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A NOTE ON WEAKLY (μ, λ)-CLOSED FUNCTIONS

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Abstract. In this paper we introduce a new class of functions called weakly (μ, λ)-closed functions with the help of generalized topology which was introduced by Á. Császár. Several characterizations and some basic properties of such functions are obtained. The connections between these functions and some other similar types of functions are given. Finally some comparisons between different weakly closed functions are discussed. This weakly (μ, λ)-closed functions enable us to facilitate the formulation of certain unified theories for different weaker forms of closed functions.

Keywords: μ-open set; weakly (μ, λ)-closed function; contra (μ, λ)-open function; strongly (μ, λ)-continuous function

MSC 2010: 54A05, 54C10

1. Introduction

The notion of generalized topology was first introduced by Á. Császár. After its introduction during the last ten years or so this area has been rapidly growing. Á. Császár had published a series of papers introducing and studying generalized topology, generalized neighbourhood systems and generalized continuity and had shown that the fundamental definitions and major part of many statements and constructions in set topology can be formulated by replacing topology with the help of generalized topology. On the other hand the notion of weakly closed functions in topological spaces was introduced by Rose and Janković. Different weak forms of closed functions have been introduced and studied in [9], [12], [13] by replacing closed sets by various weaker forms of closed sets. In this paper, in order to unify several characterizations and properties of the weak forms of closed functions we introduce a new type of function termed as weakly (μ, λ)-closed function. We obtain several characterizations and properties of such functions.

We now recall some notions defined [2]. Let X be a nonempty set and $\exp X$ be the power set of X. We call a class $\mu \subseteqq \exp X$ a generalized topology [2] (briefly, GT) if $\emptyset \in \mu$ and unions of elements of μ belong to μ. A set X with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). A GT μ is said to be a quasi topology (briefly QT) [6] if $M, M^{'} \in \mu$ implies $M \cap M^{'} \in \mu$. The pair (X, μ) is said to be a QTS if μ is a QT on X. For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteqq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [3], [5]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma\colon \exp X \to \exp X$ is said to be idempotent iff for each $A \subseteqq X$, $\gamma(\gamma(A)) = \gamma(A)$, and monotonic iff $\gamma(A) \subseteqq \gamma(B)$ whenever $A \subseteqq B \subseteqq X$. It is also well known from [5], [3] that if μ is a GT on X and $A \subseteqq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and that $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

A subset A of a topological space (X, τ) is called α-open [11] (resp. regular open [15]) if $A \subseteqq \operatorname{int}(\operatorname{cl}(\operatorname{int}(A)))$ (resp. $A = \operatorname{int}(\operatorname{cl}(A))$). The complement of an α-open set is called α-closed. The α-interior of A is the union of all α-open sets contained in A and is denoted by $\operatorname{int}_\alpha(A)$. It is known from [11] that the family of all α-open sets forms a topology τ_α larger than τ. It is also known that the collection of all regular open sets forms a base for a coarser topology τ_s than the original one τ.

2. Weakly (μ, λ)-closed function

Definition 2.1. Let (X, μ) be a GTS. The $\mu(\theta)$-closure [5], [10] (resp. $\mu(\theta)$-interior [5], [10]) of a subset A in (X, μ) is denoted by $c_{\mu(\theta)}(A)$ (resp. $i_{\mu(\theta)}(A)$) and is defined to be the set $\{x \in X\colon\ c_\mu(U) \cap A \neq \emptyset$ for each $U \in \mu(x)\}$ (resp. $\{x \in X\colon$ there exists $U \in \mu(x)$ such that $c_\mu(U) \subseteqq A\}$), where $\mu(x) = \{U \in \mu\colon\ x \in U\}$.

We define a subset A of X to be a $\mu(\theta)$-closed if $c_{\mu(\theta)}(A) = A$.

Theorem 2.2 ([10])**.** *In a GTS (X, μ), $X \setminus c_{\mu(\theta)}(A) = i_{\mu(\theta)}(X \setminus A)$ and $X \setminus i_{\mu(\theta)}(A) = c_{\mu(\theta)}(X \setminus A)$.*

Definition 2.3. Let (X, μ) and (Y, λ) be two GTS's. A function $f\colon (X, \mu) \to (Y, \lambda)$ is said to be weakly (μ, λ) closed if for each μ-closed set F of X, $c_\lambda(f(i_\mu(F))) \subseteqq f(F)$.

