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A REVERSE VIEWPOINT ON THE UPPER SEMICONTINUITY OF MULTIVALUED MAPS

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Abstract. For a multivalued map $\varphi\colon Y \multimap (X,\tau)$ between topological spaces, the upper semifinite topology $\mathcal{A}(\tau)$ on the power set $\mathcal{A}(X)=\{A\subset X\colon A\neq\emptyset\}$ is such that φ is upper semicontinuous if and only if it is continuous when viewed as a singlevalued map $\varphi\colon Y\to(\mathcal{A}(X),\mathcal{A}(\tau))$. In this paper, we seek a result like this from a reverse viewpoint, namely, given a set X and a topology Γ on $\mathcal{A}(X)$, we consider a natural topology $\mathcal{R}(\Gamma)$ on X, constructed from Γ satisfying $\mathcal{R}(\Gamma)=\tau$ if $\Gamma=\mathcal{A}(\tau)$, and we give necessary and sufficient conditions to the upper semicontinuity of a multivalued map $\varphi\colon Y\multimap(X,\mathcal{R}(\Gamma))$ to be equivalent to the continuity of the singlevalued map $\varphi\colon Y\to(\mathcal{A}(X),\Gamma)$.

Keywords: multivalued map; power set; upper semicontinuity; upper semifinite topology

MSC 2010: 54A10, 54C60

1. Introduction

For a given nonempty set X, we follow Michael [2] in defining $\mathcal{A}(X)$ to be the set of all nonempty subsets of X, that is, $\mathcal{A}(X)=\{A\subset X\colon A\neq\emptyset\}$. We will call $\mathcal{A}(X)$ the power set of X. For a given topology τ on X, we consider the *upper semifinite topology* on $\mathcal{A}(X)$, which we denote by $\mathcal{A}(\tau)$ and call the *power topology* (induced on X by τ) to shorten. Precisely, $\mathcal{A}(\tau)$ is the topology on $\mathcal{A}(X)$ generated by the basis $\mathcal{B}_\tau^{\mathcal{A}}=\{\mathcal{A}(U)\colon U\in\mathcal{B}\}$, where $\mathcal{B}$ is a basis for τ. For details we recommend the reference [2], mainly its appendix.

Let $\varphi\colon Y\multimap X$ be a multimap (i.e. a multivalued map or a set-valued map) that associates with each point y of the topological space Y a (not necessarily closed) nonempty subset $\varphi(y)$ of the topological space X. We can think of φ as a singlemap (i.e. a singlevalued map) from Y into the power set $\mathcal{A}(X)$; in this case we write it as $\check{\varphi}\colon Y\to\mathcal{A}(X)$ to avoid confusion. The notion of upper semicontinuity of the multimap φ can be defined independently of a topological structure on the power set

$\mathcal{A}(X)$. Following [1], we say that φ is *upper semicontinuous* provided that for every open $U \subset X$ the set $\varphi^{-1}(U)$ is open in Y, where $\varphi^{-1}(U) = \{y \in Y\colon \varphi(y) \subset U\}$. However, it follows from the Appendix of [2], page 179, that the multimap $\varphi\colon Y \multimap (X, \tau)$ is upper semicontinuous if and only if the singlemap $\check{\varphi}\colon Y \to (\mathcal{A}(X), \mathcal{A}(\tau))$ is continuous. In fact, this follows from the identity below that holds true for every (open or not open) subset U of X:

$$\varphi^{-1}(U) = \{y \in Y\colon \varphi(y) \subset U\} = \{y \in Y\colon \check{\varphi}^{-1}(y) \in \mathcal{A}(U)\} = \check{\varphi}^{-1}(\mathcal{A}(U)).$$

