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Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 3, 583–588

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A NOTE ON THE KERNELS OF HIGHER DERIVATIONS

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(Received February 21, 2012)

Abstract. Let $k \subseteq k'$ be a field extension. We give relations between the kernels of higher derivations on $k[X]$ and $k'[X]$, where $k[X] := k[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables over the field k. More precisely, let $D = \{D_n\}_{n=0}^{\infty}$ a higher k-derivation on $k[X]$ and $D' = \{D'_n\}_{n=0}^{\infty}$ a higher k'-derivation on $k'[X]$ such that $D'_m(x_i) = D_m(x_i)$ for all $m \geqslant 0$ and $i = 1, 2, \ldots, n$. Then (1) $k[X]^D = k$ if and only if $k'[X]^{D'} = k'$; (2) $k[X]^D$ is a finitely generated k-algebra if and only if $k'[X]^{D'}$ is a finitely generated k'-algebra. Furthermore, we also show that the kernel $k[X]^D$ of a higher derivation D of $k[X]$ can be generated by a set of closed polynomials.

Keywords: higher derivation; field extension; closed polynomial

MSC 2010: 13A50

1. INTRODUCTION

Throughout this paper, $k[X] = k[x_1, x_2, \ldots, x_n]$ is the polynomial ring in n variables over a field k.

Let R be a commutative ring and A an R-algebra. Recall that a set of R-linear endomorphisms $D = \{D_n\}_{n=0}^{\infty}$ is called a *higher R-derivation* on A if D satisfies the following conditions:

(1) D_0 is the identity map of A;
(2) $D_n(ab) = \sum\limits_{i+j=n} D_i(a)D_j(b)$ for all $n \geqslant 0$ and for any $a, b \in A$.

A higher R-derivation D is called locally finite if for any $a \in A$, there exists $n \in \mathbb{N}$ such that $D_m(a) = 0$ for any integer $m \geqslant n$. And D is called iterative if for any

This research was supported by NSF of China (No. 11071097, No. 11101176) and "211 Project" and "985 Project" of Jilin University.

$i, j \in \mathbb{N}$, $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$. For a higher R-derivation D on A, A^D is the kernel of D defined by $A^D = \bigcap_{n \geqslant 1} \mathrm{Ker} D_n$.

Higher derivations and their kernels play an important role when we deal with some curves and affine surfaces. For example, a G_a-action on an affine scheme $X = \mathrm{Spec(A)}$ can be interpreted in terms of a locally finite iterative higher derivation on A, and many things become easier to treat by observing the locally finite iterative higher derivation on A associated with the G_a-action [4]. Recently, in [2], [7], [8] the kernels of higher derivations have been studied. In [3], the structure of higher derivations is studied. In this short note, we prove the following results.

(1) Let $k \subseteq k'$ be a field extension. Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation on the polynomial ring $k[X]$ over a field k and let $D' = \{D'_n\}_{n=0}^{\infty}$ be a higher k'-derivation on $k'[X]$ such that $D'_m(x_i) = D_m(x_i)$ for all $m \geqslant 0$ and $i = 1, 2, \ldots, n$. Then

(i) $k[X]^D = k$ if and only if $k'[X]^{D'} = k'$;

(ii) $k[X]^D$ is a finitely generated k-algebra if and only if $k'[X]^{D'}$ is a finitely generated k'-algebra.

(2) Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation of $k[X]$. Then there exists a set S of closed polynomials in $k[X]$ such that $k[X]^D = k[S]$.

In the case of derivations on $k[X]$, similar results can be found in [5].

2. Higher derivations under extension of fields

Let $k \subseteq k'$ be a field extension and A a k-algebra. Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation on A. Consider the tensor product $A \otimes_k k'$, then $A \otimes_k k'$ is a k'-algebra. Consider the set of k'-linear mappings $D \otimes 1$ of $A \otimes_k k'$ given by

$$D \otimes 1 = \{D_n \otimes 1\colon\ A \otimes_k k' \to A \otimes_k k'\}_{n=0}^{\infty}.$$

Proposition 2.1. *The set of k'-linear mappings $D \otimes 1$ is a higher k'-derivation on the k'-algebra $A \otimes_k k'$. Moreover, if D is locally finite or iterative, then $D \otimes 1$ is also locally finite or iterative, respectively.*

Proof. It is obvious that $D_0 \otimes 1$ is the identity map of $A \otimes_k k'$.

