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Pseudouniform topologies on $C(X)$ given by ideals

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Abstract. Given a Tychonoff space X, a base α for an ideal on X is called *pseudouniform* if any sequence of real-valued continuous functions which converges in the topology of uniform convergence on α converges uniformly to the same limit.

This paper focuses on pseudouniform bases for ideals with particular emphasis on the ideal of compact subsets and the ideal of all countable subsets of the ground space.

Keywords: function space; topology of uniform convergence; ideal; uniformity; Lindelöf property; pseudouniform ideal; almost pseudo-ω-bounded

Classification: 54A10, 54A20, 54A25, 54C35, 54D20, 54E15

1. Introduction

Given α, a base for an ideal on a topological space X, we define, in Section 3, a uniformity on $C(X)$, the set of all real-valued continuous functions on X. The topology associated to this uniformity will be called *the topology of uniform convergence on* α and $C_{\alpha,u}(X)$ will denote the corresponding topological space. It turns out that this topology is coarser than the topology of uniform convergence on $C(X)$ and therefore the following seems like a natural property: α is *pseudouniform* if whenever a sequence $\langle f_n : n \in \omega \rangle$ in $C(X)$ converges to f in $C_{\alpha,u}(X)$, the sequence converges uniformly to f. Pseudouniform ideals are the main topic in this paper.

In Section 3 we give basic results on topologies of uniform convergence on an ideal and their corresponding natural induced functions. Some properties of pseudouniform ideals are also included and a characterization of the pseudouniformity of the ideal in terms of families of open subsets of the ground space is given.

Spaces for which the ideal generated by all compact subsets is pseudouniform are called *almost pseudo-ω-bounded*. Section 4 is dedicated to this class of spaces. We analyze the behaviour of the class under the usual topological operations and from this analysis it is deduced that arbitrary products of pseudocompact k-spaces are pseudocompact.

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When α is the collection of all countable subsets of X, $C_{\alpha,u}(X)$ is denoted by $C_s(X)$. Section 5 is mainly about cardinal functions on $C_s(X)$ and Section 6 deals with Lindelöf's property on $C_s(X)$. Finally, in the last section we list some open questions.

2. Notation and definitions

This section contains the basic definitions and notations that will be followed in this paper. All topological concepts that are not defined here should be understood as in [2] except for the following: a uniformity $\mathcal{U}$ on a set X does not have to satisfy $\bigcap \mathcal{U} = \{(x,x) : x \in X\}$. Therefore, uniform spaces are completely regular but not necessarily Hausdorff.

A property P will be called *weakly hereditary* if any closed subspace of a topological space having P also has P.

We follow the convention that all cardinal topological functions are, by definition, infinite. A cardinal function which does not appear in [2] is the *i-weight* of a topological space X: $\mathrm{iw}(X)$ is the minimum weight of a topological space Y for which there is a continuous bijection from X onto Y.

Let X and Y be two topological spaces with the same underlying set. The symbol $X \leq Y$ means that the topology of X is finer than the topology of Y. When $X \leq Y$ but $X \neq Y$ we will write $X < Y$.

As usual, given a topological space X, $C(X)$ denotes the collection of all continuous real-valued functions whose domain is X. We will use the symbol $\mathbf{0}$ to represent the zero function on X.

Unless otherwise stated, all spaces considered in this paper are Tychonoff.

Given a set X and a cardinal κ, the symbol $[X]^{<\kappa}$ will denote the collection of all subsets of X which have cardinality $< \kappa$. A similar convention will apply to $[X]^{\leq\kappa}$. Finally, $[X]^{\kappa}$ is the set of all subsets of X whose size is precisely κ.

ω is the first infinite ordinal and for this reason each integer $n \in \omega$ will be considered as an ordinal, i.e., $n = \{k \in \omega : k < n\}$. The set $\omega \setminus \{0\}$ will be denoted by $\mathbb{N}$.

The cardinality of $\mathbb{R}$, the set of real numbers, will be denoted by $\mathfrak{c}$.

A *cellular family* in a topological space is a pairwise disjoint family of open subsets of the given topological space. To simplify things, every time we refer to a collection $\{U_n : n \in \omega\}$ as a cellular family, we will be assuming that $U_m \cap U_n = \emptyset$, whenever $m \neq n$.

3. Basic results on topologies of uniform convergence

Let α be a non-empty family of subsets of X. We say that α is a *base for an ideal on* X if for any $A, B \in \alpha$ there exists $C \in \alpha$ with $A \cup B \subseteq C$.

Assume α is a base for an ideal on X. Following [6], we define, for each real number $\varepsilon > 0$ and each $A \in \alpha$, the set

$$A_\varepsilon := \{(f,g) \in C(X) \times C(X) : \forall x \in A \ (|f(x) - g(x)| < \varepsilon)\}.$$

One easily verifies that $\{A_\varepsilon : A \in \alpha \wedge \varepsilon > 0\}$ is a base for some uniformity on $C(X)$. The topology induced by this uniformity will be called the *topology of uniform convergence on* α and the resulting space will be denoted by $C_{\alpha,u}(X)$.

Given $f \in C(X)$, $A \subseteq X$, and $\varepsilon > 0$, let

$$V(f, A, \varepsilon) := \{g \in C(X) : \forall x \in A \ (|f(x) - g(x)| < \varepsilon)\}.$$

Then, for each $f \in C(X)$, the collection $\{V(f, A, \varepsilon) : A \in \alpha \wedge \varepsilon > 0\}$ forms a neighborhood base for $C_{\alpha,u}(X)$ at f.

Proposition 3.1. *If α is a base for an ideal on X, then $C_{\alpha,u}(X)$ is Tychonoff iff α has dense union in X.*

Proof: $C_{\alpha,u}(X)$ is completely regular because its topology is given by a uniformity so our only concern is Hausdorff's property.

Assume that $\bigcup \alpha$ is dense in X and let $f, g \in C_{\alpha,u}(X)$ be so that $f \neq g$. Fix $z \in X$ satisfying $f(z) \neq g(z)$ and set $\varepsilon := |f(z) - g(z)|/4$. Let U be a neighborhood of z for which $|f(x) - g(x)| > 2\varepsilon$, whenever $x \in U$. Our assumption on α implies that, for some $A \in \alpha$, there exists $y \in A \cap U$. Note that if $h \in V(f, A, \varepsilon) \cap V(g, A, \varepsilon)$, then $|f(y) - h(y)| < \varepsilon$ and $|g(y) - h(y)| < \varepsilon$, so $|f(y) - g(y)| < 2\varepsilon$; a contradiction to $y \in U$.

For the remaining implication suppose that α does not have dense union and fix $z \in X \setminus \overline{\bigcup \alpha}$ and $f \in C(X)$ in such a way that $f(z) = 1$ and $f[\overline{\bigcup \alpha}] \subseteq \{0\}$. Thus $C_{\alpha,u}(X)$ is not Hausdorff because $f \in V(\mathbf{0}, A, \varepsilon)$, for all $A \in \alpha$ and $\varepsilon > 0$. □

Definition 3.2. Let κ be an infinite cardinal. If X is a topological space and $\alpha = [X]^{<\kappa}$, then we define $C_{\kappa,u}(X) := C_{\alpha,u}(X)$. In particular,

1. (1) $C_s(X)$ is the space $C_{\omega_1,u}(X)$ and its topology will be called *the topology of pseudouniform convergence*.
2. (2) When $\kappa = \omega$, we obtain $C_p(X)$, the space of continuous functions on X equipped with the topology of pointwise convergence.

Let X be a topological space. If α is the collection of all compact subsets of X, $C_{\alpha,u}(X)$ will be denoted by $C_k(X)$. It is a consequence of [6, Theorem 1.2.3] that if one considers the cartesian product $\mathbb{R}^X$ endowed with the compact-open topology (see [2, Section 3.4]), then the relative topology of $C(X)$ coincides with the topology of $C_k(X)$.

When $\alpha = \{X\}$, we let $C_u(X) := C_{\alpha,u}(X)$. It is straightforward to verify that $C_u(X)$ is the subspace $C(X)$ of $\mathbb{R}^X$ endowed with the topology of uniform convergence (see [2, Section 2.6]). In particular, $C_u(X)$ is metrizable.

It should be clear that $C_{\alpha,u}(X) \leq C_u(X)$ for any α.

Lemma 3.3. *Let α be a base for an ideal on the topological space X. If we let $\beta := \{\overline{A} : A \in \alpha\}$, then $C_{\alpha,u}(X) = C_{\beta,u}(X)$.*

Proof: It suffices to observe that for each $f \in C(X)$, $A \in \alpha$, and $\varepsilon > 0$ one gets

$$V(f, A, \varepsilon/2) \subseteq V(f, \overline{A}, \varepsilon) \subseteq V(f, A, \varepsilon).$$

□

Theorem 3.4. *If α and β are bases for ideals on X, then the following are equivalent:*

(1) $C_{\alpha,u}(X) \le C_{\beta,u}(X)$;
(2) *for each $A \in \alpha$ there exists $B \in \beta$ so that $A \subseteq \overline{B}$.*

Proof: Suppose that (2) holds. According to Lemma 3.3 there is no loss of generality in assuming that each element of β is closed. Therefore, given $f \in C(X)$, $A \in \alpha$, and $\varepsilon > 0$ there exists $B \in \beta$ with $A \subseteq B$. In particular, $V(f, B, \varepsilon) \subseteq V(f, A, \varepsilon)$.

