in \mathbb{R}^3$. Then*

$$\int_{\mathbb{R}^3} (1+|y|)^{-1}[(1+|z-y|)\nu_1(z-y)]^{-1-\varepsilon}\,\mathrm{d}y \leqslant \mathfrak{C}(\varepsilon), \tag{2.8}$$

$$\int_{\mathbb{R}^3} (1+|y|)^{-1-\varepsilon}[(1+|z-y|)\nu_1(z-y)]^{-1}\,\mathrm{d}y \leqslant \mathfrak{C}(\varepsilon). \tag{2.9}$$

Proof. Let the left-hand side of (2.8) be denoted by $\mathfrak{A}$. We find by wirtue of (2.7) that

$$\begin{aligned} \mathfrak{A} &\leqslant \int_{\mathbb{R}^3} (1+|y|)^{-1}(1+|z-y|)^{-\varepsilon/2}[(1+|z-y|)\nu_1(z-y)]^{-1-\varepsilon/2}\,\mathrm{d}y \\ &\leqslant \mathfrak{C}\int_{\mathbb{R}^3} (1+|y_1|)^{-1}(1+|z_1-y_1|)^{-\varepsilon/2}(1+|z'-y'|)^{-2-\varepsilon}\,\mathrm{d}y \\ &\leqslant \mathfrak{C}\Big(\int_{\mathbb{R}} (1+|r|)^{-1}(1+|z_1-r|)^{-\varepsilon/2}\,\mathrm{d}r\Big)\Big(\int_{\mathbb{R}^2}(1+|\eta|)^{-2-\varepsilon}\,\mathrm{d}\eta\Big) \\ &\leqslant \mathfrak{C}(\varepsilon)\int_{\mathbb{R}} (1+|r|)^{-1}(1+|z_1-r|)^{-\varepsilon/2}\,\mathrm{d}r. \end{aligned} \tag{2.10}$$

By Hölder's inequality with exponents $1/(1-\varepsilon/4)$ and $4/\varepsilon$, we may conclude

$$\mathfrak{A} \leqslant \mathfrak{C}(\varepsilon)\Big(\int_{\mathbb{R}} (1+|r|)^{-1/(1-\varepsilon/4)}\,\mathrm{d}r\Big)^{1-\varepsilon/4}\Big(\int_{\mathbb{R}} (1+|z_1-r|)^{-2}\,\mathrm{d}r\Big)^{\varepsilon/4} \leqslant \mathfrak{C}(\varepsilon).$$

This proves (2.8). We further observe that by (2.7), the left-hand side of (2.9) is bounded by

$$\mathfrak{C}\int_{\mathbb{R}^3} (1+|y_1|)^{-1-\varepsilon/2}(1+|y'|)^{-\varepsilon/2}(1+|z'-y'|)^{-2}\,\mathrm{d}y.$$

The latter term may be estimated in a similar way as the right-hand side of the second inequality of (2.10). □

3. A scaling argument

Recall the quantities τ, S_0, σ_0, δ_0, δ_1, κ_0, S chosen in Theorem 1.1. Further recall that at the beginning of Section 2, we introduced the notation $\mathfrak{C}(\gamma_1,\ldots,\gamma_n)$ for constants that may depend on $\gamma_1,\ldots,\gamma_n \in (0,\infty)$, with $n \in \mathbb{N}$. In the sequel, if $\gamma_i = \tau$ for some $i \in \{1,\ldots,n\}$, then the symbol $\mathfrak{C}(\gamma_1,\ldots,\gamma_n)$ stands for constants that are additionally supposed to be increasing functions of τ.

We want to prove (1.8) and (1.11) on the one hand, and (1.9) and (1.12) on the other, by a single argument. To this end, we fix $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leqslant 1$. In the case $\alpha = 0$, we suppose assumption (1.7) to hold, and if $|\alpha| = 1$, we require (1.10) to be valid. Thus $a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ in both cases, $a|\overline{B_{S_0}}^c \in W^{1,1}_{\mathrm{loc}}(\overline{B_{S_0}}^c)^3$ if $|\alpha| = 1$, and

$$|\partial^\beta a(y)| \leqslant \delta_{|\alpha|}[|y|\nu_\tau(y)]^{-1-|\beta|-\kappa_0} \quad \text{for } y \in \overline{B_{S_0}}^c,\ \beta \in \{0, \alpha\} \tag{3.1}$$

in both cases. Note that (3.1) corresponds to two inequalities if $|\alpha| = 1$, and to a single one if $\alpha = 0$. We choose a function $\varphi_S \in C^\infty(\mathbb{R}^3)$ with $\varphi_S|B_{S_0+(S-S_0)/4} = 0$, $\varphi_S|B^c_{(S_0+S)/2} = 1$ and $0 \leqslant \varphi_S \leqslant 1$. This function will be kept fixed throughout. We begin our estimates by considering $\mathfrak{I}^{(\tau)}((1-\varphi_S)a)$.

Lemma 3.1. *Let $x \in B_S^c$, $t \in (0, \infty)$. Then*

$$|\partial_x^\alpha \mathfrak{I}^{(\tau)}((1-\varphi_S)a)(x,t)| \leqslant \mathfrak{C}(S_0, S, \tau)(\|a|B_{S_0}\|_1 + \delta_{|\alpha|})(|x|\nu_\tau(x))^{-1-|\alpha|/2}. \tag{3.2}$$

P r o o f. For $y \in \mathbb{R}^3$ with $(1-\varphi_S)(y) \neq 0$ we have $|y| \leqslant (S+S_0)/2$, hence with the abbreviation $S_1 := (S+S_0)/2$,

$$\begin{aligned} |x-y| &\geqslant |x| - S_1 = |x|(1 - S_1/S) + |x|S_1/S - S_1 \\ &\geqslant |x|(1-S_1/S) \geqslant S - S_1 = (S-S_0)/2, \end{aligned} \tag{3.3}$$

and by Lemma 2.5,

$$\nu_\tau(x-y)^{-1} \leqslant \mathfrak{C}(\tau)(1+|y|)\nu_\tau(x)^{-1} \leqslant \mathfrak{C}(S,\tau)\nu_\tau(x)^{-1}. \tag{3.4}$$

Now we get by Lemma 2.1, Lemma 2.4 with $K = (S-S_0)/2$, (3.3), and (3.4) that for $y \in \mathbb{R}^3$ with $(1-\varphi_S)(y) \neq 0$,

$$\begin{aligned} |\partial_x^\alpha \mathfrak{H}(x - \tau t e_1 - y, t)| &\leqslant \mathfrak{C}(S_0, S, \tau)(|x-y|\nu_\tau(x-y) + t)^{-3/2-|\alpha|/2} \\ &\leqslant \mathfrak{C}(S_0, S, \tau)(|x|\nu_\tau(x))^{-3/2-|\alpha|/2}. \end{aligned}$$

Therefore by (3.1) and (2.3),

$$\begin{aligned} |\partial_x^\alpha \mathfrak{I}^{(\tau)}((1-\varphi_S)a)(x,t)| &\leqslant \mathfrak{C}(S_0,S,\tau)(|x|\nu_\tau(x))^{-3/2-|\alpha|/2} \int_{B_{S_1}} (1-\varphi_S)(y)a(y)\,\mathrm{d}y \\ &\leqslant \mathfrak{C}(S_0,S,\tau)(|x|\nu_\tau(x))^{-3/2-|\alpha|/2}\left(\int_{B_{S_0}} |a(y)|\,\mathrm{d}y + \delta_{|\alpha|}\int_{B_{S_1}\setminus B_{S_0}} |y|^{-1-\kappa_0}\,\mathrm{d}y\right) \\ &\leqslant \mathfrak{C}(S_0,S,\tau)(\|a|B_{S_0}\|_1 + \delta_{|\alpha|})(|x|\nu_\tau(x))^{-3/2-|\alpha|/2}. \end{aligned}$$

□

Now we turn to $\varphi_S a$. Since this function vanishes on $B_{S_0+(S-S_0)/4}$, assumption (3.1) implies

Lemma 3.2. *$\varphi_S a \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$, and $\varphi_S a \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$ if $|\alpha| = 1$. Moreover,*

$$|\partial^\beta(\varphi_S a)(y)| \leqslant \mathfrak{C}(S_0, S)\delta_{|\alpha|}(|y|\nu_\tau(y))^{-1-|\beta|/2-\kappa_0} \tag{3.5}$$

for $y \in \mathbb{R}^3 \setminus \{0\}$, $\beta \in \{0, \alpha\}$.