For any $a \otimes p,\ b \otimes q \in A \otimes_k k'$,

$$\begin{aligned}(D_n \otimes 1)((a \otimes p) \cdot (b \otimes q)) &= (D_n \otimes 1)(ab \otimes pq) = D_n(ab) \otimes pq \\ &= \Big(\sum_{i+j=n} D_i(a) D_j(b) \Big) \otimes pq = \sum_{i+j=n} D_i(a) D_j(b) \otimes pq \\ &= \sum_{i+j=n} ((D_i(a) \otimes p) \cdot (D_j(b) \otimes q)) \\ &= \sum_{i+j=n} (D_i \otimes 1)(a \otimes p) \cdot (D_j \otimes 1)(b \otimes q).\end{aligned}$$

Since $D_i \otimes 1$ are k'-linear mappings of $A \otimes_k k'$, $i = 0, 1, 2, \ldots$, it follows from the above equation that $(D_n \otimes 1)(xy) = \sum\limits_{i+j=n} (D_i \otimes 1)(x) \cdot (D_j \otimes 1)(y)$ for any $x = \sum\limits_i a_i \otimes p_i$, $y = \sum\limits_j b_j \otimes q_j \in A \otimes_k k'$, where $a_i, b_j \in A$ and $p_i, q_j \in k'$. Thus, $D \otimes 1$ is a higher k'-derivation on $A \otimes_k k'$.

Moreover, D is locally finite, that is, for any $a \in A$ there exists $n \in \mathbb{N}$ such that $D_m(a) = 0$ for any $m \geqslant n$. Define $\nu_D(a) = n$ where n is the least integer such that $D_{n-1}(a) \neq 0$ and $D_m(a) = 0$ for any $m \geqslant n$. Note that any element of $A \otimes_k k'$ is a finite sum of the form $\sum\limits_i a_i \otimes p_i$, where $a_i \in A$, $p_i \in k'$. Set $N = \max\limits_i \{\nu_D(a_i)\}$. Then for any $m \geqslant N$,

$$(D_m \otimes 1)\Big(\sum_i a_i \otimes p_i\Big) = \sum_i (D_m \otimes 1)(a_i \otimes p_i) = \sum_i D_m(a_i) \otimes p_i = \sum_i 0 \otimes p_i = 0.$$

Thus, $D \otimes 1$ is a locally finite higher k'-derivation.

If D is iterative, that is, $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ for any $i, j \in \mathbb{N}$, then

$$(D_i \otimes 1) \circ (D_j \otimes 1) = (D_i \circ D_j) \otimes 1 = \left(\binom{i+j}{i} D_{i+j} \right) \otimes 1 = \binom{i+j}{i} D_{i+j} \otimes 1.$$

Thus, $D \otimes 1$ is an iterative higher k'-derivation. □

The kernel of $D \otimes 1$ is closely related to the kernel of D.

Proposition 2.2. *The k'-algebras $A^D \otimes_k k'$ and $(A \otimes_k k')^{D \otimes 1}$ are isomorphic.*

P r o o f. It is obvious that k' is flat as a k-algebra. Then the exact sequences

$$0 \longrightarrow \mathrm{Ker} D_i \xrightarrow{\mathrm{id}} A \xrightarrow{D_i} A, \quad i = 1, 2, \ldots.$$

of k-vector spaces induce the following exact sequences of k'-vector spaces:

$$0 \longrightarrow \mathrm{Ker} D_i \otimes_k k' \xrightarrow{\mathrm{id} \otimes 1} A \otimes_k k' \xrightarrow{D_i \otimes 1} A \otimes_k k', \quad i = 1, 2, \ldots.$$

Thus,

$$\mathrm{Ker}D_i \otimes_k k' \cong \mathrm{Im}\,\mathrm{id} \otimes 1 \cong \mathrm{Ker}D_i \otimes 1.$$

Note that if U, V are vector spaces and U_1, U_2 are subspaces of U, then it is known (see for instance [6, Chapter 14: Tensor Products]) that

$$(U_1 \otimes V) \cap (U_2 \otimes V) \cong (U_1 \cap U_2) \otimes V.$$

Therefore,

$$\begin{aligned}(A \otimes_k k')^{D\otimes 1} &= \bigcap_{i\geqslant 1} \mathrm{Ker}D_i \otimes 1 \cong \bigcap_{i\geqslant 1} \mathrm{Ker}D_i \otimes_k k' \cong \Big(\bigcap_{i\geqslant 1} \mathrm{Ker}D_i\Big) \otimes_k k' \\ &= A^D \otimes_k k'.\end{aligned}$$

□

Lemma 2.3 ([5]). *A is a finitely generated algebra over k if and only if $A \otimes_k k'$ is a finitely generated algebra over k'.*

As a direct corollary of Proposition 2.2 and Lemma 2.3, we have

Corollary 2.4. *Let $k \subseteq k'$ be a field extension and let $D = \{D_n\}_{n=0}^{\infty}$ and $D \otimes 1 = \{D_n \otimes 1\}_{n=0}^{\infty}$ be higher derivations on A and $A \otimes_k k'$, respectively. Then*

(1) *$A^D = k$ if and only if $(A \otimes_k k')^{D\otimes 1} = k'$;*
(2) *A^D is a finitely generated k-algebra if and only if $(A \otimes_k k')^{D\otimes 1}$ is a finitely generated k'-algebra.*

In most cases, we deal with higher derivations on polynomial rings. In that case, we can get the following theorem.