Now let us assume that (1) is true. Fix $A \in \alpha$ and $f \in C(X)$. Since f is an interior point of $V(f, A, 1)$ in $C_{\alpha,u}(X)$, there exists $B \in \beta$ and $\varepsilon > 0$ such that $V(f, B, \varepsilon) \subseteq V(f, A, 1)$. We claim that $A \subseteq \overline{B}$. Assume otherwise and fix $a \in A \setminus \overline{B}$. There is $g \in C(X)$ such that $g[\overline{B}] \subseteq \{0\}$ and $g(a) = 1$. Thus $g \in V(f, B, \varepsilon) \setminus V(f, A, 1)$. This contradiction finishes the argument. □

In particular, one gets that $C_{\alpha,u}(X) = C_u(X)$ iff there is $A \in \alpha$ satisfying $\overline{A} = X$. Similarly, $C_p(X) \le C_{\alpha,u}(X)$ is equivalent to $X = \bigcup\{\overline{A} : A \in \alpha\}$.

Corollary 3.5. *Let κ be an infinite cardinal and let α be a base for an ideal on X.*

(1) *If each element of α has density $< \kappa$ (i.e., $d(A) < \kappa$ for all $A \in \alpha$), then $C_{\alpha,u}(X) \le C_{\kappa,u}(X)$.*
(2) *If $\alpha = \{A \subseteq X : d(A) < \kappa\}$ and $\beta = \{A \in \alpha : A = \overline{A}\}$, then*
$$C_{\kappa,u}(X) = C_{\alpha,u}(X) = C_{\beta,u}(X).$$
(3) *If I is the ideal generated by α (i.e., $I = \{B \subseteq X : \exists A \in \alpha\ (B \subseteq A)\}$), then $C_{\alpha,u}(X) = C_{I,u}(X)$.*

Given a continuous map $f : X \to Y$, the *dual map* $f^\sharp : C(Y) \to C(X)$ is defined by $f^\sharp(g) := g \circ f$, for all $g \in C(Y)$.

When Y is a subspace of X, the inclusion map $i_Y : Y \to X$ is continuous so $i_Y^\sharp$ is defined. This map will be called the *restriction map* and will be denoted by π_Y. Note that $\pi_Y(g) = g \restriction Y$ for all $g \in C(X)$.

If α is a base for an ideal on a set X and $f : X \to Y$ is a function, then the collection $f[\alpha] := \{f[A] : A \in \alpha\}$ is a base for an ideal on Y.

Proposition 3.6. *Let $f : X \to Y$ be a continuous function and assume that α and β are bases for ideals on X and Y, respectively. The following statements are true for the dual map $f^\sharp : C_{\beta,u}(Y) \to C_{\alpha,u}(X)$:*

(1) *$f^\sharp$ is continuous iff $C_{f[\alpha],u}(Y) \le C_{\beta,u}(Y)$;*
(2) *if f is onto, then $f^\sharp$ is an embedding iff $C_{\beta,u}(Y) = C_{f[\alpha],u}(Y)$;*
(3) *if f is a quotient map and α covers X, then* $\operatorname{ran}(f^\sharp)$, *the range of $f^\sharp$, is a closed subset of $C_{\alpha,u}(X)$.*

Proof: Let us start with (1). Suppose that $C_{f[\alpha],u}(Y) \leq C_{\beta,u}(Y)$ and let $g \in C(Y)$, $A \in \alpha$, and $\varepsilon > 0$ be arbitrary. Fix $B \in \beta$ so that $f[A] \subseteq \overline{B}$ (Theorem 3.4) and let $h \in V(g, B, \varepsilon/2)$ be arbitrary. Then $|h(y) - g(y)| < \varepsilon$ for all $y \in \overline{B}$; in particular, $|h(f(x)) - g(f(x))| < \varepsilon$, whenever $x \in A$, i.e., $f^\sharp(h) \in V(f^\sharp(g), A, \varepsilon)$. Thus $f^\sharp$ is continuous at g.

Now assume that $C_{f[\alpha],u}(Y) \not\leq C_{\beta,u}(Y)$ and let $A \in \alpha$ be a witness to the failure of item (2) in Theorem 3.4. Then, for each $B \in \beta$, there is a point $x_B \in A$ and a map $g_B \in C(Y)$ satisfying $f(x_B) \notin \overline{B}$, $g_B(f(x_B)) = 1$, and $g_B[\overline{B}] \subseteq \{0\}$. Note that if $B_0, B \in \beta$ satisfy $B_0 \subseteq B$, then $g_B \in V(\mathbf{0}, B_0, \varepsilon)$; in other words, $\langle g_B : B \in \beta\rangle$ is a net (in the sense of [2, Section 1.6]) which converges to $\mathbf{0}$ in $C_{\beta,u}(Y)$, but $f^\sharp(g_B) \notin V(f^\sharp(\mathbf{0}), A, 1)$ for all $B \in \beta$. Thus $f^\sharp$ is not continuous and so the proof of (1) is complete.

Suppose that f is onto. One easily verifies that $f^\sharp$ is one-to-one so to prove (2) we only need to argue that the continuity of the inverse of $f^\sharp$ is equivalent to $C_{\beta,u}(Y) \leq C_{f[\alpha],u}(Y)$.

Start with the assumption $C_{\beta,u}(Y) \leq C_{f[\alpha],u}(Y)$. Given $g \in C(Y)$, $B \in \beta$, and $\varepsilon > 0$, there is $A \in \alpha$ for which $B \subseteq \overline{f[A]}$, so when $h \in C(Y)$ satisfies $f^\sharp(h) \in V(f^\sharp(g), A, \varepsilon/2)$, we have that $|h(y) - g(y)| \leq \varepsilon/2 < \varepsilon$, for all $y \in \overline{f[A]}$. Therefore $h \in V(g, B, \varepsilon)$.

For the opposite implication, let us assume that $C_{\beta,u}(Y) \not\leq C_{f[\alpha],u}(Y)$ and, as we did before, let us fix $B \in \beta$ in such a way that for each $A \in \alpha$ there are $y_A \in B$ and $g_A \in C(Y)$ satisfying $g_A(y_A) = 1$ and $g_A[\overline{f[A]}] \subseteq \{0\}$. Note that, by continuity, $g_A \circ f[\overline{A}] \subseteq \{0\}$. Therefore $\langle f^\sharp(g_A) : A \in \alpha\rangle$ is a net in the subspace $\operatorname{ran}(f^\sharp)$ which converges to $f^\sharp(\mathbf{0})$. Since $g_A \notin V(\mathbf{0}, B, 1)$, for all $A \in \alpha$, we conclude that $(f^\sharp)^{-1}$ is not continuous.

For (3), we claim that if $g \in \overline{\operatorname{ran}(f^\sharp)}$, then g is constant on each fiber of f. By contrapositive, assume that, for some $z \in Y$, there are $x, y \in f^{-1}[z]$ such that $g(x) \neq g(y)$ and fix $A \in \alpha$ for which $x, y \in A$; then set $\varepsilon := |g(x) - g(y)|/3$ and notice that $h \in V(g, A, \varepsilon)$ implies $|h(x) - h(y)| > \varepsilon$. In particular, $h \notin \operatorname{ran}(f^\sharp)$.

Since f is a quotient map, the previous claim guarantees that for each $g \in \overline{\operatorname{ran}(f^\sharp)}$ there is $h \in C(Y)$ such that $g = h \circ f$, i.e., $g \in \operatorname{ran}(f^\sharp)$. □

Remark 3.7. Let Y be a subspace of X and assume that α and β are as in the previous proposition. Then β is a base for an ideal on X too and therefore π_Y is continuous iff $C_{\beta,u}(X) \leq C_{\alpha,u}(X)$.

Assume that $\{f_n : n \in \omega\} \cup \{f\} \subseteq C(X)$. If α is a base for an ideal on X, the symbol $f_n \xrightarrow{\alpha} f$ (respectively, $f_n \xrightarrow{u} f$) means that the sequence $\langle f_n : n \in \omega\rangle$ converges to f in $C_{\alpha,u}(X)$ (respectively, $C_u(X)$). Also, the negation of $f_n \xrightarrow{\alpha} f$ will be represented by $f_n \overset{\alpha}{\not\to} f$ and similarly for $f_n \overset{u}{\not\to} f$.

Observe that $f_n \xrightarrow{u} f$ always implies $f_n \xrightarrow{\alpha} f$. We are interested in the opposite implication:

Definition 3.8. Let α be a base for an ideal on a topological space X. We say that α is *pseudouniform on* X (or simply *pseudouniform* when there is no risk of confusion) if for every sequence $\langle f_n : n \in \omega\rangle$ in $C(X)$ and all $f \in C(X)$ we have that $f_n \xrightarrow{\alpha} f$ implies that $f_n \xrightarrow{u} f$.

Note that if α is pseudouniform on X and $C_{\alpha,u}(X) \leq C_{\beta,u}(X)$, then β is pseudouniform too.

Remark 3.9. All pseudouniform bases have dense union. The proof will be by contrapositive so assume that α is a base for an ideal on X and let $z \in X \setminus \overline{\bigcup \alpha}$. Then there is a continuous map $f : X \to \mathbb{R}$ such that $f(z) = 1$ and $f[\overline{\bigcup \alpha}] \subseteq \{0\}$. Now set $f_n := f$, for each $n \in \omega$, and note that $f_n \xrightarrow{\alpha} \mathbf{0}$ but $f_n \overset{u}{\not\to} \mathbf{0}$.