In the ensuing corollary, we introduce a function $\tilde{a}_S$ which is a scaled version of $\varphi_S a$. This function $\tilde{a}_S$ vanishes in the neighbourhood $B_{S_0/\tau}$ of the origin. However, we do not want to exploit this fact because it would introduce a dependency on S_0/τ in our constants, which then would no longer be increasing functions of τ. So in Corollary 3.1 below, we estimate $\tilde{a}_S$ by an upper bound which is singular at $y = 0$. However, this singularity is weak in $\mathbb{R}^3$ and can be handled without problem. The factor $\tau^{1+|\alpha|/2+\kappa_0}$ also appearing in our bound of $\tilde{a}_S$ will be useful later on when we will return from $\mathfrak{I}^{(1)}(\tilde{a}_S)$ to $\mathfrak{I}^{(\tau)}(\varphi_S a)$ (Section 5). But when estimating $\mathfrak{I}^{(1)}(\tilde{a}_S)$, we will have to start all over again, in the sense that first we will introduce another cut-off function, this time denoted by ψ, which we will require to satisfy the equations $\psi|B_1 = 0$ and $\psi|B_2^c = 1$. Then we will evaluate $\mathfrak{I}^{(1)}((1-\psi)\tilde{a}_S)$ (Lemma 4.1), before turning to the main difficulty of our argument, that is, an estimate of $\mathfrak{I}^{(1)}(\psi\tilde{a}_S)$. We indicate that it is only in the proof of Lemma 4.1, which deals with $\mathfrak{I}^{(1)}((1-\psi)\tilde{a}_S)$, that the singularity of the upper bound of $\tilde{a}_S$ matters.

Corollary 3.1. *Put $\tilde{a}_S(y) := (\varphi_S a)(\tau^{-1}y)$ for $y \in \mathbb{R}^3$. Then $\tilde{a}_S \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$, with $\tilde{a}_S \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$ if $|\alpha| = 1$, and*

$$|\partial^\beta \tilde{a}_S(y)| \leqslant \mathfrak{C}(S_0, S)\delta_{|\alpha|}\tau^{1-|\beta|/2+\kappa_0}(|y|\nu_1(y))^{-1-|\beta|/2-\kappa_0}$$

for $y \in \mathbb{R}^3 \setminus \{0\}$, $\beta \in \{0, \alpha\}$. In particular, for y, β as before,

$$|\partial^\beta \tilde{a}_S(y)| \leqslant \mathfrak{C}(S_0, S)\delta_{|\alpha|}(1 + \tau^{-|\alpha|/2})\tau^{1+\kappa_0}(|y|\nu_1(y))^{-1-|\beta|/2-\kappa_0}. \tag{3.6}$$

Finally, by a change of variables, we scale $\mathfrak{I}^{(\tau)}(\varphi_S a)$:

Lemma 3.3. *$\mathfrak{I}^{(\tau)}(\varphi_S a)(x, t) = \mathfrak{I}^{(1)}(\tilde{a}_S)(\tau x, \tau^2 t)$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$.*

4. Decay estimates of $\mathfrak{I}^{(1)}(\tilde{a}_S)$

In this section, we study the asymptotic behaviour of $\mathfrak{I}^{(1)}(\tilde{a}_S)$. But in order to avoid the cumbersome constant in (3.6), we replace $\tilde{a}_S$ by a slightly more general function $b \in L^1_{\rm loc}(\mathbb{R}^3)^3$ with $b \in W^{1,1}_{\rm loc}(\mathbb{R}^3)^3$ if $|\alpha| = 1$, and

$$(4.1) \qquad |\partial^\beta b(y)| \leqslant \gamma(|y|\nu_1(y))^{-1-|\beta|/2-\kappa_0} \quad (y \in \mathbb{R}^3 \setminus \{0\},\ \beta \in \{0, \alpha\}),$$

for some number $\gamma \in (0, \infty)$. Recall that $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leqslant 1$ was fixed at the beginning of Section 3. We choose a cut-off function $\psi \in C^\infty(\mathbb{R}^3)$ with $\psi|B_1 = 0$, $\psi|B_2^c = 1$, $0 \leqslant \psi \leqslant 1$. For brevity, we put $\tilde{b} := \psi b$. Let $t \in (0, \infty)$ be fixed in this section.

Lemma 4.1. *Let $x \in \mathbb{R}^3 \setminus \{0\}$. Then*

$$(4.2)\ |\partial_x^\alpha \mathfrak{I}^{(1)}((1-\psi)b)(x,t)| \leqslant \mathfrak{C}\gamma(|x|\nu_1(x))^{-1-|\alpha|/2}(\chi_{(4,\infty)}(|x|) + \chi_{(0,4]}(|x|)|x|^{-\kappa_0}).$$

P r o o f. We first suppose that $|x| \geqslant 4$. Since $(1-\psi)b|B_2^c = 0$, we may then proceed similarly as in the proof of Lemma 3.1, with the parameters S_1, S replaced by the numbers 2 and 4, respectively. In particular, we obtain $|\partial^\alpha \mathfrak{H}(x - te_1 - y, t)| \leqslant \mathfrak{C}(|x|\nu_1(x))^{-3/2-|\alpha|/2}$ for $y \in \mathbb{R}^3$ with $(1-\psi)(y) \neq 0$, and thus

$$|\partial_x^\alpha \mathfrak{I}^{(1)}((1-\psi)b)(x,t)| \leqslant \mathfrak{C}(|x|\nu_1(x))^{-3/2-|\alpha|/2} \int_{B_2} |b(y)|\, \mathrm{d}y.$$

But by (4.1), $\int_{B_2} |b(y)|\, \mathrm{d}y \leqslant \gamma \int_{B_S} |y|^{-1-\kappa_0}\, \mathrm{d}y \leqslant \mathfrak{C}\gamma$, so that (4.2) is proved in the case $|x| \geqslant 4$. Now suppose that $|x| \leqslant 4$. We note that

$$(4.3) \qquad |\partial^\alpha((1-\psi)b)(y)| \leqslant \mathfrak{C}\chi_{B_2}(y)\gamma|y|^{-1-|\alpha|/2-\kappa_0}$$

by (4.1). Now we find

$$\begin{aligned}(4.4) \qquad &\int_{\mathbb{R}^3\setminus B_{|x|/2}} \mathfrak{H}(x - te_1 - y, t)|\partial^\alpha((1-\psi)b)(y)|\, \mathrm{d}y \\ &\leqslant \mathfrak{C}\gamma \int_{B_2\setminus B_{|x|/2}} \mathfrak{H}(x - te_1 - y, t)|y|^{-1-|\alpha|/2-\kappa_0}\, \mathrm{d}y \\ &\leqslant \mathfrak{C}\gamma|x|^{-1-|\alpha|/2-\kappa_0} \int_{B_2\setminus B_{|x|/2}} \mathfrak{H}(x - te_1 - y, t)\, \mathrm{d}y \leqslant \mathfrak{C}\gamma|x|^{-1-|\alpha|/2-\kappa_0},\end{aligned}$$

where the last inequality holds by Lemma 2.2. Moreover, by virtue of (4.3), Lemma 2.1 and Lemma 2.4 with $K = 6$, and because $|x-y| \geqslant |x|/2$ for $y \in B_2$,

$$
\begin{aligned}
(4.5) \qquad \int_{B_{|x|/2}} & \mathfrak{H}(x - te_1 - y, t)|\partial^\alpha((1-\psi)b)(y)| \,\mathrm{d}y \\
&\leqslant \mathfrak{C}\gamma \int_{B_{|x|/2} \cap B_2} |x-y|^{-3}|y|^{-1-|\alpha|/2-\kappa_0} \,\mathrm{d}y \\
&\leqslant \mathfrak{C}\gamma|x|^{-3} \int_{B_{|x|/2}} |y|^{-1-|\alpha|/2-\kappa_0} \,\mathrm{d}y \leqslant \mathfrak{C}\gamma|x|^{-1-|\alpha|/2-\kappa_0}.
\end{aligned}
$$

We use (2.3), perform integration by parts, and then apply (4.4) and (4.5) to obtain

$$
\begin{aligned}
|\partial_x^\alpha \mathfrak{I}^{(1)}((1-\psi)b)(x,t)| &= |\mathfrak{I}^{(1)}(\partial^\alpha[(1-\psi)b])(x,t)| \\
&\leqslant \int_{\mathbb{R}^3} \mathfrak{H}(x - te_1 - y, t)|\partial^\alpha((1-\psi)b)(y)| \,\mathrm{d}y \leqslant \mathfrak{C}\gamma|x|^{-1-|\alpha|/2-\kappa_0}.
\end{aligned}
$$

But $1 \leqslant \mathfrak{C}\nu_1(x)^{-1-|\alpha|/2-\kappa_0}$ since $|x| \leqslant 4$, so we have proved (4.2) in the case $|x| \leqslant 4$ as well. $\square$

In view of the choice of b and ψ at the beginning of this section, the ensuing lemma is obvious.