Theorem 2.4. *For a function $f\colon (X,\mu) \to (Y,\lambda)$ the following properties are equivalent:*

(a) *f is weakly (μ,λ)-closed.*

(b) *$c_\lambda(f(U)) \subseteqq f(c_\mu(U))$ for each $U \in \mu$.*

(c) *For each subset F of Y and each $U \in \mu$ with $f^{-1}(F) \subseteqq U$, there exists $V \in \lambda$ such that $F \subseteqq V$ and $f^{-1}(V) \subseteqq c_\mu(U)$.*

(d) *For each $y \in Y$ and each $U \in \mu$ with $f^{-1}(y) \subseteqq U$, there exists $V \in \lambda$ containing y such that $f^{-1}(V) \subseteqq c_\mu(U)$.*

(e) *$c_\lambda(f(i_\mu(c_\mu(A)))) \subseteqq f(c_\mu(A))$ for any subset A of X.*

(f) *$c_\lambda(f(i_\mu(c_{\mu(\theta)}(A)))) \subseteqq f(c_{\mu(\theta)}(A))$ for any subset A of X.*

Proof. (a) $\Rightarrow$ (b): Let f be weakly (μ,λ)-closed and $U \in \mu$. Then as $U \subseteqq i_\mu(c_\mu(U))$, $c_\lambda(f(U)) \subseteqq c_\lambda(f(i_\mu(c_\mu(U)))) \subseteqq f(c_\mu(U))$ (by (a)).

(b) $\Rightarrow$ (c): Let F be a subset of Y and $U \in \mu$ with the property $f^{-1}(F) \subseteqq U$. Then $f^{-1}(F) \cap c_\mu(X \setminus c_\mu(U)) = f^{-1}(F) \cap (X \setminus i_\mu(c_\mu(U))) \subseteqq f^{-1}(F) \cap (X \setminus i_\mu(U)) = f^{-1}(F) \cap (X \setminus U) = \emptyset$. Therefore, $f^{-1}(F) \cap c_\mu(X \setminus c_\mu(U)) = \emptyset$ and hence $F \cap f(c_\mu(X \setminus c_\mu(U))) = \emptyset$. Again, $F \cap c_\lambda(f(X \setminus c_\mu(U))) \subseteqq F \cap f(c_\mu(X \setminus c_\mu(U))) = \emptyset$. Hence $F \cap c_\lambda(f(X \setminus c_\mu(U))) = \emptyset$. Let $V = Y \setminus c_\lambda(f(X \setminus c_\mu(U)))$. Then V is λ-open and $F \subseteqq V$ and $f^{-1}(V) = f^{-1}(Y \setminus c_\lambda(f(X \setminus c_\mu(U)))) \subseteqq X \setminus f^{-1}(f(X \setminus c_\mu(U))) \subseteqq c_\mu(U)$.

(c) $\Rightarrow$ (d): This is trivial.

(d) $\Rightarrow$ (a): Let F be a μ-closed set in X and $y \in Y \setminus f(F)$. Since $f^{-1}(y) \subseteqq X \setminus F \in \mu$, there exists $V \in \lambda$ with $y \in V$ such that $f^{-1}(V) \subseteqq c_\mu(X \setminus F) = X \setminus i_\mu(F)$. Therefore, $V \cap f(i_\mu(F)) = \emptyset$. Then $y \in Y \setminus c_\lambda(f(i_\mu(F)))$. Hence $c_\lambda(f(i_\mu(F))) \subseteqq f(F)$. Thus f is weakly (μ,λ)-closed.

(a) $\Rightarrow$ (e): Let A be subset of X. Then $i_\mu(c_\mu(A))$ is μ-open in X and hence by (a) (as $(a) \Rightarrow (b)$), $c_\lambda(f(i_\mu(c_\mu(A)))) \subseteqq f(c_\mu(i_\mu(c_\mu(A)))) \subseteqq f(c_\mu(c_\mu(A))) = f(c_\mu(A))$.