We remark that the power topology $\mathcal{A}(\tau)$ is built *from* the topology τ given *a priori* on X. So we may view $\mathcal{A}$ as a function $\tau \mapsto \mathcal{A}(\tau)$ from the lattice of all topologies on X into the lattice of all topologies on the power set $\mathcal{A}(X)$. More generally, $\mathcal{A}$ can be viewed as the pair-function $(X, \tau) \mapsto (\mathcal{A}(X), \mathcal{A}(\tau))$. We observe that the function $\tau \mapsto \mathcal{A}(\tau)$ is injective but not surjective in general. The injectivity follows from Proposition 2.2. To illustrate the non-surjectivity, we take any topological space X with more than one point and on $\mathcal{A}(X)$ we consider the topology $\Gamma = \{\emptyset, \{X\}, \mathcal{A}(X)\}$. It is easy to see that Γ is not a power topology, that is, there is not a topology τ on X such that $\Gamma = \mathcal{A}(\tau)$.

Now, we consider the function $\sigma\colon \mathcal{A}(\mathcal{A}(X)) \to \mathcal{A}(X)$ defined by Michael in 5.5.1 of [2] by

$$\sigma(\mathcal{E}) = \bigcup_{E \in \mathcal{E}} E.$$

For convenience, we extend this function to a function from $\mathcal{A}(\mathcal{A}(X)) \cup \{\emptyset\}$ into $\mathcal{A}(X) \cup \{\emptyset\}$ defining $\sigma(\emptyset) = \emptyset$.

We call σ the *reversal function*, because $\sigma(\mathcal{A}(Z)) = Z$ for all $Z \subset X$. In particular, the composition $X \xmapsto{\mathcal{A}} \mathcal{A}(X) \xmapsto{\sigma} X$ is the identity.

If Γ is a topology on $\mathcal{A}(X)$, then it is obvious that $\Gamma \subset \mathcal{A}(\mathcal{A}(X)) \cup \{\emptyset\}$ and so we can consider its image $\sigma(\Gamma)$ by the reversal function. Unfortunately, the collection $\sigma(\Gamma)$ is not necessarily even a basis for a topology on X, but it is a subbasis, since $X = \sigma(\mathcal{A}(X))$ and certainly $\mathcal{A}(X) \in \Gamma$. We will denote the topology on X generated by the subbasis $\sigma(\Gamma)$ by $\mathcal{R}(\Gamma)$ and we will call it the *reversed topology* from Γ.

Thus, we can view $\mathcal{R}$ as a function $\Gamma \mapsto \mathcal{R}(\Gamma)$ from the lattice of all topologies on $\mathcal{A}(X)$ onto the lattice of all topologies on X. We say "onto" because this function is in fact surjective, since for each topology τ on X we have $\mathcal{R}(\mathcal{A}(\tau)) = \tau$, as we prove in Proposition 2.2. Therefore, if $\mathfrak{T}(Z)$ denotes the lattice of all topologies on a set Z, then $\mathcal{R} \circ \mathcal{A} =$ *identity* in the sequence below:

$$\mathfrak{T}(X) \xrightarrow{\mathcal{A}} \mathfrak{T}(\mathcal{A}(X)) \xrightarrow{\mathcal{R}} \mathfrak{T}(X).$$

The goal of this paper is to answer completely the following question: *Under what condition is the upper semicontinuity of a multimap $\varphi\colon Y \multimap (X, \mathcal{R}(\Gamma))$ equivalent to the continuity of the singlemap $\check{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$?*

If Γ belongs to the image of the function $\mathcal{A}$, then the answer is trivial: the required equivalence holds true under general conditions, because in this case $\Gamma = \mathcal{A}(\tau)$ for some topology τ on X and $\mathcal{R}(\Gamma) = \tau$. On the other hand, if Γ is not a power topology, that is, $\Gamma \in \mathfrak{T}(\mathcal{A}(X)) \setminus \mathcal{A}(\mathfrak{T}(X))$, then there exist restrictive conditions to the required equivalence. This condition involves a good behavior of the topology Γ (Definition 2.3) and a certain compatibility between the multimap φ and the topology Γ (Definition 2.4).