Lemma 4.2. *$\tilde{b} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$, with $\tilde{b} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)^3$ if $|\alpha| = 1$, and*

$$
(4.6) \qquad |\partial^\beta \tilde{b}(y)| \leqslant \mathfrak{C}\gamma((1+|y|)\nu_1(y))^{-1-|\beta|/2-\kappa_0} \quad \text{for } y \in \mathbb{R}^3,\ \beta \in \{0, \alpha\}.
$$

Lemma 4.3. *Let $K \in (0,\infty)$, $z \in B_K$. Then*

$$
\int_{\mathbb{R}^3} \mathfrak{H}(z-y,t)|\partial^\alpha \tilde{b}(y)| \,\mathrm{d}y \leqslant \mathfrak{C}(K)\gamma((1+|z|)\nu_1(z))^{-1-|\alpha|/2}.
$$

P r o o f. By (4.6), we have $|\partial^\alpha \tilde{b}(y)| \leqslant \mathfrak{C}\gamma$ for $y \in \mathbb{R}^3$. Since $|z| \leqslant K$, Lemma 4.3 follows by Lemma 2.2. $\square$

Lemma 4.4. *Let $x \in \mathbb{R}^3$. Then*

$$
\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t) = \int_{\mathbb{R}^3} \mathfrak{H}(y,t)\partial^\alpha \tilde{b}(x - te_1 - y) \,\mathrm{d}y.
$$

P r o o f. By (2.3) and a change of variables, we have

$$\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t) = \int_{\mathbb{R}^3} \partial_y^\alpha \mathfrak{H}(y,t)\tilde{b}(x - te_1 - y)\,\mathrm{d}y. \tag{4.7}$$

But $\mathfrak{H}(y,t)$ and $\partial^\alpha \mathfrak{H}(y,t)$ decay exponentially for $|y| \to \infty$ (Lemma 2.1), and b and $\partial^\alpha b$ are in particular bounded (Lemma 4.2). So we may integrate by parts on the right-hand side of (4.7). Lemma 4.4 then follows. $\square$

Lemma 4.5. *Let* $z \in \mathbb{R}^3 \setminus \{0\}$. *Then*

$$\left|\int_{\mathbb{R}^3} \mathfrak{H}(y,t)(1-\psi(y))\partial^\alpha \tilde{b}(z-y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}\gamma(|z|\nu_1(z))^{-1-|\alpha|/2}. \tag{4.8}$$

P r o o f. For $y \in \mathbb{R}^3$ with $(1-\psi)(y) \neq 0$ we have $|y| \leqslant 2$. Suppose that $|z| \geqslant 4$. Then, for $y \in \mathbb{R}^3$ with $(1-\psi)(y) \neq 0$, we get $|z-y| \geqslant |z|/2$ and $\nu_1(z-y)^{-1} \leqslant \mathfrak{C}\nu_1(z)^{-1}$, where we used Lemma 2.5 in the second estimate. In view of (4.6), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \mathfrak{H}(y,t)(1-\psi(y))\partial^\alpha \tilde{b}(z-y)\,\mathrm{d}y\right| &\leqslant \mathfrak{C}\gamma((1+|z|)\nu_1(z))^{-1-|\alpha|/2}\int_{B_2} \mathfrak{H}(y,t)\,\mathrm{d}y \\ &\leqslant \mathfrak{C}\gamma((1+|z|)\nu_1(z))^{-1-|\alpha|/2},\end{aligned}$$

where the last inequality is a consequence of Lemma 2.2. If $|z| \leqslant 4$, inequality (4.8) follows from Lemma 4.3 with $K = 4$. $\square$

Lemma 4.6. *Let* $z \in \mathbb{R}^3 \setminus \{0\}$ *with* $|z| - z_1 \leqslant 1$. *Then*

$$\left|\int_{\mathbb{R}^3} \mathfrak{H}(y,t)\partial^\alpha \tilde{b}(z-y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}\gamma(|z|\nu_1(z))^{-1-|\alpha|/2}. \tag{4.9}$$

P r o o f. By (4.6) we have

$$\left|\int_{\mathbb{R}^3} \mathfrak{H}(y,t)\partial^\alpha \tilde{b}(z-y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}\gamma\sum_{i=1}^{3}\mathfrak{R}_i \tag{4.10}$$

with

$$\mathfrak{R}_i := \int_{A_i} \mathfrak{H}(y,t)(1+|z-y|)^{-1-|\alpha|/2}\,\mathrm{d}y \quad \text{for } i \in \{1,2,3\},$$

with $A_1 := B_{|z|/2}$, $A_2 := B_{2|z|} \setminus B_{|z|/2}$, $A_3 := B^c_{2|z|}$. For $y \in A_1$ the relation $|z-y| \geqslant |z|/2$ holds, hence

$$|\mathfrak{R}_1| \leqslant \mathfrak{C}(1+|z|)^{-1-|\alpha|/2}\int_{B_{|z|/2}} \mathfrak{H}(y,t)\,\mathrm{d}y \leqslant \mathfrak{C}(1+|z|)^{-1-|\alpha|/2}, \tag{4.11}$$

where we used Lemma 2.2. For $y \in A_2$ we have $|y| \geqslant |z|/2$ and $|z-y| \leqslant 3|z|$, in particular $A_2 \subset \{y \in \mathbb{R}^3 \colon |z-y| \leqslant 3|z|\}$. It follows from Lemma 2.1 that

$$
\begin{aligned}
(4.12) \qquad |\mathfrak{R}_2| &\leqslant \mathfrak{C} \int_{A_2} (|y|^2+t)^{-3/2}(1+|z-y|)^{-1-|\alpha|/2}\,\mathrm{d}y \\
&\leqslant \mathfrak{C}|z|^{-3} \int_{|z-y|\leqslant 3|z|} (1+|z-y|)^{-1-|\alpha|/2}\,\mathrm{d}y \leqslant \mathfrak{C}|z|^{-1-|\alpha|/2}.
\end{aligned}
$$

Finally, for $y \in A_3$ we have $|z-y| \geqslant |y|-|z| \geqslant |y|/2$, hence Lemma 2.1 yields

$$
|\mathfrak{R}_3| \leqslant \mathfrak{C} \int_{B^c_{2|z|}} (|y|^2+t)^{-3/2}(1+|y|)^{-1-|\alpha|/2}\,\mathrm{d}y \leqslant \mathfrak{C} \int_{B^c_{2|z|}} |y|^{-4-|\alpha|/2}\,\mathrm{d}y,
$$

so that $|\mathfrak{R}_3| \leqslant \mathfrak{C}|z|^{-1-|\alpha|/2}$. Therefore, by virtue of (4.10)–(4.12),

$$
(4.13) \qquad \left| \int_{\mathbb{R}^3} \mathfrak{H}(y,t)\partial^\alpha \tilde{b}(z-y)\,\mathrm{d}y \right| \leqslant \mathfrak{C}\gamma|z|^{-1-|\alpha|/2}.
$$

Since $|z| - z_1 \leqslant 1$, we have $1 \geqslant \nu_1(z)/2$. Thus (4.9) follows from (4.13). □

Lemma 4.7. *Let $z \in \mathbb{R}^3$ with $z_1 < 0$. Suppose that $\kappa_0 > 0$. Then*

$$
\left| \int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y \right| \leqslant \mathfrak{C}(\kappa_0)\gamma|z_1|^{-2-|\alpha|}.
$$

P r o o f. For $y \in \mathbb{R}^3$ with $y_1 > 0$ we have $|z_1 - y_1| = |z_1| + y_1$. Using this equation, Lemma 2.1, (4.6) and (2.7), we get

$$
\begin{aligned}
(4.14) \qquad \left| \int_{y_1>0} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y \right| &\leqslant \mathfrak{C}\gamma \int_{y_1>0} |z-y|^{-3-|\alpha|}(1+|y'|)^{-2-2\kappa_0}\,\mathrm{d}y \\
&\leqslant \mathfrak{C}\gamma \int_{y_1>0} (|z_1|+y_1+|z'-y'|)^{-3-|\alpha|}(1+|y'|)^{-2-2\kappa_0}\,\mathrm{d}y \\
&= \mathfrak{C}\gamma \int_{\mathbb{R}^2}\int_0^\infty (|z_1|+r+|z'-\eta|)^{-3-|\alpha|}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}r\,\mathrm{d}\eta \\
&= \mathfrak{C}\gamma \int_{\mathbb{R}^2} (|z_1|+|z'-\eta|)^{-2-|\alpha|}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta \\
&\leqslant \mathfrak{C}\gamma|z_1|^{-2-|\alpha|} \int_{\mathbb{R}^2} (1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta \leqslant \mathfrak{C}(\kappa_0)\gamma|z_1|^{-2-|\alpha|}.
\end{aligned}
$$