Theorem 2.5. *Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation on the polynomial ring $k[X]$ over a field k. Let $k \subseteq k'$ be a field extension and $D' = \{D'_n\}_{n=0}^{\infty}$ a higher k'-derivation on $k'[X]$ such that $D'_m(x_i) = D_m(x_i)$ for all $m \geqslant 0$ and $i = 1, 2, \ldots, n$. Then*

(1) *$k[X]^D = k$ if and only if $k'[X]^{D'} = k'$;*
(2) *$k[X]^D$ is a finitely generated k-algebra if and only if $k'[X]^{D'}$ is a finitely generated k'-algebra.*

3. Kernels of higher k-derivations on $k[X]$

Recall that a polynomial $f \in k[X] \setminus k$ is called closed if the subalgebra $k[f]$ is a maximal element in the family $\mathfrak{L}$ of subalgebras defined by $\mathfrak{L} = \{k[f] \colon f \in k[X] \setminus k\}$. It is known that f is a closed polynomial if and only if $k[f]$ is integrally closed in $k[X]$. Define a generative polynomial h_f of a polynomial $f \in k[X] \setminus k$ as a closed polynomial such that $f = F(h_f)$ for some $F \in k[t]$. Such a generative polynomial h_f is unique up to affine transformations [1]. In [5], it is shown that if d is a derivation of $k[X]$ and $f \in k[X]^d$ is a nonconstant polynomial, then the generative polynomial h_f of f also belongs to $k[X]^d$, and hence, there is a set S of closed polynomials such that $k[X]^d = k[S]$. Next we generalize such results to the case of higher derivations.

For a higher R-derivation $D = \{D_n\}_{n=0}^{\infty}$ on an R-algebra A we can define a mapping φ_D from A to $A[[t]]$ (here, $A[[t]]$ is the formal power series ring in one variable t over A) associated with D by

$$\varphi_D(a) = \sum_{i \geqslant 0} D_i(a) t^i \quad \text{for any } a \in A.$$

Lemma 3.1 ([2]). *Let $D = \{D_n\}_{n=0}^{\infty}$ be a set of R-endomorphisms of the R-algebra A where D_0 is the identity map. Then the following statements are equivalent.*

(1) *D is a higher R-derivation on A;*

(2) *φ_D is a homomorphism of the R-algebras A and $A[[t]]$.*

Moreover, D is a locally finite higher R-derivation on A if and only if φ_D is a homomorphism of the R-algebras A and $A[t]$.

It is clear that for any $a \in A$, $a \in A^D$ if and only if $\varphi_D(a) = a$.

We are now in a position to show the main result of this section.

Theorem 3.2. *Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher k-derivation of $k[X]$. Then there exists a set S of closed polynomials in $k[X]$ such that $k[X]^D = k[S]$.*

P r o o f. If $k[X]^D = k$, then put $S = \emptyset$. Now assume that $k[X]^D \neq k$. Choose $f \in k[X]^D \setminus k$ and let h_f be the generative polynomial of f. Then h_f is a closed polynomial and there exists $F(x) = a_n x^n + \ldots + a_1 x + a_0 \in k[x]$ such that $f = F(h_f)$, where $a_i \in k$ and $a_n \neq 0$, and we can assume that F has the minimal degree. Thus,

$$f = a_n h_f^n + \ldots + a_1 h_f + a_0.$$

Applying φ_D to the above equation, we obtain

$$\varphi_D(f) = a_n \varphi_D(h_f)^n + \ldots + a_1 \varphi_D(h_f) + a_0.$$

Since $f \in k[X]^D$, it follows that $\varphi_D(f) = f$. Thus,

$$f = a_n \varphi_D(h_f)^n + \ldots + a_1 \varphi_D(h_f) + a_0.$$

If $\varphi_D(h_f) \neq h_f$, then $\varphi_D(h_f) = \sum_{i \geqslant 0} D_i(h_f) t^i$ is algebraically independent over $k[X]$, even over the field of rational functions $k(X)$. Hence $\varphi_D(h_f)$ does not satisfy a non-trivial algebraic equation over $k[X]$. This contradiction forces $\varphi_D(h_f) = h_f$. Thus, $h_f \in k[X]^D$. Therefore, it is proved that if $f \in k[X]^d$, then the generative polynomial h_f of f also belongs to $k[X]^d$.

Set $S = \{h_f \colon f \in k[X]^D \setminus k\}$. Then S is clearly a set of closed polynomials in $k[X]$ and $k[X]^D = k[S]$. □

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