(e) $\Rightarrow$ (b): Let $U \in \mu$. Then by (e), $c_\lambda(f(U)) \subseteqq c_\lambda(f(i_\mu(c_\mu(U)))) \subseteqq f(c_\mu(U))$. Hence $c_\lambda(f(U)) \subseteqq f(c_\mu(U))$.

(a) $\Rightarrow$ (f): Let A be subset of X. Then we observe that $c_{\mu(\theta)}(A)$ is μ-closed. [In fact, it is sufficient to show that $c_\mu(c_{\mu(\theta)}(A)) \subseteqq c_{\mu(\theta)}(A)$. Let $x \in c_\mu(c_{\mu(\theta)}(A))$ and V be any μ-open set containing x. Then $V \cap c_{\mu(\theta)}(A) \neq \emptyset$. Let $y \in V \cap c_{\mu(\theta)}(A) \neq \emptyset$. Then $y \in V$ and $y \in c_{\mu(\theta)}(A)$. Thus $c_\mu(V) \cap A \neq \emptyset$. Thus $x \in c_{\mu(\theta)}(A)$]. Hence by (a) we have $c_\lambda(f(i_\mu(c_{\mu(\theta)}(A)))) \subseteqq f(c_{\mu(\theta)}(A))$.

(f) $\Rightarrow$ (b): We shall first observe that for any μ-open set A in a GTS (X,μ), $c_\mu(A) = c_{\mu(\theta)}(A)$. [For any subset A of X, $c_\mu(A) \subseteqq c_{\mu(\theta)}(A)$. Next, let A be any μ-open subset of X and $x \notin c_\mu(A)$. Then there exists $V \in \mu(x)$ such that $A \cap V = \emptyset$. Then $c_\mu(V) \subseteqq c_\mu(X \setminus A) = X \setminus A$. So that $x \notin c_{\mu(\theta)}(A)$]. Let $U \in \mu$. Then we have $c_\mu(U) = c_{\mu(\theta)}(U)$ and $c_\lambda(f(U)) \subseteqq c_\lambda(f(i_\mu(c_\mu(U)))) = c_\lambda(f(i_\mu(c_{\mu(\theta)}(U)))) \subseteqq f(c_{\mu(\theta)}(U)) = f(c_\mu(U))$. □

E x a m p l e 2.5. The concept of weak BR-closedness as a natural dual to weak BR-continuity due to Ekici [8] was introduced and studied in [1].

Definition 2.6. Let (X,μ) and (Y,λ) be two GTS's. A function $f\colon (X,\mu)\to(Y,\lambda)$ is said to be (i) (μ,λ)-closed [14] if $f(F)$ is λ-closed in Y for each μ-closed set F of X.

(ii) (μ,λ)-open if $f(U)$ is λ-open in Y for each μ-open set U of X.

R e m a r k 2.7. Every (μ,λ)-closed function is weakly (μ,λ)-closed, in fact suppose that f is a (μ,λ)-closed function and let F be any μ-closed subset of X. Then $c_\lambda(f(i_\mu(F)))\subseteqq c_\lambda(f(F))=f(F)$. Thus f is weakly (μ,λ)-closed.

E x a m p l e 2.8. Let $X=\{a,b,c\}$, $\mu=\{\emptyset,X,\{a\},\{b\},\{a,b\}\}$ and $\lambda=\{\emptyset,\{a\},\{a,b\}\}$. Then it is easy to check that the identity map $f\colon (X,\mu)\to(X,\lambda)$ is weakly (μ,λ)-closed but not (μ,λ)-closed.

Definition 2.9. Let (X,μ) be a GTS and $G\subseteqq X$.

(i) G is called μ-dense [7] if $c_\mu(G)=X$.

(ii) (X,μ) is said to be hyperconnected [7] if every nonempty μ-open set is μ-dense in X.