The answer for the proposed problem is our main result, namely, Theorem 3.1. Before we present it, we explore, in Section 2, some properties involving the reversal function and the reversed topology. Also we set the conditions that appear as assumptions in the main theorem. We present several examples to illustrate the main results. We awaken the curiosity for the fact that all examples used to show the essentiality of the assumption in the results are given considering topological spaces with no more than four points.

2. The assumptions for the main theorem

As said in the introduction, given a nonempty set X, we consider, as in [2], the so-called *reversal function*

$$\sigma\colon \mathcal{A}(\mathcal{A}(X)) \cup \{\emptyset\} \to \mathcal{A}(X) \cup \{\emptyset\} \quad \text{given by} \quad \sigma(\mathcal{E}) = \bigcup_{E \in \mathcal{E}} E.$$

For future use, it is important to note that σ has the following properties:

Lemma 2.1. *If $\{\mathcal{E}_\lambda\}_\lambda$ is a collection of elements of $\mathcal{A}(\mathcal{A}(X))$, then we have*

$$\sigma\Big(\bigcup_\lambda \mathcal{E}_\lambda\Big) = \bigcup_\lambda \sigma(\mathcal{E}_\lambda) \quad \textit{and} \quad \sigma\Big(\bigcap_\lambda \mathcal{E}_\lambda\Big) \subset \bigcap_\lambda \sigma(\mathcal{E}_\lambda).$$

It is easy to prove this lemma and to provide examples in which the inclusion $\sigma\Big(\bigcap_\lambda \mathcal{E}_\lambda\Big) \subset \bigcap_\lambda \sigma(\mathcal{E}_\lambda)$ is strict, that is, it is not an equality.

Given a topology Γ on $\mathcal{A}(X)$, we have seen that $\sigma(\Gamma)$ is a subbasis for a topology on X, which we denote by $\mathcal{R}(\Gamma)$ and call the *reversed topology* from Γ. Thus $\mathcal{R}(\Gamma)$ is the collection of all unions of finite intersections of elements of $\sigma(\Gamma)$, and so the collection of all finite intersections of elements of $\sigma(\Gamma)$ is a basis for $\mathcal{R}(\Gamma)$. It may happen that the collection $\sigma(\Gamma)$ is itself a topology on X. If this is the case, then it is obvious that $\mathcal{R}(\Gamma) = \sigma(\Gamma)$. But, in general, $\sigma(\Gamma)$ is not even a basis for $\mathcal{R}(\Gamma)$, as in the following example: Let $X = \{a, b, c\}$ and on $\mathcal{A}(X)$ consider the topology

$$\Gamma = \{\emptyset, \{\{a\}, \{b\}\}, \{\{a, c\}\}, \{\{a\}, \{b\}, \{a, c\}\}, \mathcal{A}(X)\}.$$

Note that $\sigma(\Gamma) = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ is not a topology on X. Also, $\sigma(\Gamma)$ is not a basis for $\mathcal{R}(\Gamma)$. In fact: Put $\mathcal{E}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{E}_2 = \{\{a, c\}\}$. Then $\mathcal{E}_1, \mathcal{E}_2 \in \Gamma$ and we have $\sigma(\mathcal{E}_1) \cap \sigma(\mathcal{E}_2) = \{a, b\} \cap \{a, c\} = \{a\}$. Thus $a \in \sigma(\mathcal{E}_1) \cap \sigma(\mathcal{E}_2)$, but there is no $\mathcal{E}$ in Γ such that $a \in \sigma(\mathcal{E}) \subset \sigma(\mathcal{E}_1) \cap \sigma(\mathcal{E}_2)$.