Abbreviating $H := \{y \in \mathbb{R}^3 \colon y_1 \leqslant 0\}$, we get by integration by parts that

$$
(4.15) \qquad \int_H \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y = \mathfrak{A}_1 + \mathfrak{A}_2,
$$

with

$$\mathfrak{A}_1 := \int_H \mathfrak{H}(z-y,t)\partial^\alpha \tilde{b}(y)\,\mathrm{d}y, \quad \mathfrak{A}_2 := -\int_{\partial H} \mathfrak{H}(z-y,t)\tilde{b}(y)(e_1\cdot\alpha)\,\mathrm{d}o_y.$$

This integration by parts is possible because $\tilde{b}$ and $\partial^\alpha \tilde{b}$ are bounded according to (4.6), and $\mathfrak{H}(z-y,t)$ and $\partial^\alpha\mathfrak{H}(z-y,t)$ decay exponentially for $|y|\to\infty$. Observe that for $y\in H$ we have $\nu_1(y)\geqslant 1+|y|$. Therefore, using (4.6) and Lemma 2.1, we get in the case $|z|\geqslant 1$ that

$$\begin{aligned}|\mathfrak{A}_1| &\leqslant \int_H \mathfrak{H}(z-y,t)(1+|y|)^{-2-|\alpha|-2\kappa_0}\,\mathrm{d}y \\ &\leqslant \mathfrak{C}\gamma\Big(\int_{B_{|z|/2}} |z-y|^{-3}(1+|y|)^{-2-|\alpha|-2\kappa_0}\,\mathrm{d}y \\ &\qquad + \int_{B^c_{|z|/2}} \mathfrak{H}(z-y,t)(1+|y|)^{-2-|\alpha|}\,\mathrm{d}y\Big) \\ &\leqslant \mathfrak{C}\gamma\Big(|z|^{-3}\int_{B_{|z|/2}} (1+|y|)^{-2-|\alpha|-2\kappa_0}\,\mathrm{d}y \\ &\qquad + |z|^{-2-|\alpha|}\int_{B^c_{|z|/2}} \mathfrak{H}(z-y,t)\,\mathrm{d}y\Big).\end{aligned} \tag{4.16}$$

In the case $\alpha=0$, we observe that

$$\int_{B_{|z|/2}} (1+|y|)^{-2-|\alpha|-2\kappa_0}\,\mathrm{d}y \leqslant \int_{B_{|z|/2}} (1+|y|)^{-2}\,\mathrm{d}y \leqslant \mathfrak{C}|z|.$$

If $|\alpha|=1$, we use the estimate

$$\int_{B_{|z|/2}} (1+|y|)^{-2-|\alpha|-2\kappa_0}\,\mathrm{d}y \leqslant \int_{\mathbb{R}^3} (1+|y|)^{-3-2\kappa_0}\,\mathrm{d}y \leqslant \mathfrak{C}(\kappa_0).$$

Recalling Lemma 2.2, we thus deduce from (4.16) that

$$|\mathfrak{A}_1| \leqslant \mathfrak{C}(\kappa_0)\gamma|z|^{-2-|\alpha|}. \tag{4.17}$$

Noting that $\nu_1(y)=1+|y|$ for $y\in\partial H$, we further find by Lemma 2.1 and (4.6) that

$$\begin{aligned}|\mathfrak{A}_2| &\leqslant |\alpha|\mathfrak{C}\gamma\int_{\partial H} |z-y|^{-3}(1+|y|)^{-2-2\kappa_0}\,\mathrm{d}o_y \\ &\leqslant |\alpha|\mathfrak{C}\gamma\int_{\mathbb{R}^2} (|z_1|+|z'-\eta|)^{-3}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta \\ &\leqslant |\alpha|\mathfrak{C}\gamma|z_1|^{-3}\int_{\mathbb{R}^2} (1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta \leqslant |\alpha|\mathfrak{C}(\kappa_0)\gamma|z_1|^{-3} \\ &\leqslant \mathfrak{C}(\kappa_0)\gamma|z_1|^{-2-|\alpha|}.\end{aligned} \tag{4.18}$$

Combining (4.14), (4.15), (4.17) and (4.18), we obtain the lemma. □

Lemma 4.8. *Let $z \in \mathbb{R}^3$ with $z_1 < 0$. Then*

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|z_1|^{-2-|\alpha|}.$$

Note that contrary to Lemma 4.7, we do not suppose in Lemma 4.8 that $\kappa_0 > 0$.

P r o o f of Lemma 4.8. Proceed as in the proof of Lemma 4.7, but use Lemma 2.6 with $\varepsilon = 2\sigma_0$ instead of Lemma 2.1. Thus, when revisiting (4.14) and (4.18), we may exploit the fact that

$$\int_{\mathbb{R}^2} (1+|z'-\eta|)^{-2\sigma_0}(1+|\eta|)^{-2}\,\mathrm{d}\eta \leqslant \mathfrak{C}(\sigma_0),$$

as follows by Hölder's inequality. Concerning (4.16), we observe that by Lemma 2.6,

$$\begin{aligned}
&\int_{B_{|z|/2}} \mathfrak{H}(z-y,t)(1+|y|)^{-2-|\alpha|}\,\mathrm{d}y \\
&\qquad \leqslant \mathfrak{C}(\sigma_0)(1+t)^{\sigma_0}\int_{B_{|z|/2}} |z-y|^{-3}(1+|z-y|)^{-2\sigma_0}(1+|y|)^{-2-|\alpha|}\,\mathrm{d}y \\
&\qquad \leqslant \mathfrak{C}(\sigma_0)(1+t)^{\sigma_0}|z|^{-3-2\sigma_0}\int_{B_{|z|/2}} (1+|y|)^{-2-|\alpha|}\,\mathrm{d}y.
\end{aligned}$$

In view of this inequality, the estimate in (4.16) may be modified in an obvious way. $\square$

Lemma 4.9. *Let $z \in \mathbb{R}^3 \setminus \{0\}$ with $|z'| \geqslant |z|/2$. Suppose that $\kappa_0 > 0$. Then*

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}(\kappa_0)\gamma|z|^{-2-|\alpha|}.$$

P r o o f. Put $H := \{y \in \mathbb{R}^3\colon\ |y'| \leqslant |z|/4\}$. Then, by integration by parts,

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \sum_{i=1}^{3}\mathfrak{B}_i, \tag{4.19}$$

with

$$\mathfrak{B}_1 := \left|\int_H \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right|, \quad \mathfrak{B}_2 := \left|\int_{\mathbb{R}^3\setminus H} \mathfrak{H}(z-y,t)\partial^\alpha\tilde{b}(y)\,\mathrm{d}y\right|,$$
$$\mathfrak{B}_3 := \left|\int_{\partial H} \mathfrak{H}(z-y,t)\tilde{b}(y)((0,y')\cdot\alpha)/|y'|\,\mathrm{d}o_y\right|.$$

Due to the decay properties of $\mathfrak{H}(\cdot,t)$ and $\tilde{b}$, this integration by parts is possible although H is unbounded. Now (4.6), Lemma 2.1 and (2.7) yield

$$
\begin{aligned}
(4.20)\qquad |\mathfrak{B}_1| &\leqslant \mathfrak{C}\gamma \int_H |z-y|^{-3-|\alpha|}((1+|y|)\nu_1(y))^{-1-\kappa_0}\,\mathrm{d}y \\
&\leqslant \mathfrak{C}\gamma \int_H (|z_1-y_1|+|z'-y'|)^{-3-|\alpha|}(1+|y'|)^{-2-2\kappa_0}\,\mathrm{d}y \\
&= \mathfrak{C}\gamma \int_{\{\eta\in\mathbb{R}^2:\ |\eta|\leqslant|z|/4\}} \int_0^\infty (r+|z'-\eta|)^{-3-|\alpha|}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}r\,\mathrm{d}\eta \\
&\leqslant \mathfrak{C}\gamma \int_{\{\eta\in\mathbb{R}^2:\ |\eta|\leqslant|z|/4\}} |z'-\eta|^{-2-|\alpha|}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta.
\end{aligned}
$$