Theorem 3.10. *If α is a base for an ideal on X with dense union, the following statements are equivalent.*

1. *α is pseudouniform.*
2. *For every family $\{U_n : n \in \omega\}$ of nonempty open sets in X there exists $A \in \alpha$ such that the set $\{n \in \omega : U_n \cap A \neq \emptyset\}$ is infinite.*
3. *For every cellular family $\{U_n : n \in \omega\}$ in X there exists $A \in \alpha$ such that the set $\{n \in \omega : U_n \cap A \neq \emptyset\}$ is infinite.*

Proof: To prove that (1) implies (2) assume that $\mathcal{U} = \{U_n : n \in \omega\}$ is a family of nonempty open subsets of X such that for each $A \in \alpha$ the set $\{n \in \omega : A \cap U_n \neq \emptyset\}$ is finite.

For each $n \in \omega$ fix a point $x_n \in U_n$ and $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n[X \setminus U_n] \subseteq \{0\}$. Then $f_n \notin V(\mathbf{0}, X, 1)$ for all n. Therefore, in order to prove that α is not pseudouniform, it suffices to show that $f_n \xrightarrow{\alpha} \mathbf{0}$.

Let $A \in \alpha$ and $\varepsilon > 0$ be arbitrary. Our assumption on $\mathcal{U}$ implies the existence of an integer m such that $A \cap U_n = \emptyset$ for each $n \geq m$. Hence $f_n \in V(\mathbf{0}, A, \varepsilon)$ for all $n \geq m$.

Clearly (3) is a consequence of (2) so we only need to show that the negation of (1) implies the negation of (3).

Suppose that α is not pseudouniform and fix $\{f_n : n \in \omega\} \cup \{f\} \subseteq C(X)$ so that $f_n \xrightarrow{\alpha} f$ but $f_n \overset{u}{\not\to} f$. Thus there is $\varepsilon > 0$ so that $b := \{n \in \omega : f_n \notin V(f, X, 2\varepsilon)\}$ is infinite. For each $n \in b$ fix a point $x_n \in X$ satisfying $|f_n(x_n) - f(x_n)| > \varepsilon$.

We face two cases. First, assume that $\{x_n : n \in b\}$ is infinite. Then use the fact that X is completely regular to get an infinite set $a \subseteq b$ and a cellular family $\{W_n : n \in a\}$ in X such that $x_n \in W_n$, for each $n \in a$. To finish this case we will show that each member of α has nonempty intersection with only finitely many elements of $\{W_n : n \in a\}$. Let $A \in \alpha$ be arbitrary. For some integer m we have $\{f_n : n \geq m\} \subseteq V(f, A, \varepsilon)$ and therefore $A \cap W_n = \emptyset$, for all $n \in a \setminus m$.

When the first case fails, there is $z \in X$ so that $c := \{n \in b : x_n = z\}$ is infinite. For each $n \in c$ let U_n be a neighborhood of z such that $|f_n(y) - f(y)| > \varepsilon$, for all $y \in U_n$. Notice that if z were an isolated point, there would be $A \in \alpha$ with $z \in A$ (recall that α has dense union) and hence $\{f_n : n \in c\} \cap V(f, A, \varepsilon) = \emptyset$; a contradiction to $f_n \xrightarrow{\alpha} f$. Therefore z is not isolated and so we inductively construct a sequence $\langle y_n : n \in c\rangle$ in such a way that $y_n \in U_n \setminus \{y_k : k \in c \cap n\}$, for each $n \in c$. Clearly $\{y_n : n \in c\}$ is infinite and $|f_n(y_n) - f(y_n)| > \varepsilon$, whenever $n \in c$; thus the arguments used for the first case work here and this ends the proof. □

Recall that a subset A of a topological space X is *bounded* if, for each $f \in C(X)$, $f[A]$ is a bounded subset of $\mathbb{R}$.

Corollary 3.11. *If α is a pseudouniform base for an ideal on X, the following are equivalent:*

(1) *X is pseudocompact;*
(2) *each element of α is bounded; and*
(3) *$C_{\alpha,u}(X)$ is a topological vector space with the usual operations.*

Proof: Clearly, (2) follows from (1). Now, to show that (2) implies (1), assume that X is not pseudocompact and let $f \in C(X)$ be an unbounded function. For each $n \in \omega$ define $U_n := f^{-1}[(n, n+1)]$. Then $\{U_n : n \in \omega\}$ is a family of nonempty open sets and hence, for some $A \in \alpha$, the set $\{n \in \omega : A \cap U_n \neq \emptyset\}$ is infinite. Thus A is not bounded.

Finally, [6, Theorem 1.1] states that (3) and (2) are equivalent (even if α is not pseudouniform). □

The proof of our next result is a routine argument so we omit it.

Proposition 3.12. *If α is a base for an ideal on X, then $(C_{\alpha,u}(X), +)$ is a topological group and therefore $w(C_{\alpha,u}(X)) = \chi(C_{\alpha,u}(X)) \cdot d(C_{\alpha,u}(X))$.*

Proposition 3.13. *Let α be a countable base for an ideal on X. If α is pseudouniform, then $C_{\alpha,u}(X) = C_u(X)$.*

Proof: Let $\alpha = \{A_n : n \in \omega\}$ and assume that $C_u(X) \not\leq C_{\alpha,u}(X)$. For each integer n fix $x_n \in X \setminus \overline{A_n}$ (Theorem 3.4) and $f_n \in C(X)$ in such a way that $f_n(x_n) = 1$ and $f_n[\overline{A_n}] \subseteq \{0\}$.

Thus $f_n \xrightarrow{\alpha} \mathbf{0}$, but $\{f_n : n \in \omega\} \cap V(\mathbf{0}, X, 1) = \emptyset$, i.e., α is not pseudouniform. □

Our previous proposition and Corollary 3.5-(3) guarantee that if α is pseudouniform and the ideal generated by α has a countable base, then $C_{\alpha,u}(X) = C_u(X)$. On the other hand, $I := [\mathbb{R}]^{\leq\omega}$ is an ideal on $\mathbb{R}$ with $C_{I,u}(\mathbb{R}) = C_u(\mathbb{R})$ (Theorem 3.4), but no countable subset of I is a base for I.

Theorem 3.14. *If α is a pseudouniform base for an ideal on X, the following are equivalent:*

(1) *$C_{\alpha,u}(X) = C_u(X)$,*
(2) *$C_{\alpha,u}(X)$ is metrizable,*
(3) *$C_{\alpha,u}(X)$ is first countable,*
(4) *$C_{\alpha,u}(X)$ is Fréchet-Urysohn, and*
(5) *$C_{\alpha,u}(X)$ is sequential.*

Proof: $C_u(X)$ is metrizable and therefore (2) follows from (1). Implications $(2) \to (3) \to (4) \to (5)$ hold for any space so we only need to show that (1) is a consequence of (5).

Suppose that (1) fails and fix F, a closed subset of $C_u(X)$ which is not closed in $C_{\alpha,u}(X)$. To prove that (5) fails it suffices to show that F is sequentially closed, so let $\{f_n : n \in \omega\} \subseteq F$ and $f \in C(X)$ be so that $f_n \xrightarrow{\alpha} f$. Since α is pseudouniform, $f_n \xrightarrow{u} f$ and therefore $f \in F$. □

4. Almost pseudo-ω-bounded spaces

Let us recall that a topological space X is *ω-bounded* if the closure of any countable subset of X is compact. Also, X is *pseudo-ω-bounded* if for each countable cellular family there is a compact set which intersects all members of the family.

Definition 4.1. X will be called *almost pseudo-ω-bounded* if the collection of all compact subsets of X is pseudouniform on X.

According to Theorem 3.10, X is almost pseudo-ω-bounded iff for any countable cellular family there is a compact subset of X which intersects infinitely many elements of the family. This remark justifies the name we adopted for this notion.

A simple consequence of Corollary 3.11 is that every almost pseudo-ω-bounded space is pseudocompact. This argument provides an alternative proof to the one given in [4].

Proposition 4.2. *Every almost pseudo-ω-bounded space is pseudocompact.*

Proposition 4.3. *There are pseudocompact spaces which are not almost pseudo-ω-bounded.*

Proof: Let X be an infinite pseudocompact space such that any countable subset of it is closed discrete (such a space is constructed in [9]). Since X is Hausdorff and infinite, it possesses an infinite cellular family $\{U_n : n \in \omega\}$. Suppose that A is a subset of X for which $b := \{n \in \omega : A \cap U_n \neq \emptyset\}$ is infinite. Fix, for each $n \in b$, a point $x_n \in A \cap U_n$. Hence $\{x_n : n \in b\}$ is a closed discrete subset of A. In particular, A is not compact. □

Notice that the following implications hold trivially:

compactness → ω-boundedness → pseudo ω-boundedness → almost pseudo-ω-boundedness

To show that the last arrow cannot be reversed some concepts are needed.

A *MAD family* is an infinite collection $\mathcal{A} \subseteq [\omega]^\omega$ such that (1) $a \cap b$ is finite for all $a, b \in \mathcal{A}$ with $a \neq b$ and (2) for each infinite set $a \subseteq \omega$ there exists $b \in \mathcal{A}$ so that $a \cap b$ is infinite.

The Isbell-Mrówka space $\Psi(\mathcal{A})$ associated with the MAD family $\mathcal{A}$ is $\omega \cup \mathcal{A}$ endowed with the topology in which $\{n\}$ is open for each $n \in \omega$ and a neighborhood base for $a \in \mathcal{A}$ is given by $\{\{a\} \cup (a \setminus n) : n \in \omega\}$.