Despite the above, there are situations in which $\mathcal{R}(\Gamma) = \sigma(\Gamma)$. The more important of them is the following one (which we state in the introduction of the paper):

Proposition 2.1. *Let τ be a topology on X. Then $\mathcal{R}(\mathcal{A}(\tau)) = \sigma(\mathcal{A}(\tau)) = \tau$.*

Proof. It is sufficient to prove that $\sigma(\mathcal{A}(\tau)) = \tau$. It is clear that $\tau \subset \sigma(\mathcal{A}(\tau))$, since for each $E \in \tau$, we have $\mathcal{A}(E) \in \mathcal{A}(\tau)$ and $E = \sigma(\mathcal{A}(E))$. Now, if $Y \in \mathcal{A}(\tau)$, we can write $Y = \bigcup\limits_{E \in \varrho} \mathcal{A}(E)$, where ϱ is a subset of τ, since $\mathcal{B}_\tau^{\mathcal{A}}$ is a basis for $\mathcal{A}(\tau)$. Hence, by Lemma 2.1, we have $\sigma(Y) = \bigcup\limits_{E \in \varrho} E$, which implies that $\sigma(Y) \in \tau$. □

Next we define the condition on the topology Γ on $\mathcal{A}(X)$ that appears as assumption in the main theorem.

Definition 2.1. We say that a topology Γ on $\mathcal{A}(X)$ is a *reversible topology* on $\mathcal{A}(X)$ if $\mathcal{R}(\Gamma) = \sigma(\Gamma)$ or, equivalently, if $\sigma(\Gamma)$ is a topology on X.

In order to introduce the other condition that appears in the main theorem, we remark that the reversal map $\sigma\colon \mathcal{A}(\mathcal{A}(X)) \to \mathcal{A}(X)$ is not injective if X has more than one point. In fact, if x_1 and x_2 are different points in X, then the sets $\{\{x_1\}, \{x_2\}\}$ and $\{\{x_1, x_2\}\}$ are different points of $\mathcal{A}(\mathcal{A}(X))$ and we have $\sigma(\{\{x_1\}, \{x_2\}\}) = \sigma(\{\{x_1, x_2\}\})$. We will take advantage of this non-injectivity.

Definition 2.2. Let $\varphi\colon Y \multimap X$ be a multimap and let Γ be a topology on $\mathcal{A}(X)$. We say that φ *is semicompatible with* Γ if for every $\emptyset \neq \mathcal{V} \in \Gamma$ there exists $\mathcal{V}' \in \Gamma$ such that $\sigma(\mathcal{V}) = \sigma(\mathcal{V}')$ and $\varphi^{-1}(\sigma(\mathcal{V})) = \check{\varphi}^{-1}(\mathcal{V}')$. If, additionally, $\mathcal{V}'$ always can be chosen to be $\mathcal{V}$ itself, that is, if for every $\mathcal{V} \in \Gamma$ we have $\varphi^{-1}(\sigma(\mathcal{V})) = \check{\varphi}^{-1}(\mathcal{V})$, then we say that φ *is compatible with* Γ.

It is obvious that if φ is compatible with Γ, then also φ is semicompatible with Γ. Next we present examples in which the reciprocal is not true.

To avoid any confusion, we set $\varphi^{-1}(\emptyset) = \emptyset$, and so $\varphi^{-1}(\emptyset) = \check{\varphi}^{-1}(\emptyset)$.

R e m a r k 2.1. Given a multimap $\varphi\colon Y \multimap X$ and a topology Γ on $\mathcal{A}(X)$, for every $\mathcal{V} \in \Gamma$ the inclusion $\check{\varphi}^{-1}(\mathcal{V}) \subset \varphi^{-1}(\sigma(\mathcal{V}))$ holds true, since $\check{\varphi}(y) \in \mathcal{V}$ implies that $\varphi(y) \in \mathcal{V}$, and so $\varphi(y) \subset \sigma(\mathcal{V})$, since $\sigma(\mathcal{V}) = \bigcup\limits_{V\in\mathcal{V}} V$. Then it follows that φ is semicompatible with Γ if and only if for every $\mathcal{V} \in \Gamma$ there exists $\mathcal{V}' \in \Gamma$ with $\sigma(\mathcal{V}) = \sigma(\mathcal{V}')$ such that for all $y \in Y$, $\varphi(y) \subset \sigma(\mathcal{V})$ implies $\varphi(y) \in \mathcal{V}'$. Analogously, φ is compatible with Γ if and only if for every $\mathcal{V} \in \Gamma$ and for every $y \in Y$, $\varphi(y) \subset \sigma(\mathcal{V})$ implies $\varphi(y) \in \mathcal{V}$.