Theorem 2.10. *Let $f\colon (X,\mu)\to(Y,\lambda)$ be a weakly (μ,λ)-closed map. If $X\in\mu$ then, $f(X)$ is λ-closed. If f is (μ,λ)-open and Y is hyperconnected, then f is surjective.*

P r o o f. Since $X\in\mu$ and f is (μ,λ)-closed by Theorem 2.4 ((a) $\Leftrightarrow$ (b)), $c_\lambda(f(X))\subseteqq f(c_\mu(X))=f(X)$. Thus $f(X)$ is λ-closed. Also, as X is μ-closed, $c_\lambda(f(i_\mu(X)))\subseteqq f(X)$ (by Definition 2.3). By (μ,λ)-openness of f, $f(i_\mu(X))$ is λ-open and hence $c_\lambda(f(i_\mu(X)))=Y$ (as (Y,λ) is hyperconnected). Thus from (i), $Y\subseteqq f(X)$, i.e., f is surjective. $\square$

Definition 2.11. Let (X,μ) and (Y,λ) be two GTS's. A function $f\colon (X,\mu)\to(Y,\lambda)$ is said to be contra (μ,λ)-open if $f(U)$ is λ-closed for each μ-open set U of X.

Theorem 2.12. *If $f\colon (X,\mu)\to(Y,\lambda)$ is contra (μ,λ)-open, then f is weakly (μ,λ)-closed.*

P r o o f. Let F be a μ-closed subset of X. Then $i_\mu(F)$ is a μ-open subset of X. Thus we have $c_\lambda(f(i_\mu(F)))=f(i_\mu(F))\subseteqq f(F)$. Hence f is weakly (μ,λ)-closed. $\square$

E x a m p l e 2.13. If we consider Example 2.8, then it is easy to verify that the function is weakly (μ,λ)-closed but not contra (μ,λ)-open.

Definition 2.14. Let (X,μ) and (Y,λ) be two GTS's. A function $f\colon (X,\mu)\to (Y,\lambda)$ is said to be strongly (μ,λ)-continuous if $f(c_\mu(A))\subseteqq f(A)$ for every subset A of X.

Theorem 2.15. *If the function $f\colon (X,\mu)\to(Y,\lambda)$ is strongly (μ,λ)-continuous then the following statements are equivalent:*
(i) *f is weakly (μ,λ)-closed.*
(ii) *f is contra (μ,λ)-open.*

Proof. (i) $\Rightarrow$ (ii): Let U be a μ-open subset of X. Then by hypothesis and Theorem 2.4, we have $c_\lambda(f(U))\subseteqq f(c_\mu(U))\subseteqq f(U)$. Hence $f(U)$ is λ-closed.

(ii) $\Rightarrow$ (i): Follows from Theorem 2.12. $\square$

Theorem 2.16. *If the function $f\colon (X,\mu)\to(Y,\lambda)$ is weakly (μ,λ)-closed and a strongly (μ,λ)-continuous bijection, then $f(U)$ is λ-open as well as λ-closed for each $U\in\mu$.*

Proof. Let U be a μ-open subset of X. Since f is weakly (μ,λ) closed, $c_\lambda(f(i_\mu(X\setminus U)))\subseteqq f(X\setminus U)$. Hence by hypothesis, $f(U)\subseteqq i_\lambda(f(c_\mu(U)))\subseteqq i_\lambda(f(U))$ (as f is strongly (μ,λ)-continuous). So $f(U)$ is λ-open. But by Theorem 2.15, f is contra (μ,λ)-open and hence $f(U)$ is λ-closed. $\square$

Theorem 2.17. *Let $f\colon (X,\mu)\to(Y,\lambda)$ be a weakly (μ,λ)-closed bijection. Then for each subset B of Y and each $U\in\mu$ with $f^{-1}(B)\subseteqq U$ there exists a μ-closed set F in Y such that $B\subseteqq F$ and $f^{-1}(F)\subseteqq c_\mu(U)$.*

Proof. Let B be a subset of Y and $U\in\mu$ with $f^{-1}(B)\subseteqq U$. Put $F=c_\lambda(f(i_\mu(c_\mu(U))))$. Then $B\subseteqq F$ (as $B\subseteqq f(U)\subseteqq f(i_\mu(c_\mu(U)))\subseteqq c_\lambda(f(i_\mu(c_\mu(U))))=F$) and F is λ-closed. Since f is weakly (μ,λ)-closed, $f^{-1}(F)=f^{-1}(c_\lambda(f(i_\mu(c_\mu(U)))))\subseteqq f^{-1}(f(c_\mu(U)))$ (by Theorem 2.4) $=c_\mu(U)$. $\square$

Definition 2.18. A GTS (X,μ) is said to be μ-regular [14] if for each μ-closed set F with $x\notin F$, there exist two disjoint μ-open sets U and V such that $x\in U$ and $F\subseteqq V$.