Proposition 4.4. *There are almost pseudo-ω-bounded spaces which fail to be pseudo-ω-bounded.*

PROOF: Assume that $\mathcal{A}$ is a MAD family and set $X := \Psi(\mathcal{A})$. Since X is locally compact and pseudocompact, Proposition 4.9 below applies and therefore X is almost pseudo-ω-bounded.

Now let K be an arbitrary compact subspace of X. Then $K \setminus \omega$ is finite because $K \setminus \omega$ is a closed discrete subspace of K. Fix $a \in \mathcal{A} \setminus K$ and $m \in \omega$ in such a way that $K \cap (a \setminus m) = \emptyset$. Hence $K \cap \{n\} = \emptyset$, for all $n \in a \setminus m$, and, in particular, K does not meet all members of the cellular family $\{\{n\} : n \in \omega\}$. Thus X is not pseudo-ω-bounded. $\square$

Observe that if $\mathcal{A}$ is a MAD family, then $\mathcal{A}$ is an infinite closed discrete subspace of $\Psi(\mathcal{A})$. Thus we have the following.

Remark 4.5. There are almost pseudo-ω-bounded spaces which are not countably compact and being almost pseudo-ω-bounded is not a weakly hereditary property.

On the other hand, we also have:

Proposition 4.6. *There are countably compact spaces which are not almost pseudo-ω-bounded.*

PROOF: One can find in [12, 2.13] the construction of an infinite countably compact space in which all compact subsets are finite. Thus this space is not almost pseudo-ω-bounded. $\square$

Theorem 4.7. *Almost pseudo-ω-boundedness is a productive property which is preserved by continuous maps.*

PROOF: The argument needed to show preservation under continuous maps is straightforward so we will omit it.

The remaining part of our theorem will be proved by transfinite induction on the number of factors.

If X and Y are almost pseudo-ω-bounded, then so is $X \times Y$. Indeed, having Theorem 3.10 in mind, let us suppose that $\{U_n : n \in \omega\}$ is a family of nonempty basic open sets in $X \times Y$. Denote by π_X and π_Y the corresponding projections. Then there is a compact set K_0 in X for which $b_0 := \{n \in \omega : K_0 \cap \pi_X[U_n] \neq \emptyset\}$ is infinite (Theorem 3.10). Similarly, Y has a compact subset K_1 for which the set $b_1 := \{n \in b_0 : K_1 \cap \pi_Y[U_n] \neq \emptyset\}$ is infinite. Thus $K_0 \times K_1$ is a compact subset of $X \times Y$ which has nonempty intersection with U_n, for each $n \in b_1$.

Let us assume that for some infinite cardinal κ the product of fewer than κ factors, each one of them an almost pseudo-ω-bounded space, is almost pseudo-ω-bounded (note that the preceding paragraph takes care of all finite cardinals).

Let X be the topological product of $\{X_\xi : \xi < \kappa\}$, where each X_ξ is almost pseudo-ω-bounded. We consider two cases.

First, if κ has cofinality ω, there exists $\{\delta_n : n \in \omega\}$, an increasing cofinal sequence in κ with $\delta_0 = 0$. For each integer n define the product $Y_n = \prod\{X_\xi : \delta_n \leq$

$\xi < \delta_{n+1}\}$ and denote by $\pi_n : Y \to Y_n$ the nth projection map. By our inductive hypothesis, each Y_n is almost pseudo-ω-bounded. Since X is homeomorphic to $Y := \prod_n Y_n$, we only need to show that Y is almost pseudo-ω-bounded so let $\{U_n : n \in \omega\}$ be a sequence of nonempty open sets in Y (we will use Theorem 3.10). Without loss of generality let us assume that each U_n is a basic open set.

We claim that there are two sequences, $\{b_n : n \in \omega\}$ and $\{C_n : n \in \omega\}$, such that the following holds for all $n \in \omega$: b_n is an infinite subset of ω; $b_{n+1} \subseteq b_n$; C_n is a compact subset of Y_n; and $C_n \cap \pi_n[U_m] \neq \emptyset$, for all $m \in b_n$. Indeed, there is a compact set $C_0 \subseteq Y_0$ such that $b_0 := \{m < \omega : C_0 \cap \pi_0[U_m] \neq \emptyset\}$ is infinite. Assuming we have defined $\{b_n : n < \ell\}$ and $\{C_n : n < \ell\}$ for some positive integer ℓ, we apply Theorem 3.10 to $\{\pi_\ell[U_m] : m \in b_{\ell-1}\}$ to obtain a compact set $C_\ell \subseteq Y_\ell$ for which $b_\ell := \{m \in b_{\ell-1} : C_\ell \cap \pi_\ell[U_m] \neq \emptyset\}$ is infinite.

Fix $b \in [\omega]^\omega$ in such a way that $\{b \setminus b_n : n \in \omega\} \subseteq [\omega]^{<\omega}$. Now, given $n \in \omega$ and $m \in b \setminus b_n$, let $x_n^m \in \pi_n[U_m]$ be an arbitrary point. Notice that $K_n := C_n \cup \{x_n^k : k \in b \setminus b_n\}$ is a compact subset of Y_n which has nonempty intersection with $\pi_n[U_k]$, for all $k \in b$. Hence $K := \prod_n K_n$ is a compact subset of Y satisfying $K \cap U_k \neq \emptyset$, for each $k \in b$. This concludes the first case.

The remaining case is $\mathrm{cf}(\kappa) > \omega$. Let $\{U_n : n \in \omega\}$ be a family of nonempty basic open sets in X. Our assumption on the cofinality of κ implies the existence of $\delta < \kappa$ such that $\pi_\xi[U_n] = X_\xi$, whenever $n \in \omega$ and $\delta \leq \xi < \kappa$. Define $Z_0 := \prod_{\xi<\delta} X_\xi$ and $Z_1 := \prod_{\xi \in \kappa \setminus \delta} X_\xi$.

For each $n \in \omega$ let $W_n := \{x \restriction \delta : x \in U_n\}$ (each $x \in X$ is considered as a function with domain κ so $x \restriction \delta$, the restriction of x to δ, makes sense). A slight abuse of notation gives $X = Z_0 \times Z_1$ and $U_n = W_n \times Z_1$, for all $n \in \omega$. According to our inductive hypothesis, Z_0 is almost pseudo-ω-bounded so there is a compact set $C \subseteq Z_0$ for which $b := \{n \in \omega : C \cap W_n \neq \emptyset\}$ is infinite. If we fix a point $z \in Z_1$, then $K := C \times \{z\}$ is a compact subset of X and $K \cap U_n \neq \emptyset$, for all $n \in b$. □

A topological space X is *Frolík* if for any pseudocompact space Y the product $X \times Y$ is pseudocompact. For example, all compact spaces are Frolík.

Theorem 4.8. *All almost pseudo-ω-bounded spaces are Frolík.*

Proof: Let X be an almost pseudo-ω-bounded space and let Y be a pseudocompact space. Seeking a contradiction let us assume that there exist $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ such that

(1) for each $n \in \omega$, U_n and V_n are nonempty open subsets of X and Y, respectively,
(2) $\{U_n \times V_n : n \in \omega\}$ is locally finite in $X \times Y$, and
(3) $m \neq n$ implies $U_m \times V_m \neq U_n \times V_n$.

Let K be a compact subset of X for which the set $b := \{n \in \omega : K \cap U_n \neq \emptyset\}$ is infinite. Then $\{(K \cap U_n) \times V_n : n \in b\}$ is an infinite locally finite family of nonempty open subsets of $K \times Y$, contradicting the fact that this space is pseudocompact. □

For k-spaces the reverse implication is also valid.

Proposition 4.9. *If X is a k-space, the following are equivalent:*

(1) *X is almost pseudo-ω-bounded,*
(2) *X is pseudocompact,*
(3) *X is Frolík.*

PROOF: (3) → (2) is trivial and (1)→(3) is Theorem 4.8, so let us assume that X is pseudocompact. We will use Theorem 3.10-(3) to prove that X is almost pseudo-ω-bounded.

Let $\mathcal{U} = \{U_n : n \in \omega\}$ be a cellular family in X and fix, for each integer n, a nonempty open set V_n with $\overline{V_n} \subseteq U_n$. Since X is pseudocompact, there is a point $x \in X$ so that any neighborhood of it intersects infinitely many V_n's. Clearly x is an accumulation point of the set $A := \bigcup_{n\in\omega} \overline{V_n}$. Moreover, the fact that $\mathcal{U}$ is a cellular family implies that $x \notin A$. This shows that A is not closed in X. Let K be a compact subset of X for which $K \cap A$ is not closed in K. Hence $|\{n \in \omega : K \cap \overline{V_n} \neq \emptyset\}| = \omega$ and therefore K has nonempty intersection with infinitely many members of $\mathcal{U}$. □

The following result is a consequence of Theorem 4.7 and Proposition 4.9.

Corollary 4.10. *The product of any family of pseudocompact k-spaces is pseudocompact.*

5. The topology of pseudouniform convergence

Note that $C_p(X) \leq C_s(X) \leq C_u(X)$, for any space X.