E x a m p l e 2.1. Let $\varphi\colon Y \multimap X$ be a multimap. We have:

(1) *φ is compatible with the indiscrete topology $\{\emptyset, \mathcal{A}(X)\}$ on $\mathcal{A}(X)$.*

In fact, we have $\varphi^{-1}(\sigma(\mathcal{A}(X))) = \varphi^{-1}(X) = Y = \check{\varphi}^{-1}(\mathcal{A}(X))$.

(2) *φ is semicompatible with the discrete topology $\mathcal{A}(\mathcal{A}(X)) \cup \{\emptyset\}$ on $\mathcal{A}(X)$.*

In fact: Let $\mathcal{V} \subset \mathcal{A}(X)$; we need to find $\mathcal{V}' \subset \mathcal{A}(X)$ such that $\sigma(\mathcal{V}) = \sigma(\mathcal{V}')$ and $\varphi^{-1}(\sigma(\mathcal{V})) = \check{\varphi}^{-1}(\mathcal{V}')$. For this, we take $\mathcal{V}' = \{U \in \mathcal{A}(X)\colon U \subset \sigma(\mathcal{V})\}$, or equivalently, $\mathcal{V}' = \mathcal{A}(\sigma(\mathcal{V}))$. Then $\mathcal{V}'$ is a subset of $\mathcal{A}(X)$ and we have $\sigma(\mathcal{V}') = \sigma(\mathcal{A}(\sigma(\mathcal{V}))) = \sigma(\mathcal{V})$ and, moreover, if $\varphi(y) \subset \sigma(\mathcal{V})$, then, by construction, $\varphi(y) \in \mathcal{V}'$. By Remark 2.5, this proves that φ is semicompatible with the discrete topology on $\mathcal{A}(X)$.

(3) *If φ is compatible with the discrete topology on $\mathcal{A}(X)$, then $\varphi(y)$ is a unitary set for all $y \in Y$.*

In fact: Suppose that $\varphi(y_0)$ is not unitary and let $\mathcal{V} = \{\{x\}\colon x \in \varphi(y_0)\} \subset \mathcal{A}(X)$. Considering $\mathcal{A}(X)$ with the discrete topology, the set $\mathcal{V}$ is open in $\mathcal{A}(X)$ and we have $\sigma(\mathcal{V}) = \varphi(y_0)$. It follows that $\varphi(y_0) \subset \sigma(\mathcal{V})$, but since $\varphi(y_0)$ is not single, we have $\varphi(y_0) \notin \mathcal{V}$, which proves that φ is not compatible with the discrete topology on $\mathcal{A}(X)$.

We note that if Γ is the indiscrete or the discrete topology on $\mathcal{A}(X)$, then $\mathcal{R}(\Gamma)$ is, respectively, the indiscrete or the discrete topology on X. Thus, the previous example presents conditions for a multimap $\varphi\colon Y \multimap X$ to be semicompatible and/or compatible with the largest and smallest topologies on $\mathcal{A}(X)$. But what about intermediary topologies? The next two facts about this are interesting.