But for $\eta\in\mathbb{R}^2$ with $|\eta|\leqslant|z|/4$, we find by our assumption on z that $|z'-\eta|\geqslant |z'|-|\eta|\geqslant|z|/2-|\eta|\geqslant|z|/4$. Therefore we may conclude

$$
(4.21)\qquad |\mathfrak{B}_1| \leqslant \mathfrak{C}\gamma|z|^{-2-|\alpha|}\int_{\mathbb{R}^2}(1+|\eta|)^{-2-2\kappa_0}\,\mathrm{d}\eta \leqslant \mathfrak{C}(\kappa_0)\gamma|z|^{-2-|\alpha|}.
$$

For $y\in\mathbb{R}^3\setminus H$ we have $|y'|\geqslant|z|/4$. Therefore, by (4.6) and (2.7),

$$
\begin{aligned}
(4.22)\qquad |\mathfrak{B}_2| &\leqslant \gamma\int_{\mathbb{R}^3\setminus H}\mathfrak{H}(z-y,t)(1+|y'|)^{-2-|\alpha|}\,\mathrm{d}y \\
&\leqslant \mathfrak{C}\gamma|z|^{-2-|\alpha|}\int_{\mathbb{R}^3\setminus H}\mathfrak{H}(z-y,t)\,\mathrm{d}y \leqslant \mathfrak{C}\gamma|z|^{-2-|\alpha|},
\end{aligned}
$$

where the last inequality follows from Lemma 2.2. For $y\in\partial H$, the equation $|y'|=|z|/4$ holds. Recalling (4.6) and (2.7), we thus get

$$
\begin{aligned}
|\mathfrak{B}_3| &\leqslant |\alpha|\mathfrak{C}\gamma\int_{\partial H}\mathfrak{H}(z-y,t)(1+|y'|)^{-2}\,\mathrm{d}o_y \\
&\leqslant |\alpha|\mathfrak{C}\gamma(1+|z|)^{-2}\int_{\partial H}\mathfrak{H}(z-y,t)\,\mathrm{d}o_y.
\end{aligned}
$$

Next we use Lemma 2.1 to conclude that

$$
\begin{aligned}
|\mathfrak{B}_3| &\leqslant |\alpha|\mathfrak{C}\gamma|z|^{-2}\int_{\partial H}|z-y|^{-3}\,\mathrm{d}o_y \\
&\leqslant |\alpha|\mathfrak{C}\gamma|z|^{-2}\int_{\partial H}(|z_1-y_1|+|z'-y'|)^{-3}\,\mathrm{d}o_y \\
&= |\alpha|\mathfrak{C}\gamma|z|^{-1}\int_{\{\eta\in\mathbb{R}^2:\ |\eta|=1\}}\int_0^\infty (r+|z'-(|z|/4)\eta|)^{-3}\,\mathrm{d}r\,\mathrm{d}o_\eta \\
&\leqslant |\alpha|\mathfrak{C}\gamma|z|^{-1}\int_{\{\eta\in\mathbb{R}^2:\ |\eta|=1\}}|z'-(|z|/4)\eta|^{-2}\,\mathrm{d}o_\eta.
\end{aligned}
$$

But for $\eta \in \mathbb{R}^2$ with $|\eta| = 1$, the estimate $|z' - (|z|/4)\eta| \geqslant |z'| - |z|/4 \geqslant |z|/4$ holds. (Recall we have assumed $|z'| \geqslant |z|/2$.) Thus we arrive at the inequality

$$|\mathfrak{B}_3| \leqslant |\alpha|\mathfrak{C}\gamma|z|^{-3} \leqslant \mathfrak{C}\gamma|z|^{-2-|\alpha|}. \tag{4.23}$$

The lemma follows from (4.19) and (4.21)–(4.23). $\square$

Lemma 4.10. *Let $z \in \mathbb{R}^3 \setminus \{0\}$ with $|z'| \geqslant |z|/2$. Then*

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|z|^{-2-|\alpha|}. \tag{4.24}$$

P r o o f. The proof is almost identical to that of Lemma 4.9; only the estimate of the term $\mathfrak{B}_1$ must be modified. In fact, by (4.6), (2.7), and using Lemma 2.6 with $\varepsilon = 2\sigma_0$ instead of Lemma 2.1, we get

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}\int_H |z-y|^{-3-|\alpha|}(1+|z-y|)^{-2\sigma_0}(1+|y'|)^{-2}\,\mathrm{d}y, \tag{4.25}$$

with H defined as in the proof of Lemma 4.9. By starting with (4.25), and then proceeding as in (4.20) and (4.21), we find

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|z|^{-2-|\alpha|}\int_{\mathbb{R}^2}(1+|z'-\eta|)^{-2\sigma_0}(1+|\eta|)^{-2}\,\mathrm{d}\eta.$$

But the last integral is bounded by a constant only depending on σ_0, as follows from Hölder's inequality. Combining this estimate of $\mathfrak{B}_1$ with (4.19), (4.22) and (4.23), we arrive at (4.24). $\square$

Corollary 4.1. *Let $z \in \mathbb{R}^3$ with $z_1 < 0$. If $\kappa_0 > 0$, then*

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}(\kappa_0)\gamma|z|^{-2-|\alpha|}. \tag{4.26}$$

In any case, we have

$$\left|\int_{\mathbb{R}^3} \partial_z^\alpha \mathfrak{H}(z-y,t)\tilde{b}(y)\,\mathrm{d}y\right| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|z|^{-2-|\alpha|}. \tag{4.27}$$

P r o o f. For any $z \in \mathbb{R}^3 \setminus \{0\}$, at least one of the two relations $|z_1| \geqslant |z|/2$ and $|z'| \geqslant |z|/2$ is valid. Therefore inequality (4.26) follows from Lemma 4.7 (if $|z_1| \geqslant |z|/2$) or from Lemma 4.9 (if $|z'| \geqslant |z|/2$), whereas (4.27) is a consequence of Lemma 4.8 and 4.10. $\square$

Now we are in a position to prove a decay estimate of $|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)|$.

Theorem 4.1. *Let $x \in \mathbb{R}^3$. If $\kappa_0 > 0$, we have*

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\kappa_0)\gamma(|x|\nu_1(x))^{-1-|\alpha|/2}. \tag{4.28}$$

Otherwise,

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}(|x|\nu_1(x))^{-1-|\alpha|/2}. \tag{4.29}$$

Proof. Suppose that $\kappa_0 > 0$. Abbreviate $x_t := x - te_1$. We first determine a bound of $\mathfrak{I}^{(1)}(\tilde{b})(x,t)$ in terms of $|x_t|\nu_1(x_t)$. Then we use this bound in order to establish (4.28). We distinguish several cases. First assume that $x_{t1} \geqslant 0$ and $|x_t| - x_{t1} \geqslant 1$. In this case, we start from Lemma 4.4, observing that

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant |\mathfrak{M}_1| + |\mathfrak{M}_2|, \tag{4.30}$$

with

$$\begin{aligned} \mathfrak{M}_1 &:= \int_{\mathbb{R}^3} \mathfrak{H}(y,t)\psi(y)\partial^\alpha \tilde{b}(x_t - y)\,\mathrm{d}y, \\ \mathfrak{M}_2 &:= \int_{\mathbb{R}^3} \mathfrak{H}(y,t)(1-\psi(y))\partial^\alpha \tilde{b}(x_t - y)\,\mathrm{d}y. \end{aligned} \tag{4.31}$$

We observe that the conditions $x_{t1} \geqslant 0$, $|x_t| - x_{t1} \geqslant 1$ together with (2.7) imply $|x_t'|^2 = (|x_t| + x_{t1})(|x_t| - x_{t1}) \geqslant |x_t|$, so that $|x_t'| \geqslant |x_t|^{1/2}$. On the other hand, it is obvious that $|x_t'| \leqslant |x_t|$. Thus there is a $\sigma \in [1/2, 1]$ such that $|x_t'| = |x_t|^\sigma$. It follows by (2.7) and the assumption $x_{t1} \geqslant 0$ that

$$|x_t|^{2\sigma} = |x_t'|^2 \geqslant |x_t|(|x_t| - x_{t1}) \geqslant |x_t|\nu_1(x_t)/2, \tag{4.32}$$

where the last inequality holds because of the assumption $|x_t| - x_{t1} \geqslant 1$.