Proposition 5.1. *For any topological space X, we have:*

(1) *$C_s(X)$ is a Tychonoff space;*
(2) *$[X]^{\leq\omega}$ is pseudouniform (in other words, every converging sequence in $C_s(X)$ converges uniformly);*
(3) *$C_s(X) = C_p(X)$ iff X is finite.*

PROOF: Since $[X]^{\leq\omega}$ covers X, (1) is true. (2) is an immediate consequence of Theorem 3.10 and (3) is a corollary of Theorem 3.4. □

It is tempting to think that if α is a pseudouniform base for X, then $C_s(X) \leq C_{\alpha,u}(X)$, but this is not the case. Indeed, consider a topological space X possessing two dense subsets D and E in such a way that D is countable and E is not a separable subspace (for example, the product 2^{ω_1} is separable and any Σ-product of it is dense and non-separable). Using the density of E and Theorem 3.10 it is easy to show that $\alpha := [E]^{\leq\omega}$ is pseudouniform on X. We will use Theorem 3.4 to prove that $C_s(X) \not\leq C_{\alpha,u}(X)$: if $A \in \alpha$ satisfies $D \subseteq \overline{A}$, then A would be a countable dense subset of E, contradicting its non-separability.

An immediate consequence of Corollary 3.11 is that a topological space is pseudocompact iff each countable subset of it is bounded.

Recall that $C^*(X)$ is the family of all bounded continuous functions from X into $\mathbb{R}$.

Proposition 5.2. *For any topological space X, $C^*(X)$ is dense in $C_s(X)$ iff X is pseudocompact.*

PROOF: When X is pseudocompact, $C^*(X) = C(X)$. On the other hand, if X is not pseudocompact, there are $A \in [X]^\omega$ and $f \in C(X)$ such that $f \restriction A$ is unbounded. Hence $V(f, A, 1) \cap C^*(X) = \emptyset$. □

Observe that a topological space X is ω-bounded if and only if for each $A \in [X]^{\leq\omega}$ there is a compact set B with $A \subseteq B$. Thus we can use Theorem 3.4 to obtain the following.

Proposition 5.3. *For any topological space X,*

1. *$C_s(X) \leq C_k(X)$ iff X is ω-bounded,*
2. *$C_k(X) \leq C_s(X)$ iff every compact subspace of X is contained in a separable subspace of X.*

Let X be the topological sum of ω_1 copies of $\mathbb{R}$. Then X is not almost pseudo-ω-bounded and as a consequence of the previous proposition and Theorem 3.4 we obtain $C_k(X) < C_s(X) < C_u(X)$.

In this paragraph we show that there is a space Z for which $C_s(Z)$ is a topological vector space but $C_k(Z)$ and $C_s(Z)$ are incomparable. Let X be a countably compact space which is not ω-bounded (Proposition 4.6) and let Y be a compact non-separable space. Then Z, the topological sum of X and Y, is countably compact (therefore pseudocompact) and hence $C_s(X)$ is a topological vector space (Corollary 3.11). On the other hand, Z is not ω-bounded because X is a closed subspace of it; thus $C_s(Z) \not\leq C_k(Z)$. Finally, $C_k(Z) \not\leq C_s(Z)$ because Y is a compact subset of X which is not contained in any separable subspace of X.

Now we turn our attention to the analysis of some cardinal functions.

Theorem 5.4. *The following are equivalent for any space X:*

1. *$d(X) = \omega$;*
2. *$C_s(X) = C_u(X)$;*
3. *$C_s(X)$ is metrizable;*
4. *$\chi(C_s(X)) = \omega$;*
5. *$C_s(X)$ is Fréchet-Urysohn;*
6. *$C_s(X)$ is sequential.*

PROOF: The fact that (1) and (2) are equivalent is a straightforward application of Theorem 3.4. For the remaining implications we only need to invoke Theorem 3.14. □

The following result is [8, Theorem 4.2.4].

Lemma 5.5. *For any topological space X, $d(C_u(X)) = w(\beta X)$.*

Theorem 5.6. *The following conditions are equivalent for any space X:*

(1) $w(C_s(X)) = \omega$;
(2) $d(C_s(X)) = \omega$;
(3) $d(C_u(X)) = \omega$;
(4) $w(C_u(X)) = \omega$.
(5) *X is compact metrizable.*

Proof: (2) is a trivial consequence of (1).

Let D be a countable dense subset of $C_s(X)$. Seeking a contradiction let us assume that D is not dense in $C_u(X)$. There exist $f \in C(X)$ and $\varepsilon > 0$ such that $V(f, X, \varepsilon) \cap D = \emptyset$. Fix, for each $d \in D$, a point $x_d \in X$ such that $|d(x_d) - f(x_d)| \geq \varepsilon$. Since $A = \{x_d : d \in D\}$ is countable, there exists $g \in V(f, A, \varepsilon) \cap D$. In particular, $|g(x_g) - f(x_g)| < \varepsilon$. This contradiction shows that (2) implies (3).

Since $C_u(X)$ is metrizable, (3) and (4) are equivalent.

If we assume (4), then $w(X) \leq w(\beta X) = d(C_u(X)) \leq w(C_u(X)) = \omega$ (Lemma 5.5) and therefore X is metrizable. Suppose that X is not compact, i.e., that X possesses an infinite closed discrete subset $Y = \{y_n : n \in \omega\}$. Let $D = \{d_n : n \in \omega\}$ be a dense subset of $C_u(X)$. Since Y is C-embedded, the function $g : Y \to \mathbb{R}$ defined by $g(y_n) = d_n(y_n) + 1$ has an extension $f \in C(X)$. Hence $V(f, X, 1)$ and D are disjoint. This is a contradiction that shows that X is compact.

Now suppose that X is compact metrizable. Since $d(C_u(X)) = w(\beta X) = w(X) = \omega$ (Lemma 5.5) and $C_u(X)$ is metrizable, $C_u(X)$ is second countable. On the other hand, Theorem 5.4 gives $C_u(X) = C_s(X)$. So $w(C_s(X)) = \omega$. □

Given an infinite set E, a subset $S \subseteq [E]^\omega$ will be called *cofinal in* $[E]^\omega$ if for each $a \in [E]^\omega$ there is $b \in S$ such that $a \subseteq b$. The *cofinality* of $[E]^\omega$ is defined as the minimum cardinality of a cofinal subset of $[E]^\omega$ and it will be denoted by $\mathrm{cf}([E]^\omega)$. If E is countable, $\mathrm{cf}([E]^\omega) = 1$; when E is uncountable, we get $|E| \leq \mathrm{cf}([E]^\omega) \leq |E|^\omega$.

Note that the collection of all infinite initial segments in ω_1 is cofinal in $[\omega_1]^\omega$ and therefore $\mathrm{cf}([E]^\omega) = \omega_1$, whenever $|E| = \omega_1$. By finite induction one shows that the same is true when one replaces ω_1 with ω_n, $n \in \omega$, but the cofinality of $[\omega_\omega]^\omega$ cannot be decided within ZFC.

Let X be an infinite topological space. We shall denote by $\varphi(X)$ the least cardinality of an infinite family $\mathcal{S} \subseteq [X]^{\leq\omega}$ such that

$$(\dagger) \qquad \forall A \in [X]^{\leq\omega}\ \exists S \in \mathcal{S}\ (A \subseteq \overline{S}).$$

Thus, for any space X, we get (1) $\varphi(X) \leq \mathrm{cf}([X]^\omega)$ and (2) $\varphi(X) = \omega$ iff X is separable. Also, when X is discrete, $\varphi(X) = \mathrm{cf}([X]^\omega)$.

Proposition 5.7. *For any infinite topological space X,*

(1) $\mathrm{iw}(X) \leq d(C_s(X)) \leq w(\beta X)$;
(2) $\chi(C_s(X)) = \varphi(X)$; *and*
(3) $d(X) = \psi(C_s(X)) = \mathrm{iw}(C_s(X))$.

Proof: Since $C_p(X) \leq C_s(X) \leq C_u(X)$, we apply Lemma 5.5 and the equality $d(C_p(X)) = \mathrm{iw}(X)$ [1, Theorem I.1.5] to obtain $d(C_s(X)) \leq w(\beta X)$ and $\mathrm{iw}(C_s(X)) \leq d(C_s(X))$, respectively.

Let us prove (2). According to Proposition 3.12, $\kappa := \chi(C_s(X)) = \chi(\mathbf{0}, C_s(X))$ so assume that $\{V(\mathbf{0}, A_\alpha, \varepsilon_\alpha) : \alpha < \kappa\}$ is a local neighborhood base for $C_s(X)$ at $\mathbf{0}$, where $\mathcal{A} := \{A_\alpha : \alpha < \kappa\} \subseteq [X]^{\leq\omega}$ and $\{\varepsilon_\alpha : \alpha < \kappa\}$ is a collection of positive real numbers. To show that $\varphi(X) \leq \kappa$, we shall argue that $\mathcal{A}$ satisfies (†). Given $A \in [X]^{\leq\omega}$, fix $\alpha < \kappa$ in such a way that $V(\mathbf{0}, A_\alpha, \varepsilon_\alpha) \subseteq V(\mathbf{0}, A, 1)$. Note that the existence of a point $z \in A \setminus \overline{A_\alpha}$ would lead to the existence of a map $f \in C(X)$ satisfying $f[\overline{A_\alpha}] \subseteq \{0\}$ and $f(z) = 1$; in particular, $f \in V(\mathbf{0}, A_\alpha, \varepsilon_\alpha) \setminus V(\mathbf{0}, A, 1)$. Hence $A \subseteq \overline{A_\alpha}$.