Proposition 2.2. *Let $\varphi\colon Y \multimap X$ be a multimap and let Γ be a topology on the power set $\mathcal{A}(X)$. If Γ is finer than the power topology $\mathcal{A}(\mathcal{R}(\Gamma))$, then φ is semicompatible, but not necessarily compatible, with Γ.*

P r o o f. Suppose that Γ is finer than $\mathcal{A}(\mathcal{R}(\Gamma))$ and let $\mathcal{V} \in \Gamma$ be nonempty. Then $\mathcal{A}(\sigma(\mathcal{V}))$ belongs to Γ, since $\sigma(\mathcal{V})$ belongs to $\sigma(\Gamma) \subset \mathcal{R}(\Gamma)$ and Γ contains $\mathcal{A}(\mathcal{R}(\Gamma))$. By the same argument as that used in the proof of (2) in Example 2.6, it follows that φ is semicompatible with Γ.

In order to show that φ is not necessarily compatible with Γ, we consider an arbitrary set X with at least two points and on $\mathcal{A}(X)$ we consider the topology $\Gamma = \{\emptyset, \{X\}, \mathcal{A}(X)\}$. Then $\mathcal{R}(\Gamma) = \{\emptyset, X\}$ and $\mathcal{A}(\mathcal{R}(\Gamma)) = \{\emptyset, \mathcal{A}(X)\}$. It is obvious that Γ is (strictly) finer than $\mathcal{A}(\mathcal{R}(\Gamma))$. Now, let Y be a nonempty set and consider the constant multimap $\varphi\colon Y \multimap X$ given by $\varphi(y) = \{x_0\}$ for all $y \in Y$, where x_0 is a fixed point in X. We have $\varphi^{-1}(\sigma(\{X\})) = \varphi^{-1}(X) = Y$, but $\check{\varphi}^{-1}(\{X\}) = \emptyset$. Therefore, φ is not compatible with Γ. $\square$

We observe that, since the collection $\sigma(\Gamma)$ is a subbasis for the topology $\mathcal{R}(\Gamma)$, for Γ to be finer than $\mathcal{A}(\mathcal{R}(\Gamma))$ it is necessary and sufficient that $\Gamma \supset \mathcal{A}(\sigma(\Gamma))$.

Proposition 2.3. *If τ is a topology on X, then every multimap $\varphi\colon Y \multimap X$ is semicompatible, but not necessarily compatible, with the power topology $\mathcal{A}(\tau)$.*

P r o o f. That φ is semicompatible with $\mathcal{A}(\tau)$ follows directly from the previous proposition and from the equality $\mathcal{A}(\tau) = \mathcal{A}(\mathcal{R}(\mathcal{A}(\tau)))$. That φ is not, in general, compatible with $\mathcal{A}(\tau)$ is a consequence of the fact that, in general, $\bigcup\limits_{\lambda} \mathcal{A}(E_\lambda) \not\supseteq \mathcal{A}\Big(\bigcup\limits_{\lambda} E_\lambda\Big)$. In fact, the identity $\bigcup\limits_{\lambda} \mathcal{A}(E_\lambda) = \mathcal{A}\Big(\bigcup\limits_{\lambda} E_\lambda\Big)$ holds true if and only if some E_λ contains all the others. An example is presented below. $\square$

E x a m p l e 2.2. Let us consider the space of three points $X = \{a, b, c\}$ endowed with the topology $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Denote $E_1 = \{a, b\}$ and $E_2 = \{b, c\}$. Then $a \in E_1 \setminus E_2$ and $c \in E_2 \setminus E_1$. Let Y be an arbitrary nonempty set and consider the constant multimap $\varphi\colon Y \multimap X$ given by $\varphi(y) = \{a, c\}$ for all y in Y. We state that φ is not compatible with the power topology $\mathcal{A}(\tau)$ on $\mathcal{A}(X)$. In fact, the set $\mathcal{V} = \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$ belongs to $\mathcal{A}(\tau)$ and we have $\sigma(\mathcal{V}) = E_1 \cup E_2 = X$. Now, although $\check{\varphi}^{-1}(\mathcal{V}) = \emptyset$, since $\{a, c\} \notin \mathcal{V}$, we have $\varphi^{-1}(\sigma(\mathcal{V})) = Y$.