Now put $G := \{y \in \mathbb{R}^3\colon |y'| > |x_t|^\sigma/2\}$. Using (4.32), we find for $y \in \mathbb{R}^3 \setminus G$ (hence $|y'| \leqslant |x_t|^\sigma/2$) that

$$|x_t' - y'| \geqslant |x_t'| - |y'| = |x_t|^\sigma - |y'| \geqslant |x_t|^\sigma/2 \geqslant \mathfrak{C}(|x_t|\nu_1(x_t))^{1/2},$$

so that by virtue of (2.7),

$$(1 + |x_t - y|)\nu_1(x_t - y) \geqslant \mathfrak{C}|x_t' - y'|^2 \geqslant \mathfrak{C}|x_t|^{2\sigma} \geqslant \mathfrak{C}|x_t|\nu_1(x_t). \tag{4.33}$$

Now we may conclude using (4.6) that

$$\left|\int_{\mathbb{R}^3\setminus G}\mathfrak{H}(y,t)\psi(y)\partial^\alpha\tilde{b}(x_t-y)\,\mathrm{d}y\right| \tag{4.34}$$
$$\leqslant \mathfrak{C}\gamma\int_{\mathbb{R}^3\setminus G}\mathfrak{H}(y,t)((1+|x_t-y|)\nu_1(x_t-y))^{-1-|\alpha|/2}\,\mathrm{d}y$$
$$\leqslant \mathfrak{C}\gamma(|x_t|\nu_1(x_t))^{-1-|\alpha|/2}\int_{\mathbb{R}^3\setminus G}\mathfrak{H}(y,t)\,\mathrm{d}y \leqslant \mathfrak{C}\gamma(|x_t|\nu_1(x_t))^{-1-|\alpha|/2},$$

where the last inequality follows from Lemma 2.2. We further obtain by integration by parts that

$$\int_G \mathfrak{H}(y,t)\psi(y)\partial^\alpha\tilde{b}(x_t-y)\,\mathrm{d}y = \sum_{i=1}^{3}\mathfrak{B}_i, \tag{4.35}$$

with

$$\mathfrak{B}_1 := \int_G \partial_y^\alpha\mathfrak{H}(y,t)\psi(y)\tilde{b}(x_t-y)\,\mathrm{d}y,$$
$$\mathfrak{B}_2 := |\alpha|\int_G \mathfrak{H}(y,t)\partial^\alpha\psi(y)\tilde{b}(x_t-y)\,\mathrm{d}y,$$
$$\mathfrak{B}_3 := \int_{\partial G}\mathfrak{H}(y,t)\psi(y)\tilde{b}(x_t-y)((0,y')\cdot\alpha)2|x_t|^{-\sigma}\,\mathrm{d}y.$$

For $y\in\mathbb{R}^3$ with $\psi(y)\neq 0$ we have $|y|\geqslant 1$, so that $|y|\geqslant\mathfrak{C}(1+|y|)$. As a consequence, we may deduce from Lemma 2.1 and (4.6) that

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}\gamma\int_G (1+|y|)^{-3-|\alpha|}((1+|x_t-y|)\nu_1(x_t-y))^{-1-\kappa_0}\,\mathrm{d}y. \tag{4.36}$$

But for $y\in G$, the inequality $1+|y|\geqslant|y|\geqslant|y'|\geqslant|x_t|^\sigma/2$ holds. Hence

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}\gamma|x_t|^{-\sigma(2+|\alpha|)}\int_G (1+|y|)^{-1}((1+|x_t-y|)\nu_1(x_t-y))^{-1-\kappa_0}\,\mathrm{d}y.$$

Therefore, (2.8) and (4.32), imply

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}(\kappa_0)\gamma|x_t|^{-\sigma(2+|\alpha|)} \leqslant \mathfrak{C}(\kappa_0)\gamma(|x_t|\nu_1(x_t))^{-1-|\alpha|/2}. \tag{4.37}$$

Since $|\tilde{b}(x_t-y)|\leqslant\mathfrak{C}\gamma$ for $y\in\mathbb{R}^3$ by (4.6), and because $|y|\geqslant|y'|\geqslant|x_t|^\sigma/2$ for $y\in G$, hence $\mathfrak{H}(y,t)\leqslant\mathfrak{C}|x_t|^{-3\sigma}$ for such y (Lemma 2.1), we see that

$$|\mathfrak{B}_2| \leqslant |\alpha|\mathfrak{C}\gamma|x_t|^{-3\sigma}\int_G|\partial^\alpha\psi(y)|\,\mathrm{d}y.$$

But if $|\alpha| = 1$, we have $\partial^\alpha \psi(y) = 0$ for $y \in B_2^c$, so that

$$
\begin{aligned}
(4.38) \qquad |\mathfrak{B}_2| &\leqslant |\alpha| \mathfrak{C} \gamma |x_t|^{-3\sigma} \int_{B_2} |\partial^\alpha \psi(y)| \, \mathrm{d}y \leqslant |\alpha| \mathfrak{C} \gamma |x_t|^{-3\sigma} \\
&\leqslant |\alpha| \mathfrak{C} \gamma (|x_t| \nu_1(x_t))^{-3/2} \leqslant \mathfrak{C} \gamma (|x_t| \nu_1(x_t))^{-1-|\alpha|/2},
\end{aligned}
$$

where we have again used (4.32). Once more referring to (4.6) and Lemma 2.1, we get

$$
|\mathfrak{B}_3| \leqslant |\alpha| \mathfrak{C} \gamma \int_{\partial G} |y|^{-3} ((1 + |x_t - y|) \nu_1(x_t - y))^{-1} \, \mathrm{d}o_y.
$$

But for $y \in \partial G$, the relation $|y'| = |x_t|^\sigma / 2$ holds, so that

$$
|y| \geqslant \mathfrak{C}(|y_1| + |y'|) \geqslant \mathfrak{C}(|y_1| + |x_t|^\sigma).
$$

Moreover, inequality (4.33) holds for $y \in \partial G$. Therefore we may conclude that

$$
(4.39) \qquad |\mathfrak{B}_3| \leqslant |\alpha| \mathfrak{C} \gamma |x_t|^{-2\sigma} \int_{\partial G} (|y_1| + |x_t|^\sigma)^{-3} \, \mathrm{d}o_y.
$$

On the other hand,

$$
\begin{aligned}
\int_{\partial G} (|y_1| + |x_t|^\sigma)^{-3} \, \mathrm{d}o_y &= \int_0^\infty \int_{\{\eta \in \mathbb{R}^2 : \, |\eta| = 1\}} (|x_t|^\sigma / 2)(r + |x_t|^\sigma)^{-3} \, \mathrm{d}o_\eta \, \mathrm{d}r \\
&\leqslant \mathfrak{C} |x_t|^\sigma \int_0^\infty (r + |x_t|^\sigma)^{-3} \, \mathrm{d}r = \mathfrak{C} |x_t|^{-\sigma},
\end{aligned}
$$

so that from (4.39) and (4.32),

$$
\begin{aligned}
(4.40) \qquad |\mathfrak{B}_3| &\leqslant |\alpha| \mathfrak{C} \gamma |x_t|^{-3\sigma} \leqslant |\alpha| \mathfrak{C} \gamma (|x_t| \nu_1(x_t))^{-3/2} \\
&\leqslant \mathfrak{C} \gamma (|x_t| \nu_1(x_t))^{-1-|\alpha|/2}.
\end{aligned}
$$

By Lemma 4.5 we have $|\mathfrak{M}_2| \leqslant \mathfrak{C} \gamma (|x_t| \nu_1(x_t))^{-1-|\alpha|/2}$, so we may conclude from (4.30), (4.34), (4.35), (4.37), (4.38) and (4.40) that

$$
(4.41) \qquad |\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\kappa_0) \gamma (|x_t| \nu_1(x_t))^{-1-|\alpha|/2}.
$$

This estimate was shown under the assumptions that $x_{t1} \geqslant 0$ and $|x_t| - x_{t1} \geqslant 1$. If these assumptions do not hold, we may use the previous lemmas. In fact, if $|x_t| - x_{t1} \leqslant 1$, Lemma 4.4 and 4.6 yield inequality (4.41) even with a constant that does not depend on κ_0. In the case $x_{t1} < 0$ and $|x_t| \geqslant 1$, estimate (4.41) follows from (2.3), (4.26) and the inequality

$$
|x_t| \geqslant \tfrac{1}{2}(1 + |x_t|) \geqslant \tfrac{1}{2}(1 + \tfrac{1}{2}(|x_t| - x_{t1})) \geqslant \tfrac{1}{4} \nu_1(x_t).
$$

Finally, if $|x_t| \leqslant 1$, we may use Lemma 4.4 and Lemma 4.3 with $K = 1$, $z = x_t$ in order to obtain (4.41), again with a constant independent of κ_0. Thus we have shown (4.41) for all cases.