To prove the remaining inequality assume that $\mathcal{S} \subseteq [X]^{\leq\omega}$ satisfies (†) and $|\mathcal{S}| = \varphi(X)$. It suffices to show that $\{V(\mathbf{0}, S, 1/n) : S \in \mathcal{S} \wedge n \in \mathbb{N}\}$ is a local neighborhood base for $C_s(X)$ at $\mathbf{0}$. Given $A \in [X]^{\leq\omega}$ and $\varepsilon > 0$, there are $S \in \mathcal{S}$ and $n \in \mathbb{N}$ such that $A \subseteq \overline{S}$ and $1/n < \varepsilon$. Thus $V(\mathbf{0}, S, 1/n) \subseteq V(\mathbf{0}, A, \varepsilon)$.

To prove (3) we will show that $\mathrm{iw}(C_s(X)) \leq d(X) \leq \psi(C_s(X)) \leq \mathrm{iw}(C_s(X))$. First, the fact $C_p(X) \leq C_s(X)$ implies that $\mathrm{iw}(C_s(X)) \leq \mathrm{iw}(C_p(X)) = d(X)$ (see [1, Theorem I.1.4]). Now set $\kappa := \psi(C_s(X))$ and let $\{U_\alpha : \alpha < \kappa\}$ be a family of open subsets of $C_s(X)$ such that $\bigcap\{U_\alpha : \alpha < \kappa\} = \{\mathbf{0}\}$ (see Proposition 3.12). Then, for each $\alpha < \kappa$, there exist $A_\alpha \in [X]^{\leq\omega}$ and $\varepsilon_\alpha > 0$ such that $V(\mathbf{0}, A_\alpha, \varepsilon_\alpha) \subseteq U_\alpha$. If $D := \bigcup\{A_\alpha : \alpha < \kappa\}$ were not dense in X, there would be a point $z \in X \setminus \overline{D}$ and therefore, for some continuous map $f : X \to \mathbb{R}$, $f(z) = 1$ and $f[\overline{D}] \subseteq \{0\}$; in particular, $f \in \bigcap\{U_\alpha : \alpha < \kappa\} \setminus \{\mathbf{0}\}$. This absurdity guarantees that D is dense in X and hence $d(X) \leq |D| \leq \psi(C_s(X))$. Finally note that the inequality $\psi(Y) \leq \mathrm{iw}(Y)$ holds for any topological space Y. □

Notice that if X is ω-bounded, then $C_s(X) \leq C_k(X)$ (Proposition 5.3) and therefore $\mathrm{iw}(X) \leq d(C_s(X)) \leq d(C_k(X)) = \mathrm{iw}(X)$ (see [8, Theorem 4.2.1] for the last equality), i.e., $d(C_s(X)) = \mathrm{iw}(X)$.

Definition 5.8. We will say that a topological space X *satisfies* $(\star)$ if the closure of any countable subset of X is C-embedded in X.

For example, if X is normal or ω-bounded, then X satisfies $(\star)$.

Let us recall that a cardinal number λ is a *caliber* for the topological space X if any family of λ nonempty open subsets of X contains a collection of size λ with nonempty intersection. If we relax the requirement and only ask for a subfamily of size λ with the finite intersection property, then λ will be called a *precaliber* for X. Therefore, if λ^+ is a precaliber for X, then $c(X) \leq \lambda$.

Proposition 5.9. *For any space X satisfying $(\star)$, we have the following:*

1. *if X is Fréchet-Urysohn, then $(2^{\mathfrak{c}})^+$ is a precaliber for $C_s(X)$ and hence $c(C_s(X)) \leq 2^{\mathfrak{c}}$ and*
2. *if $\mathcal{S} \subseteq [X]^{\leq\omega}$ satisfies (†),*

$$\sup\{w(\beta\overline{S}) : S \in \mathcal{S}\} \leq d(C_s(X)) \leq \mathfrak{c} \cdot \chi(C_s(X)).$$

PROOF: We will prove (1). Let $\theta := (2^{\mathfrak{c}})^+$ and assume that $\{U_\alpha : \alpha < \theta\}$ is a family of non-empty open subsets of $C_s(X)$ such that $U_\alpha \neq U_\beta$ whenever $\alpha < \beta < \theta$. For each $\alpha < \theta$, let $f_\alpha \in C_s(X)$, $A_\alpha \in [X]^{\leq\omega}$, and $\varepsilon_\alpha > 0$ be so that $V(f_\alpha, A_\alpha, \varepsilon_\alpha) \subseteq U_\alpha$.

Our assumption on X guarantees that each member of $\mathcal{A} := \{\overline{A_\alpha} : \alpha < \theta\}$ has size at most $\mathfrak{c}$; a straightforward application of [7, II Theorem 1.6] produces $H \in [\theta]^\theta$ and $A \subseteq X$ so that $\{\overline{A_\alpha} : \alpha \in H\}$ is a Δ-system with root A (if $|\mathcal{A}| < \theta$, there is $\gamma < \theta$ for which $H := \{\alpha < \theta : \overline{A_\alpha} = \overline{A_\gamma}\}$ has size θ). Thus $\overline{A_\alpha} \cap \overline{A_\beta} = A$, whenever α and β are distinct members of H.

Since $|\mathbb{R}^A| \leq 2^{\mathfrak{c}} < |H|$, there is $H_0 \in [H]^\theta$ in such a way that if $\alpha, \beta \in H_0$, then $f_\alpha \restriction A = f_\beta \restriction A$. We claim that $\{U_\alpha : \alpha \in H_0\}$ has the finite intersection property. Indeed, suppose that F is a finite subset of H_0 and define $f := \bigcup\{f_\alpha \restriction \overline{A_\alpha} : \alpha \in F\}$ to obtain a continuous map from $B := \bigcup\{\overline{A_\alpha} : \alpha \in F\}$ into $\mathbb{R}$; note that B is the closure of $\bigcup\{A_\alpha : \alpha \in F\}$ and use $(\star)$ to get $g \in C_s(X)$ so that $g \restriction B = f$. Hence $g \in \bigcap\{V(f_\alpha, A_\alpha, \varepsilon_\alpha) : \alpha \in F\}$.

In order to prove the inequality on the right of (2), fix a family $\mathcal{S}_0 \subseteq [X]^\omega$ of cardinality $\varphi(X)$ which satisfies (†). Let $S \in \mathcal{S}_0$ be arbitrary. There exists D_S, a dense subset of $C_s(\overline{S})$, such that $|D_S| = w(\beta\overline{S})$ (Theorem 5.4-(2) and Lemma 5.5). Observe that $\beta\overline{S}$ is separable and therefore $|D_S| \leq \mathfrak{c}$ ([2, Theorem 1.5.3]). Now, for each $g \in D_S$ let $\widehat{g} \in C_s(X)$ be so that $\widehat{g} \restriction \overline{S} = g$ (X satisfies $(\star)$).

Define $E := \{\widehat{g} : S \in \mathcal{S}_0 \wedge g \in D_S\}$. We only need to show that E is dense in $C_s(X)$ because $|E| \leq \varphi(X) \cdot \mathfrak{c}$. Given $f \in C_s(X)$, $A \in [X]^{\leq\omega}$, and $\varepsilon > 0$ let $S \in \mathcal{S}$ be so that $A \subseteq \overline{S}$. Since D_S is dense in $C_s(\overline{S})$, there is $g \in D_S$ satisfying $|g(x) - f(x)| < \varepsilon$, for all $x \in A$, and therefore $\widehat{g} \in V(f, A, \varepsilon)$.

For the remaining inequality, fix $S \in \mathcal{S}$. Let D be a dense subset of $C_s(X)$ of minimum cardinality. Theorem 5.4-(2) and Lemma 5.5 imply that $d(C_s(\overline{S})) = w(\beta\overline{S})$ so we only need to show that $\{f \restriction \overline{S} : f \in D\}$ is a dense subset of $C_s(\overline{S})$. Given $g \in C_s(\overline{S})$, $A \in [S]^{\leq\omega}$, and $\varepsilon > 0$, there is $\widehat{g} \in C(X)$ satisfying $\widehat{g} \restriction \overline{S} = g$ and therefore $f \in D \cap V(\widehat{g}, A, \varepsilon)$ implies $f \restriction \overline{S} \in V(g, A, \varepsilon)$. □

Proposition 5.10. *If X is an infinite discrete topological space, then*

(1) $d(C_s(X)) \leq \mathfrak{c} \cdot \mathrm{cf}([X]^\omega)$,
(2) $\mathfrak{c}^+$ *is a caliber for* $C_s(X)$,
(3) $c(C_s(X)) = \mathfrak{c}$,
(4) $\chi(C_s(X)) = \omega \cdot \mathrm{cf}([X]^\omega)$, *and*
(5) $\psi(C_s(X)) = \mathrm{iw}(C_s(X)) = |X|$.

PROOF: For (1) and (4), note that $\varphi(X) = \omega \cdot \mathrm{cf}([X]^\omega)$ and apply Propositions 5.7 and 5.9.

Now set $\theta := \mathfrak{c}^+$ and assume that $\{U_\alpha : \alpha < \theta\}$ is a family of non-empty open subsets of $C_s(X)$ such that $U_\alpha \neq U_\beta$ whenever $\alpha < \beta < \theta$. Proceeding as we did in the proof of Proposition 5.9-(1), fix $H_0 \in [\theta]^\theta$ and $A \in [X]^{\leq\omega}$ in such a way that for each $\alpha \in H_0$ there are $f_\alpha \in C_s(X)$, $A_\alpha \in [X]^{\leq\omega}$, and $\varepsilon_\alpha > 0$ satisfying (i) $V(f_\alpha, A_\alpha, \varepsilon_\alpha) \subseteq U_\alpha$, (ii) $\{A_\xi : \xi \in H_0\}$ is a Δ-system with root A, and (iii) $f_\alpha \restriction A = f_\beta \restriction A$, for all $\alpha, \beta \in H_0$.