3. The main theorem

Theorem 3.1. *Let $\varphi\colon Y \multimap X$ be a multimap and let Γ be a topology on $\mathcal{A}(X)$. Then the following assertions hold:*

(a) *If φ is compatible with Γ and $\varphi\colon Y \to (X, \mathcal{R}(\Gamma))$ is upper-semicontinuous, then $\check{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is continuous.*

(b) *If φ is semicompatible with Γ, Γ is a reversible topology and $\check{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is continuous, then $\varphi\colon Y \to (X, \mathcal{R}(\Gamma))$ is upper-semicontinuous,*

Moreover, to obtain assertions (a) *and* (b), *all the assumptions are essential.*

P r o o f. First we prove (a): Suppose that φ is compatible with Γ and that $\check{\varphi}$ is not continuous. Then there exists $\mathcal{V} \in \Gamma$ such that $\check{\varphi}^{-1}(\mathcal{V})$ is not open in Y. From the compatibility between φ and Γ it follows that $\varphi^{-1}(\sigma(\mathcal{V}))$ is not open in Y. But since $\sigma(\mathcal{V}) \in \sigma(\Gamma) \subset \mathcal{R}(\Gamma)$, this shows that φ is not upper-semicontinuous.

Now we prove (b): Assume that Γ is a reversible topology and φ is semicompatible with Γ. Suppose that $\check{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is continuous and let $U \in \sigma(\Gamma) = \mathcal{R}(\Gamma)$ be a nonempty open subset of the topological space $(X, \mathcal{R}(\Gamma))$. We need to prove that $\varphi^{-1}(U)$ is open in Y. But since U belongs to the collection $\sigma(\Gamma)$, there exists $\mathcal{V} \in \Gamma$ such that $U = \sigma(\mathcal{V})$. It follows from the semicompatibility between φ and Γ that $\varphi^{-1}(U) = \check{\varphi}^{-1}(\mathcal{V}')$ for some $\mathcal{V}' \in \Gamma$ with $\sigma(\mathcal{V}) = \sigma(\mathcal{V}')$. It follows from the continuity of $\check{\varphi}$ that $\check{\varphi}^{-1}(\mathcal{V}')$ and so $\varphi^{-1}(U)$ are open in Y, as we wanted to prove.

The next three examples show that all assumptions required in the statement of the theorem are essential. $\square$

Example 3.1. If φ is not compatible with Γ, then assertion (a) in Theorem 3.1 is not necessarily true. In fact, define $X = \{a, b\}$ and on the set $\mathcal{A}(X)$ consider the topology

$$\Gamma = \{\emptyset, \{X\}, \{\{a\}, X\}, \mathcal{A}(X)\}.$$

Then we have

$$\mathcal{R}(\Gamma) = \sigma(\Gamma) = \{\emptyset, X\}.$$

Note that Γ is a reversible topology. Consider $Y = \{1, 2\}$ with the indiscrete topology $\tau = \{\emptyset, Y\}$ and let $\varphi\colon Y \multimap X$ be the multimap given by $\varphi(1) = \{a\}$ and $\varphi(2) = \{b\}$. Then φ is not compatible with Γ, since

$$\varphi^{-1}(\sigma(\{X\})) = \varphi^{-1}(X) = Y \neq \emptyset = \check{\varphi}^{-1}(\{X\}),$$

although it is semicompatible with Γ, since $\sigma(\{X\}) = \sigma(\{\{a\}, X\}) = \sigma(\mathcal{A}(X)) = X$ and $\check{\varphi}^{-1}(\mathcal{A}(X)) = Y$. Now, since $\mathcal{R}(\Gamma)$ is the indiscrete topology on X, it is clear that the multimap $\varphi\colon Y \multimap (X, \mathcal{R}(\Gamma))$ is upper semicontinuous, although the singlemap $\check{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is not continuous, since $\check{\varphi}^{-1}(\{\{a\}, X\}) = \{1\}$ and Y has the indiscrete topology.