Now we turn to estimates in terms of $|x|\nu_1(x)$. First suppose that $|x| - x_1 \geqslant 1$, $x_1 \geqslant 0$. Then (2.7), implies

$$\begin{aligned}|x_t|\nu_1(x_t) &\geqslant \tfrac{1}{2}(|x_t| + x_{t1})(|x_t| - x_{t1}) = \tfrac{1}{2}|x_t'|^2 = \tfrac{1}{2}|x'|^2 \\ &= \tfrac{1}{2}(|x| + x_1)(|x| - x_1) \geqslant \tfrac{1}{2}|x|(|x| - x_1) \geqslant \mathfrak{C}|x|\nu_1(x);\end{aligned}$$

note that $|x| - x_1 \geqslant \frac{1}{2}\nu_1(x)$ because $|x| - x_1 \geqslant 1$. It thus follows from (4.41) that (4.28) holds. Next suppose that $x_1 < 0$. Then $x_{t1} = x_1 - t < 0$, so that by (2.3) and (4.26),

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\kappa_0)\gamma|x_t|^{-2-|\alpha|}. \tag{4.42}$$

Moreover, we have $x_{1t} = x_1 - t < x_1 < 0$, so that $|x_t| \geqslant |x|$. In the case $|x| \geqslant 1$, we additionally observe that $|x| \geqslant \frac{1}{2}(1 + |x|) \geqslant \frac{1}{4}\nu_1(x)$. Thus, under our assumption $x_1 < 0$, and if $|x| \geqslant 1$, we see that inequality (4.28) follows from (4.42). In the case $x_1 < 0$, $|x| \leqslant 1$, we still have $|x_t| \geqslant |x|$, and inequality (4.42) continues to hold. If $|x_t| \geqslant 1$, we may conclude from that latter estimate and the relation $|x_t| \geqslant |x|$ that $|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\kappa_0)\gamma|x|^{-1-|\alpha|/2}$. The same inequality follows from Lemma 4.4 and Lemma 4.3 with $K = 1$, $z = x_t$ if $|x_t| \leqslant 1$. On the other hand, the relation $|x| \leqslant 1$ implies that $1 \geqslant \frac{1}{3}\nu_1(x)$. Thus we see that inequality (4.28) holds also in the case $x_1 < 0$, $|x| \leqslant 1$. Suppose that $|x| - x_1 \leqslant 1$ and $|x_t| \geqslant \frac{1}{2}|x|$. Then

$$|x_t|\nu_1(x_t) \geqslant |x_t| \geqslant \tfrac{1}{2}|x| \geqslant \tfrac{1}{4}|x|\nu_1(x).$$

Hence inequality (4.28) follows from (4.41). Consider the case $|x| - x_1 \leqslant 1$, $|x_t| \leqslant \frac{1}{2}|x|$, $|x| \leqslant 1$. The first of these three relations yields $1 \geqslant \nu_1(x)/2$. The second and the third imply $|x_t| \leqslant \frac{1}{2}$. Thus, by Lemma 4.4 and Lemma 4.3 with $K = \frac{1}{2}$, $z = x_t$, we have

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}\gamma \leqslant \mathfrak{C}\gamma(|x|\nu_1(x))^{-1-|\alpha|/2},$$

so that (4.28) is valid once more. This leaves us to consider the situation that the conditions $|x| - x_1 \leqslant 1$, $|x_t| \leqslant |x|/2$ and $|x| \geqslant 1$ hold. The inequalities $|x_t| \leqslant \frac{1}{2}|x|$ and $|x| \geqslant 1$ yield

$$t = |x - x_t| \geqslant |x| - |x_t| \geqslant \tfrac{1}{2}|x| \geqslant \tfrac{1}{2}. \tag{4.43}$$

Thus, by (2.3), (4.6) and Lemma 2.1,

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}\gamma \int_{\mathbb{R}^3} (|x_t - y|^2 + t)^{-3/2-|\alpha|/2}((1+|y|)\nu_1(y))^{-1-\kappa_0}\,\mathrm{d}y \tag{4.44}$$
$$\leqslant \mathfrak{C}\gamma t^{-1-|\alpha|/2} \int_{\mathbb{R}^3} (|x_t - y| + \sqrt{t})^{-1}((1+|y|)\nu_1(y))^{-1-\kappa_0}\,\mathrm{d}y$$
$$\leqslant \mathfrak{C}\gamma |x|^{-1-|\alpha|/2} \int_{\mathbb{R}^3} (|x_t - y| + 1)^{-1}((1+|y|)\nu_1(y))^{-1-\kappa_0}\,\mathrm{d}y,$$

where we have used (4.43) in the last inequality. Now (2.8) implies the estimate $|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\kappa_0)\gamma|x|^{-1-|\alpha|/2}$. Since $1 \geqslant \nu_1(x)/2$ because of the condition $|x| - x_1 \leqslant 1$, we see that (4.28) is valid in the present situation as well. This completes the proof of (4.28). Turning to the proof of (4.29), we suppose that $x_{t1} \geqslant 0$, $|x_t| - x_{t1} \geqslant 1$. Then, using Lemma 2.6 instead of Lemma 2.1, we may replace (4.36) by

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0} \int_G (1+|y|)^{-3-|\alpha|-2\sigma_0}((1+|x_t - y|)\nu_1(x_t - y))^{-1}\,\mathrm{d}y.$$

Starting from this estimate, we continue as in the passage following (4.36), but refer to (2.9) instead of (2.8). In this way we obtain

$$|\mathfrak{B}_1| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}(|x_t|\nu_1(x_t))^{-1-|\alpha|/2}. \tag{4.45}$$

In the case $x_{t1} < 0$, $|x_t| \geqslant 1$, we apply (4.27) instead of (4.26) to obtain

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}(|x_t|\nu_1(x_t))^{-1-|\alpha|/2}. \tag{4.46}$$

Similarly, if $x_1 < 0$ (hence $x_{t1} = x_1 - t < 0$), we again refer to (4.27) instead of (4.26) in order to replace (4.42) by

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|x_t|^{-2-|\alpha|}. \tag{4.47}$$

We finally have to modify (4.44). To this end, we suppose as in (4.44) that $|x| - x_1 \leqslant 1$, $|x_t| \leqslant \frac{1}{2}|x|$ and $|x| \geqslant 1$, and then use Lemma 2.6 instead of Lemma 2.1, to obtain by (4.43) that

$$|\partial_x^\alpha \mathfrak{I}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0} \int_{\mathbb{R}^3} (|x_t - y|^2 + t)^{-3/2-|\alpha|/2}$$
$$\times (1+|x_t - y|)^{-2\sigma_0}((1+|y|)\nu_1(y))^{-1}\,\mathrm{d}y$$
$$\leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|x|^{-1-|\alpha|/2} \int_{\mathbb{R}^3} (1+|x_t - y|)^{-1-2\sigma_0}((1+|y|)\nu_1(y))^{-1}\,\mathrm{d}y.$$

Now (2.9) implies

$$|\partial_x^\alpha \mathfrak{J}^{(1)}(\tilde{b})(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}|x|^{-1-|\alpha|/2}. \tag{4.48}$$

All the other estimates used in the proof of (4.28) need not be modified. These estimates combined with (4.45)–(4.48) imply (4.29). □

Theorem 4.1 and Lemma 4.1 imply

Corollary 4.2. *Let $x \in \mathbb{R}^3 \setminus \{0\}$. If $\kappa_0 > 0$, we have*

$$|\partial_x^\alpha \mathfrak{J}^{(1)}(b)(x,t)| \leqslant \mathfrak{C}(\kappa_0)\gamma(|x|\nu_1(x))^{-1-|\alpha|/2}(\chi_{(4,\infty)}(|x|) + \chi_{(0,4]}(|x|)|x|^{-\kappa_0}).$$

Otherwise, $|\partial_x^\alpha \mathfrak{J}^{(1)}(b)(x,t)| \leqslant \mathfrak{C}(\sigma_0)\gamma(1+t)^{\sigma_0}(|x|\nu_1(x))^{-1-|\alpha|/2}$.