It is also well known from [14] that a GTS (X,μ) is μ-regular iff $x\in X$ and $U_x\in\mu$ imply that there is $V_x\in\mu$ such that $x\in V_x\subseteqq c_\mu(V_x)\subseteqq U_x$.

Theorem 2.19. *Let $f\colon (X,\mu)\to(Y,\lambda)$ be a weakly (μ,λ)-closed function. If (X,μ) is μ-regular, then f is (μ,λ)-closed.*

P r o o f. Let F be a μ-closed subset of X and $y \in Y \setminus f(F)$. Then $f^{-1}(y) \cap F = \emptyset$ and hence $f^{-1}(y) \in X \setminus F$. Thus by μ-regularity, there exists a μ-open set U in X such that $f^{-1}(y) \subseteqq U \subseteqq c_\mu(U) \subseteqq X \setminus F$. Since f is weakly (μ, λ)-closed, there exists a λ-open set V containing y such that $f^{-1}(V) \subseteqq c_\mu(U)$ (by Theorem 2.4). Therefore $f^{-1}(V) \cap F = \emptyset$ and hence $V \cap f(F) = \emptyset$. Thus $y \notin c_\lambda(f(F))$ and hence $f(F) = c_\lambda(f(F))$. So $f(F)$ is λ-closed in Y. Hence f is (μ, λ)-closed. $\square$

Definition 2.20. A GTS (X, μ) is said to be μ-T_2 [7] if for any pair of distinct points $x, y \in X$, there exist two disjoint μ-open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.21. Two nonempty subsets A and B of a GTS (X, μ) are said to be strongly μ-separated if there exist μ-open sets U and V such that $A \subseteqq U$, $B \subseteqq V$ and $c_\mu(U) \cap c_\mu(V) = \emptyset$.

It follows from Definitions 2.9 and 2.21 that in any hyperconneced GTS, strongly μ-separated sets do not exist.

Theorem 2.22. *If $f\colon (X, \mu) \to (Y, \lambda)$ is a weakly (μ, λ)-closed surjection and each pair of distinct fibers are strongly μ-separated, then (Y, λ) is λ-T_2.*

P r o o f. Let y_1 and y_2 be two distinct points of Y. Let $U_1, U_2 \in \mu$ be such that $f^{-1}(y_1) \subseteqq U_1$ and $f^{-1}(y_2) \subseteqq U_2$ with $c_\mu(U_1) \cap c_\mu(U_2) = \emptyset$. Since f is weakly (μ, λ)-closed, by Theorem 2.4, there exist λ-open sets V_1, V_2 in Y such that $y_i \in V_i$ and $f^{-1}(V_i) \subseteqq c_\mu(U_i)$ for $i = 1, 2$. Therefore $V_1 \cap V_2 = \emptyset$ (as f is surjective). Hence (Y, λ) is λ-T_2. $\square$

Definition 2.23. Let (X, μ) and (Y, λ) be two GTS's. A function $f\colon (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-continuous [2] if for each $x \in X$ and each $V \in \lambda$ containing $f(x)$, there exists $U \in \mu$ such that $f(U) \subseteqq V$.

Definition 2.24. A GTS (X, μ) is said to be μ-normal [4] if for each pair of disjoint μ-closed sets F_1 and F_2 of X there exist disjoint μ-open sets U_1 and U_2 such that $F_1 \subseteqq U_1$ and $F_2 \subseteqq U_2$.

It is also well known that (X, μ) is μ-normal [4] if F is μ-closed and $F \subseteqq U \in \mu$, then there exists $V \in \mu$ such that $F \subseteqq V \subseteqq c_\mu(V) \subseteqq U$.