Conditions (ii) and (iii) above guarantee that there is a function $g : X \to \mathbb{R}$ such that $g \upharpoonright A_\alpha = f_\alpha \upharpoonright A_\alpha$, for all $\alpha \in H_0$. Thus condition (i) gives $g \in \bigcap\{U_\alpha : \alpha \in H_0\}$ and this completes the proof of (2).

An immediate corollary of (2) is $c(C_s(X)) \leq \mathfrak{c}$. For the remaining inequality: let Y be an infinite countable subset of X and for each $A \subseteq Y$ let $\chi_A : X \to \{0,1\}$ be the characteristic function of A. Then $\{\text{int}(V(\chi_A, Y, 1/2)) : A \subseteq Y\}$ is a cellular family in $C_s(X)$ of size $\mathfrak{c}$.

(5) is a consequence of Proposition 5.7(3). □

The bounds for the density given in Proposition 5.7-(1) are not optimal: if X is the discrete space of size $\mathfrak{c}$, then $\mathfrak{c} = c(C_s(X)) \leq d(C_s(X)) \leq \mathfrak{c} \cdot \text{cf}([X]^\omega) = \mathfrak{c}$. On the other hand, $\text{iw}(X) = d(C_p(X)) = d(\mathbb{R}^\mathfrak{c}) = \omega$ (see [1, Theorem I.1.5]) and therefore $\text{iw}(X) < d(C_s(X)) < w(\beta X)$.

Corollary 5.11. *If $c(C_s(X)) < \mathfrak{c}$, then X is pseudocompact.*

PROOF: Assume that X is not pseudocompact and let Y be a countable discrete subspace of X which is C-embedded in X. Since the restriction map $\pi_Y : C_s(X) \to C_s(Y)$ is continuous and onto (Proposition 3.6 and the fact that Y is C-embedded), we obtain $\mathfrak{c} = c(C_s(Y)) \leq c(C_s(X))$. □

Theorem 5.12. *Let κ be a regular uncountable cardinal. If $X = [0,\kappa)$ or $X = [0,\kappa]$, then*

(1) $d(C_s(X)) = \text{iw}(C_s(X)) = \kappa$,
(2) $w(C_s(X)) = \chi(C_s(X))$, *and*
(3) $e(C_s(X)) \geq \kappa$.

PROOF: Our assumptions on κ imply that X is ω-bounded and $\beta X = [0,\kappa]$. Therefore (see Proposition 5.7 and the paragraph after it)

$$\text{iw}(X) = d(C_s(X)) \leq w(\beta X) = \kappa.$$

Hence we only need to show that $\kappa \leq d(C_s(X))$ to complete the proof of (1).

Let $Y \in [C_s(X)]^{<\kappa}$ be arbitrary. To prove that Y is not dense in $C_s(X)$ start by noticing that for each $f \in Y$ there is $\alpha_f < \kappa$ such that $f \upharpoonright (X \setminus [0,\alpha_f))$ is constant. Now set $\beta := \sup\{\alpha_f : f \in Y\}$ and observe that $V(\chi_{[0,\beta]}, \{\beta, \beta+1\}, 1/2)$ is disjoint from Y.

According to Propositions 3.12 and 5.7, to prove (2) it suffices to show that $\varphi(X) \geq \kappa$; so assume that $\mathcal{S} \subseteq [X]^{\leq\omega}$ satisfies $|\mathcal{S}| < \kappa$. Then $\beta := \sup \bigcup \mathcal{S} < \kappa$ and, in particular, $\beta + 1 \notin \overline{S}$, for all $S \in \mathcal{S}$.

For each $\alpha < \kappa$, define $f_\alpha := \chi_{[0,\alpha]}$ and set $Y := \{f_\xi : \xi < \kappa\}$. Note that, for each $\alpha < \kappa$, $V(f_\alpha, \{\alpha, \alpha+1\}, 1/2) \cap Y = \{f_\alpha\}$ and therefore (3) will be proved if we show that Y is closed.

Fix $g \in C_s(X) \setminus Y$. If there is $\beta \in X$ with $g(\beta) \notin \{0,1\}$, then Y is disjoint from $V(g, \{\beta\}, \min\{|g(\beta)|, |1 - g(\beta)|\})$; so let us assume that $g : X \to \{0,1\}$. Set $\delta := \min\{\xi < \kappa : g \upharpoonright (X \setminus [0,\xi)) \text{ is constant}\}$. If $g(\delta) \neq 1$, then $V(g, \{\delta\}, |1 - g(\delta)|)$ is disjoint from Y so suppose that $g(\delta) = 1$. Since $g \neq f_\delta$, there is $\beta < \delta$ with

$g(\beta) = 1$. The way δ was chosen guarantees that $g(\gamma) = 0$ for some $\beta < \gamma < \delta$ and therefore $V(g, \{\beta, \gamma\}, 1/2) \cap Y = \emptyset$. □

6. Lindelöf property in $C_s(X)$

Since Lindelöf's property and separability are equivalent for metric spaces, the following result is a consequence of Theorem 5.4.

Proposition 6.1. *If X is a separable space, then $C_s(X)$ is Lindelöf iff $C_s(X)$ is separable.*

Theorem 6.2. *If $C_s(X)$ is Lindelöf, then X is pseudocompact.*

Proof: Assume that X is not pseudocompact. Then there is a discrete family $\mathcal{U} = \{U_n : n \in \omega\}$ of nonempty open subsets of X. For each $n \in \omega$ let $x_n \in U_n$. We will show that if $A := \{x_n : n \in \omega\}$ then $\{V(f, A, 1) : f \in C(X)\}$ has no countable subcover.

Let $\{f_n : n \in \omega\} \subseteq C(X)$ be arbitrary. For each $n \in \omega$ let $g_n \in C(X)$ be so that $|g_n(x_n) - f_n(x_n)| \geq 1$ and $g_n[X \setminus U_n] \subseteq \{0\}$. Since $\mathcal{U}$ is discrete, $g := \sum_{n \in \omega} g_n$ is a continuous function; moreover, for any $n \in \omega$, $g(x_n) = g_n(x_n)$ and therefore $g \notin V(f_n, A, 1)$. □

Note that if $C_s(X)$ is Lindelöf, then so is $C_p(X)$. The converse fails: Zenor-Veličko's theorem [1, Theorem II.5.10] implies that $C_p(\mathbb{R})$ is hereditarily Lindelöf but, according to Theorem 6.2, $C_s(\mathbb{R})$ is not Lindelöf.

Lemma 6.3. *Let X be such that $C_s(X)$ is Lindelöf. If A is a countable subset of X such that $\overline{A}$ is C-embedded in X, then $\overline{A}$ is compact metrizable.*

Proof: Define $Y = \overline{A}$ and let $\pi_Y : C_s(X) \to C_s(Y)$ be the restriction map. Note that π_Y is continuous (Proposition 3.6) and onto because Y is C-embedded in X. Hence $C_s(Y)$ is Lindelöf and metrizable (Theorem 5.4); in particular, $C_s(Y)$ is separable and thus Y is compact metrizable (Theorem 5.6). □

To simplify writing, we will say that a topological space *is* ckm if the closure of any countable subset of it is compact metrizable.

Proposition 6.4. *The property of being* ckm *is countably productive, preserved under continuous mappings, and weakly hereditary.*

Proof: Let $\{X_n : n \in \omega\}$ be a family of ckm spaces and set $X := \prod_n X_n$. If A is a countable subset of X, then we can project A into X_n, for each $n \in \omega$, to obtain K_n, a compact metrizable subspace of X_n, in such a way that $A \subseteq \prod_n K_n$. Therefore, $\overline{A}$ is compact metrizable.

Assume that X is ckm and that $f : X \to Y$ is continuous and onto. If B is a countable subset of Y, there is a countable set $A \subseteq X$ for which $f[A] = B$. Hence $\overline{A}$ is compact metrizable and therefore $f[\overline{A}]$ is a compact metrizable subspace of Y ([2, Theorem 3.1.22]) which contains the closure of B.

It is immediate to prove that any closed subspace of a ckm space is ckm. □

Proposition 6.5. *If X satisfies $(\star)$ and $C_s(X)$ is Lindelöf, then X is* ckm *and all its finite powers are Fréchet-Urysohn.*

Proof: The fact that X is ckm is a consequence of Lemma 6.3. To prove the second assertion: let $n \in \omega$, $A \subseteq X^n$, and $x \in \overline{A}$ be arbitrary. Our assumption on $C_s(X)$ implies that $C_p(X)$ is Lindelöf so, by Asanov's theorem [1, Theorem I.4.1], X^n has countable tightness. Thus there is a countable set $B \subseteq A$ with $x \in \overline{B}$. Since $\overline{B}$ is metrizable, there is a sequence in B converging to x. □

A result of Nakhmanson [1, Theorem IV.10.1] establishes that $L(C_p(X)) = w(X)$ for any linearly ordered compactum X; therefore if X is an Aronszajn continuum (see [10, Section 3]), then X is ckm and $\chi(X) = \omega$, but $C_s(X)$ is not Lindelöf.