Comparing (4.1) with (3.6), we may deduce the following result from Corollary 4.2:

Corollary 4.3. *Let $x \in \mathbb{R}^3 \setminus \{0\}$. If $\kappa_0 > 0$, we have*

$$\begin{aligned}|\partial_x^\alpha \mathfrak{J}^{(1)}(\tilde{a}_S)(x,t)| \leqslant \mathfrak{C}(S_0,S,\kappa_0)\delta_{|\alpha|}&(1+\tau^{-|\alpha|/2})\tau^{1+\kappa_0}(|x|\nu_1(x))^{-1-|\alpha|/2}\\ &\times (\chi_{(4,\infty)}(|x|) + \chi_{(0,4]}(|x|)|x|^{-\kappa_0}),\end{aligned}$$

else

$$|\partial_x^\alpha \mathfrak{J}^{(1)}(\tilde{a}_S)(x,t)| \leqslant \mathfrak{C}(S_0,S,\kappa_0)\delta_{|\alpha|}(1+\tau^{-|\alpha|/2})\tau(1+t)^{\sigma_0}(|x|\nu_1(x))^{-1-|\alpha|/2}.$$

5. Proof of Theorem 1.1

The first statement of Theorem 1.1 is true according to Lemma 2.3. Let $t \in (0,\infty)$, $x \in B_S^c$. Suppose that $\kappa_0 > 0$. Then, by Lemma 3.3 and Corollary 4.3,

$$\begin{aligned}|\partial_x^\alpha \mathfrak{J}^{(\tau)}(\varphi_s a)(x,t)| &= \tau^{|\alpha|}|\partial_y^\alpha \mathfrak{J}^{(1)}(\tilde{a}_S)(y,\tau^2 t)_{|y=\tau x}|\\ &\leqslant \mathfrak{C}(S_0,S,\kappa_0)\delta_{|\alpha|}(\tau^{|\alpha|/2}+1)(|x|\nu_\tau(x))^{-1-|\alpha|/2}\\ &\quad\times(\chi_{(4,\infty)}(|\tau x|)\tau^{\kappa_0} + \chi_{(0,4]}(|\tau x|)|x|^{-\kappa_0}).\end{aligned} \tag{5.1}$$

If $|\tau x| \leqslant 4$, we have $\chi_{(4,\infty)}(|\tau x|)\tau^{\kappa_0} + \chi_{(0,4]}(|\tau x|)|x|^{-\kappa_0} = |x|^{-\kappa_0} \leqslant \mathfrak{C}(S)$. In the case $|\tau x| \geqslant 4$, the equation $\chi_{(4,\infty)}(|\tau x|)\tau^{\kappa_0} + \chi_{(0,4]}(|\tau x|)|x|^{-\kappa_0} = \tau^{\kappa_0}$ holds. Thus we may deduce from (5.1) that

$$|\partial_x^\alpha \mathfrak{J}^{(\tau)}(\varphi_s a)(x,t)| \leqslant \mathfrak{C}(S_0,S,\kappa_0,\tau)\delta_{|\alpha|}(|x|\nu_\tau(x))^{-1-|\alpha|/2}.$$

Hence by virtue of Lemma 3.1,

$$|\partial_x^\alpha \mathfrak{I}^{(\tau)}(a)(x,t)| \leqslant \mathfrak{C}(S_0, S, \kappa_0, \tau)(\delta_{|\alpha|} + \|a|B_{S_0}\|_1)(|x|\nu_\tau(x))^{-1-|\alpha|/2}. \tag{5.2}$$

Since we chose α as an arbitrary multiindex with $|\alpha| \leqslant 1$ (see at the beginning of Section 3), inequalities (1.8) and (1.11) follow from (5.2). The estimates in (1.9) and (1.12) may be deduced in a similar way from Corollary 4.3 and Lemma 3.1.

References

[1] *K. I. Babenko, M. M. Vasil'ev*: On the asymptotic behavior of a steady flow of viscous fluid at some distance from an immersed body. J. Appl. Math. Mech. *37* (1973), 651–665; translation from Prikl. Mat. Mekh. *37* (1973), 690–705. (In Russian.)

[2] *H.-O. Bae, B. J. Jin*: Estimates of the wake for the 3D Oseen equations. Discrete Contin. Dyn. Syst., Ser. B *10* (2008), 1–18.

[3] *H.-O. Bae, J. Roh*: Stability for the 3D Navier-Stokes equations with nonzero far field velocity on exterior domains. J. Math. Fluid Mech. *14* (2012), 117–139.

[4] *P. Deuring*: Exterior stationary Navier-Stokes flows in 3D with nonzero velocity at infinity: asymptotic behaviour of the velocity and its gradient. IASME Trans. *2* (2005), 900–904.

[5] *P. Deuring*: The single-layer potential associated with the time-dependent Oseen system. In: Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics. Chalkida, Greece, May 11–13, 2006, pp. 117–125.

[6] *P. Deuring*: On volume potentials related to the time-dependent Oseen system. WSEAS Trans. Math. *5* (2006), 252–259.

[7] *P. Deuring*: On boundary-driven time-dependent Oseen flows. Parabolic and Navier-Stokes Equations. Part 1. (Rencławowicz, Joanna et al., ed.). Polish Acad. Sci. Inst. Math., Banach Center Publications 81, Warsaw, 2008, pp. 119–132.

[8] *P. Deuring*: A potential theoretic approach to the time-dependent Oseen system. Advances in Mathematical Fluid Mechanics (R. Rannacher, A. Sequeira, eds.). Dedicated to Giovanni Paolo Galdi on the occasion of his 60th birthday. Selected papers of the international conference on mathematical fluid mechanics, Estoril, Portugal, May 21–25, 2007. Springer, Berlin, 2010, pp. 191–214.

[9] *P. Deuring*: Spatial decay of time-dependent Oseen flows. SIAM J. Math. Anal. *41* (2009), 886–922.

[10] *P. Deuring*: A representation formula for the velocity part of 3D time-dependent Oseen flows. To appear in J. Math. Fluid Mech.

[11] *P. Deuring*: Pointwise spatial decay of time-dependent Oseen flows: the case of data with noncompact support. Discrete Contin. Dyn. Syst. Ser. A *33* (2013), 2757–2776.

[12] *P. Deuring*: Spatial decay of time-dependent incompressible Navier-Stokes flows with nonzero velocity at infinity. SIAM J. Math. Anal. *45* (2013), 1388–1421.

[13] *P. Deuring, S. Kračmar*: Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: Approximation by flows in bounded domains. Math. Nachr. *269–270* (2004), 86–115.

[14] *Y. Enomoto, Y. Shibata*: Local energy decay of solutions to the Oseen equation in the exterior domains. Indiana Univ. Math. J. *53* (2004), 1291–1330.

[15] *Y. Enomoto, Y. Shibata*: On the rate of decay of the Oseen semigroup in exterior domains and its application to Navier-Stokes equation. J. Math. Fluid Mech. *7* (2005), 339–367.

[16] *R. Farwig*: The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. Math. Z. *211* (1992), 409–447.
[17] *A. Friedman*: Partial Differential Equations of Parabolic Type. Prentice-Hall, Englewood Cliffs N.J., 1964.
[18] *G. P. Galdi*: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. II. Nonlinear Steady Problems. Springer Tracts in Natural Philosophy 39. Springer, New York, 1994.
[19] *J. G. Heywood*: The exterior nonstationary problem for the Navier-Stokes equations. Acta Math. *129* (1972), 11–34.
[20] *J. G. Heywood*: The Navier-Stokes equations: On the existence, regularity and decay of solutions. Indiana Univ. Math. J. *29* (1980), 639–681.
[21] *G. H. Knightly*: A Cauchy problem for the Navier-Stokes equations in $\mathbb{R}^n$. SIAM J. Math. Anal. *3* (1972), 506–511.
[22] *G. H. Knightly*: Some decay properties of solutions of the Navier-Stokes equations. Approximation Methods for Navier-Stokes Problems (R. Rautmann, ed.). Proc. Sympos., Univ. Paderborn 1979, Lecture Notes in Math. 771, Springer, Berlin, 1980, pp. 287–289.
[23] *T. Kobayashi, Y. Shibata*: On the Oseen equation in three dimensional exterior domains. Math. Ann. *310* (1998), 1–45.
[24] *S. Kračmar, A. Novotný, M. Pokorný*: Estimates of Oseen kernels in weighted L^p spaces. J. Math. Soc. Japan *53* (2001), 59–111.
[25] *K. Masuda*: On the stability of incompressible viscous fluid motions past objects. J. Math. Soc. Japan *27* (1975), 294–327.
[26] *T. Miyakawa*: On nonstationary solutions of the Navier-Stokes equations in an exterior domain. Hiroshima Math. J. *12* (1982), 115–140.
[27] *R. Mizumachi*: On the asymptotic behavior of incompressible viscous fluid motions past bodies. J. Math. Soc. Japan *36* (1984), 497–522.
[28] *Y. Shibata*: On an exterior initial-boundary value problem for Navier-Stokes equations. Q. Appl. Math. *57* (1999), 117–155.
[29] *V. A. Solonnikov*: A priori estimates for second order parabolic equations. Am. Math. Soc., Transl., II. Ser. *65* (1967), 51–137.
[30] *V. A. Solonnikov*: Estimates for solutions of nonstationary Navier-Stokes equations. Boundary value problems of mathematical physics and related questions in the theory of functions 7. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) *38* (1973), 153–231 (In Russian.); English translation, J. Sov. Math. *8* (1977), 467–529.
[31] *S. Takahashi*: A weighted equation approach to decay rate estimates for the Navier-Stokes equations. Nonlinear Anal., Theory Methods Appl. *37* (1999), 751–789.

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