Theorem 2.25. *If $f\colon (X, \mu) \to (Y, \lambda)$ is a weakly (μ, λ)-closed, (μ, λ)-continuous surjection and (X, μ) is μ-normal, then (Y, λ) is λ-normal.*

P r o o f. Let F_1 and F_2 be any two disjoint λ-closed sets of (Y, λ). Since f is (μ, λ)-continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint μ-closed sets in X. Now by

μ-normality of (X,μ), there exist two disjoint μ-open sets U_1 and U_2 such that $f^{-1}(F_1) \subseteqq U_1$ and $f^{-1}(F_2) \subseteqq U_2$. Moreover, we have $G_i \in \mu$ such that $f^{-1}(F_i) \subseteqq G_i \subseteqq c_\mu(G_i) \subseteqq U_i$ for $i = 1,2$. Since f is weakly (μ,λ)-closed, there exist $V_i \in \mu$ such that $F_i \subseteqq V_i$ and $f^{-1}(V_i) \subseteqq c_\mu(G_i) \subseteqq U_i$ for $i = 1,2$. Since f is surjective and $U_1 \cap U_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$. Thus (Y,λ) is λ-normal. □

Definition 2.26. A subset F of a GTS (X,μ) is said to be μ_θ-compact if for each cover Ω of F by μ-open sets in X, there is a finite family $U_1, U_2, \ldots, U_n$ in Ω such that $F \subseteqq \{i_\mu(\bigcup c_\mu(U_i))\colon\ i = 1,2,\ldots,n\}$.

Theorem 2.27. *Let (X,μ) be a QTS and (Y,λ) be a GTS. If $f\colon\ (X,\mu) \to (Y,\lambda)$ is weakly (μ,λ)-closed with all fibers $\mu(\theta)$-closed in X, then $f(F)$ is λ-closed for each θ-compact set F in X.*

P r o o f. Let F be μ_θ-compact and $y \notin f(F)$. Then $f^{-1}(y) \cap F = \emptyset$ and for each $x \in F$ there is a μ-open set U_x in X containing x such that $c_\mu(U_x) \cap f^{-1}(y) = \emptyset$. Clearly, $\Omega = \{U_x\colon\ x \in F\}$ is a collection of μ-open sets in X that covers F and since F is a μ_θ-compact, there is a finite family $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ in Ω such that $F \subseteqq i_\mu(A)$ where $A = \bigcup\{c_\mu(U_{x_i})\colon\ i = 1,2,\ldots,n\}$. Since f is weakly (μ,λ)-closed there exists a λ-open set B in Y such that $f^{-1}(y) \subseteqq f^{-1}(B) \subseteqq c_\mu(X \setminus A) \subseteqq X \setminus F$. Therefore $y \in B$ and $B \cap f(F) = \emptyset$. Thus $y \notin c_\lambda(f(F))$. So $f(F)$ is λ-closed. □

3. Comparison of weakly closed functions

Theorem 3.1. *Suppose that μ_1 and μ_2 are two GT's on X such that $\mu_1 \subseteqq \mu_2$ and let (Y,λ) be a GTS. If $f\colon\ (X,\mu_2) \to (Y,\lambda)$ is weakly (μ_2,λ)-closed then $f\colon\ (X,\mu_1) \to (Y,\lambda)$ is weakly (μ_1,λ)-closed.*

P r o o f. Suppose that $f\colon\ (X,\mu_2) \to (Y,\lambda)$ is a weakly (μ_2,λ)-closed function. Let F be a μ_1-closed set in (X,μ_1). Then F is μ_2-closed in (X,μ_2) and hence $c_\lambda(f(i_{\mu_2}(F))) \subseteqq f(F)$. Moreover, $i_{\mu_1}(F) \subseteqq i_{\mu_2}(F)$ and hence $c_\lambda(f(i_{\mu_1}(F))) \subseteqq c_\lambda(f(i_{\mu_2}(F))) \subseteqq f(F)$. Thus $f\colon\ (X,\mu_1) \to (Y,\lambda)$ is weakly (μ_1,λ)-closed. □

Theorem 3.2. *Suppose that λ_1 and λ_2 are two GT's on Y such that $\lambda_1 \subseteqq \lambda_2$ and let (X,μ) be a GTS. If $f\colon\ (X,\mu) \to (Y,\lambda_1)$ is weakly (μ,λ_1)-closed then $f\colon\ (X,\mu) \to (Y,\lambda_2)$ is weakly (μ,λ_2)-closed.*