Corollary 6.6. *If $C_s(X)$ is Lindelöf, then X is normal iff X is ω-bounded.*

Proof: Proposition 6.5 shows that, under our assumptions, normal implies ω-bounded and since $C_p(X)$ is Lindelöf, the reverse implication is [1, Corollary I.4.14]. □

Given a set E, a collection $C \subseteq [E]^\omega$ is called a *club in* $[E]^\omega$ (see [5, Section 8]) if C is cofinal in $[E]^\omega$ and for each increasing sequence $\{a_n : n \in \omega\} \subseteq C$ we get $\bigcup_n a_n \in C$.

Definition 6.7. Assume that S is the Σ-product of the family of topological spaces $\{M_\alpha : \alpha < \kappa\}$ about the point z.

(1) For each $a \subseteq \kappa$, the *natural retraction* $r_a : S \to S$ is defined by

$$r_a(x) := (x \restriction a) \cup (z \restriction (\kappa \setminus a)).$$

(2) A subspace $X \subseteq S$ will be called ω-*invariant* if there is a club C in $[\kappa]^\omega$ such that $r_a[X] \subseteq X$, for all $a \in C$.

The proof of [3, Lemma 1] shows that the following is true.

Remark 6.8. Any closed subspace of a Σ-product of separable metric spaces is ω-invariant.

We also have the following.

Proposition 6.9. *If S is as in Definition* 6.7, *then the union of countably many ω-invariant subspaces of S is ω-invariant too.*

Proof: For each $n \in \omega$ let X_n be an ω-invariant subspace of S as witnessed by the club $C_n \subseteq [\kappa]^\omega$. Then $C := \bigcap_n C_n$ is a club (see [5, Theorem 8.3]) and if $a \in C$, then $r_a[X_n] \subseteq X_n$, for all $n \in \omega$, so $r_a[\bigcup_n X_n] \subseteq \bigcup_n X_n$. □

Definition 6.10. Let η be a nonempty family of subsets of a topological space X. We will denote by $C_\eta(X)$ the space that results of endowing $C(X)$ with the topology which has the collection of all sets of the form $\{f \in C(X) : f[S] \subseteq U\}$, where U is an open subset of $\mathbb{R}$ and $S \in \eta$, as a subbase.

It should be noticed that if X is ckm and we let $\alpha := \{\overline{A} : A \in [X]^{\leq\omega}\}$, then Lemma 3.3 and [6, Theorem 1.2.3] imply that $C_s(X) = C_\alpha(X)$.

Remark 6.11. If $C_k(X)$ is Lindelöf, then all compact subspaces of X are metrizable. Indeed, suppose that K is a compact subset of X. Thus $C_k(K) = C_u(K)$ (Proposition 3.4) and since $\pi_K : C_k(X) \to C_k(K)$, the restriction map, is continuous and onto (Proposition 3.6 and [2, Exercise 3.2.J]), we have that $C_u(K)$ is Lindelöf. Therefore K is metrizable (Theorem 5.6).

Theorem 6.12. *Let X be an ω-invariant subspace of a Σ-product of separable metric spaces. Then:*

(1) *$C_k(X)$ is Lindelöf iff each compact subspace of X is metrizable and*
(2) *if X is ckm, then $C_s(X)$ is Lindelöf.*

PROOF: The direct implication in (1) is Remark 6.11. For the converse more tools are needed.

Let us denote by α the collection of all compact metrizable subspaces of X. A minor modification of the proof of [3, Lemma 2] shows that $C_\alpha(X)$ (recall Definition 6.10) is Lindelöf. On the other hand, [8, Theorem 1.2.3] implies that $C_{\alpha,u}(X) = C_\alpha(X)$ and therefore $C_{\alpha,u}(X)$ is Lindelöf.

To complete the proof of (1) note that when all compact subsets of X are metrizable, one has $C_k(X) = C_{\alpha,u}(X)$.

Finally, if X is ckm, Theorem 3.4 gives $C_s(X) \leq C_{\alpha,u}(X)$ and so (2) is proved. □

Corollary 6.13. *If X is a Corson compactum (i.e., a compact subspace of a Σ-product of separable metric spaces), then $C_s(X)$ is Lindelöf.*

PROOF: If A is an arbitrary subset of X, then $\overline{A}$ is a Corson compactum and therefore $w(\overline{A}) = d(\overline{A})$ (see, for example, [11, Corollary on p. 158]). So when A is countable, $\overline{A}$ is compact and second countable, i.e., $\overline{A}$ is compact metrizable. This suffices in view of Remark 6.8. □

Corollary 6.14. *Let Z be the topological product of a family of compact metric spaces. If X is a closed subspace of Z which is contained in a Σ-product of Z, then all finite powers of $C_s(X)$ are Lindelöf.*

PROOF: Let $n \in \mathbb{N}$ be arbitrary. A routine argument shows that, in general, $C_s(X)^n$ is homeomorphic to $C_s(X \times n)$ (note that $X \times n$ is the topological sum of n copies of X). Thus, if X is as in the statement of our corollary, $X \times n$ is a closed subspace of $Z \times n$ which is contained in a Σ-product of $Z \times n$. Hence it suffices to prove that $C_s(X)$ itself is Lindelöf.

Assume that $\{M_\alpha : \alpha < \kappa\}$ is the family of compact metrizable spaces whose product is Z. We will use Theorem 6.12, so let $A \in [X]^{\leq\omega}$ be arbitrary. For each $a \in A$ let $s(a)$ be its support. Then $N := \bigcup\{s(a) : a \in A\}$ is a countable subset of κ; moreover, $\mathrm{cl}_X(A)$ is homeomorphic to a subspace of the product $\prod_{\alpha\in N} M_\alpha$ and therefore it is metrizable. On the other hand, $\mathrm{cl}_X(A)$ is clearly a compact subspace of Z. □

Proposition 6.15. *If X is a dense subspace of Y, then $L(C_s(\beta Y)) \leq L(C_s(\beta X))$.*

Proof: Our assumption on X and Y implies that there is a continuous map f from βX onto βY. f is a quotient map because f is closed ([2, Corollary 2.4.8]). Moreover, Proposition 3.6 applies and therefore $C_s(\beta Y)$ is homeomorphic to a closed subspace of $C_s(\beta X)$. □

Note that if X and Y are as in the previous proposition and Y is pseudocompact, then the restriction map $\pi_Y : C_s(\beta Y) \to C_s(Y)$ is continuous and onto. Hence $L(C_s(Y)) \leq L(C_s(\beta X))$.

The following result is probably well-known.

Lemma 6.16. *If $w(Y) = \omega$, then $hL(X \times Y) = hL(X)$ for any space X.*

Proof: Set $\kappa := hL(X)$. Fix $\mathcal{B} = \{B_n : n \in \omega\}$, a base for Y, and assume that Z is a subspace of $X \times Y$. In order to prove that $L(Z) \leq \kappa$ it suffices to show that if $\mathcal{U} = \{V_\xi \times W_\xi : \xi < \theta\}$ covers Z, where, for each $\xi < \theta$, V_ξ is an open subset of X and $W_\xi \in \mathcal{B}$, then $\mathcal{U}$ has a subset of size $\leq \kappa$ which covers Z.

Let $n \in \omega$. Define $S_n := \{\xi < \theta : W_\xi = B_n\}$ and set $X_n := \bigcup\{V_\xi : \xi \in S_n\}$. Thus there is $H_n \in [S_n]^{\leq\kappa}$ in such a way that $\{V_\xi : \xi \in H_n\}$ covers X_n.

Now note that $\{V_\xi \times B_n : n \in \omega \wedge \xi \in H_n\} \subseteq \mathcal{U}$ has size at most κ and covers Z.

On the other hand, X embeds as a closed subspace of $X \times Y$ and therefore $hL(X \times Y) \geq \kappa$. □

Let us note that if $\alpha < \omega_1$, then $[0, \alpha]$ is compact metrizable and therefore Theorem 5.6 implies that $hL(C_s([0, \alpha])) = \omega$.

Proposition 6.17. $hL(C_s([0, \omega_1])) = \omega_1$.

Proof: Let $X := [0, \omega_1]$. For each $\alpha < \omega_1$ define the following subspace of $C_s(X)$,

$$Y_\alpha := \{f \in C(X) : f \restriction (\alpha, \omega_1] \text{ is constant}\}.$$

Then the map $h : Y_\alpha \to C_s([0, \alpha]) \times \mathbb{R}$ given by $h(f) = (f \restriction [0, \alpha], f(\alpha + 1))$ is a homeomorphism, so Lemma 6.16 gives $hL(Y_\alpha) = \omega$.

Since $C_s(X) = \bigcup\{Y_\alpha : \alpha < \omega_1\}$, we obtain $hL(C_s(X)) \leq \omega_1$. The other inequality follows from the fact that $C_s(X)$ is not Lindelöf (Proposition 6.5). □

7. Questions

In the following questions X represents an arbitrary Tychonoff space.

(1) Does the inequality $w(X) \leq d(C_s(X))$ always hold?
(2) Assuming that X is normal, Fréchet-Urysohn and ckm, is there an upper bound for the Lindelöf degree of $C_s(X)$ which does not depend on X (e.g. $L(C_s(X)) \leq \omega_1$)?
(3) Does there exist an X satisfying the hypotheses of Proposition 5.9 for which $c(C_s(X)) = 2^{\mathfrak{c}}$?
(4) Does there exist a Frolík space which is not almost pseudo-ω-bounded?
(5) Does the equality $L(C_s(X)) = \omega$ imply any of the following statements?

(a) X is Frolík.
(b) X is a k-space.
(c) X is almost pseudo-ω-bounded.

(6) Does the equality $d(C_s(X)) = L(C_s(X))$ always hold?

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