P r o o f. Suppose that $f\colon\ (X,\mu) \to (Y,\lambda_1)$ is weakly (μ,λ_1)-closed. Let F be μ-closed in (X,μ). Then we have $c_{\lambda_1}(f(i_\mu(F))) \subseteqq f(F)$. Since $\lambda_1 \subseteqq \lambda_2$, $c_{\lambda_2}(B) \subseteqq c_{\lambda_1}(B)$ for every $B \subseteqq Y$. Hence $c_{\lambda_2}(f(i_\mu(F))) \subseteqq c_{\lambda_1}(f(i_\mu(F))) \subseteqq f(F)$. Hence $f\colon\ (X,\mu) \to (Y,\lambda_2)$ is weakly (μ,λ_2)-closed. □

E x a m p l e 3.3. (a) Let $X = \{a, b, c\}$, $\mu_1 = \{\emptyset, \{a\}, \{a, b\}\}$, $\mu_2 = \{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\lambda = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then μ_1, μ_2, λ are three GT's on X. Consider the identity functions $f\colon (X, \mu_2) \to (X, \lambda)$ and $f\colon (X, \mu_1) \to (X, \lambda)$. It can be checked that $f\colon (X, \mu_1) \to (X, \lambda)$ is weakly (μ_1, λ)-closed but $f\colon (X, \mu_2) \to (X, \lambda)$ is not weakly (μ_2, λ)-closed.

(b) Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{a, b\}\}$, $\lambda_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\lambda_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then λ_1, λ_2, μ are three GT's on X. Consider the identity functions $f\colon (X, \mu) \to (X, \lambda_1)$ and $f\colon (X, \mu) \to (X, \lambda_2)$. It can be checked that $f\colon (X, \mu) \to (X, \lambda_1)$ is not weakly (μ, λ_1)-closed but $f\colon (X, \mu) \to (X, \lambda_2)$ is weakly (μ, λ_2)-closed.

Theorem 3.4. *Let (X, τ) be a topological space and (Y, λ) be a GTS. Then the following properties are equivalent:*

(a) $f\colon (X, \tau_s) \to (Y, \lambda)$ *is weakly* (τ_s, λ) *closed.*
(b) $f\colon (X, \tau) \to (Y, \lambda)$ *is weakly* (τ, λ) *closed.*
(c) $f\colon (X, \tau_\alpha) \to (Y, \lambda)$ *is weakly* (τ_α, λ) *closed.*

P r o o f. Since $\tau_s \subseteqq \tau \subseteqq \tau_\alpha$, by Theorem 3.1 we have $(c) \Rightarrow (b) \Rightarrow (a)$.

(a) $\Rightarrow$ (c): Let F be a α-closed subset of (X, τ). Then $\mathrm{int}_\alpha(F) = F \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(F))) \subseteqq \mathrm{int}(\mathrm{cl}(\mathrm{int}(F))) \subseteqq \mathrm{cl}(\mathrm{int}(F)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(F)))$. Therefore, we have $c_\lambda(f(\mathrm{int}_\alpha\,(F))) \subseteqq f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(F)))) \subseteqq f(F)$. □

Let (X, τ) and (Y, σ) be two topological spaces. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be weakly closed [13] if $\mathrm{cl}(f(\mathrm{int}(F))) \subseteqq f(F)$ (instead of weakly (τ, σ)-closed) for every closed set F of X.

Corollary 3.5. *Let (X, τ) and (Y, σ) be two topological spaces. Then the following statements are equivalent:*

(a) $f\colon (X, \tau_s) \to (Y, \sigma)$ *is weakly closed.*
(b) $f\colon (X, \tau) \to (Y, \sigma)$ *is weakly closed.*
(c) $f\colon (X, \tau_\alpha) \to (Y, \sigma)$ *is weakly closed.*

P r o o f. Follows from Theorem 3.4. □

C o n c l u s i o n. The definitions of various types of weakly closed functions may be introduced from the definition of weakly (μ, λ)-closed function by replacing the generalized topologies μ and λ (on X and Y, respectively) suitably.

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