Example 3.2. If φ is not semicompatible with Γ, then assertion (b) in Theorem 3.1 is not necessarily true. In fact, define $X = \{a, b, c, d\}$ and on $\mathcal{A}(X)$ consider the topology

$$\begin{aligned}\Gamma = \{&\emptyset, \{\{a, b\}\}, \{\{a\}, \{b, c\}\}, \{\{a, b\}, \{c, d\}\},\\ &\{\{a\}, \{a, b\}, \{b, c\}\}, \{\{a\}, \{a, b\}, \{b, c\}, \{c, d\}\}, \mathcal{A}(X)\}.\end{aligned}$$

Then we have

$$\mathcal{R}(\Gamma) = \sigma(\Gamma) = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}.$$

Note that Γ is a reversible topology. Consider $Y = \{1,2\}$ with the indiscrete topology $\tau = \{\emptyset, Y\}$ and let $\varphi\colon Y \multimap X$ be the multimap given by $\varphi(1) = \{a\}$ and $\varphi(2) = \{b,c\}$. Then φ is not semicompatible with Γ, since

$$\varphi^{-1}(\sigma(\{\{a,b\}\})) = \varphi^{-1}(\{a,b\}) = \{1\} \neq \breve{\varphi}^{-1}(\mathcal{V}') \text{ for all } \mathcal{V}' \in \Gamma.$$

It follows from the identity $\varphi^{-1}(\sigma(\{\{a,b\}\})) = \{1\}$ that $\varphi\colon Y \multimap (X, \mathcal{R}(\Gamma))$ is not upper semicontinuous. However, it is easy to check that for every $\mathcal{V} \in \Gamma$, we have either $\breve{\varphi}^{-1}(\mathcal{V}) = \emptyset$ or $\breve{\varphi}^{-1}(\mathcal{V}) = Y$. Therefore, $\breve{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is continuous.

It remains to prove that the hypothesis of Γ being a reversible topology on $\mathcal{A}(X)$ is essential for assertion (b) in Theorem 3.1. To show this, we present an example where we use the following technical result:

Lemma 3.1. *Let $\varphi\colon Y \multimap X$ be a multimap and let Γ be a topology on $\mathcal{A}(X)$. Suppose that $\mathfrak{B}$ is a basis for Γ and that for each collection $\{\mathcal{B}_\lambda\}_\lambda \subset \mathfrak{B}$ the identity $\varphi^{-1}\Big(\bigcup\limits_\lambda \sigma(\mathcal{B}_\lambda)\Big) = \bigcup\limits_\lambda \varphi^{-1}(\sigma(\mathcal{B}_\lambda))$ holds true. Then φ is semicompatible with Γ if for every $\mathcal{B} \in \mathfrak{B}$ there exists $\mathcal{B}' \in \mathfrak{B}$ such that $\sigma(\mathcal{B}) = \sigma(\mathcal{B}')$ and $\varphi^{-1}(\sigma(\mathcal{B})) = \breve{\varphi}^{-1}(\mathcal{B}')$.*

P r o o f. Let $\mathcal{V} = \bigcup\limits_\lambda \mathcal{B}_\lambda$, with each $\mathcal{B}_\lambda \in \mathfrak{B}$, an arbitrary element of Γ. By assumption, for each λ there is $\mathcal{B}'_\lambda \in \mathfrak{B}$ such that $\sigma(\mathcal{B}_\lambda) = \sigma(\mathcal{B}'_\lambda)$ and $\varphi^{-1}(\sigma(\mathcal{B}_\lambda)) = \breve{\varphi}^{-1}(\mathcal{B}'_\lambda)$. Let $\mathcal{V}' = \bigcup\limits_\lambda \mathcal{B}'_\lambda$. Then $\mathcal{V}' \in \Gamma$ and, by Lemma 2.1, $\sigma(\mathcal{V}) = \bigcup\limits_\lambda \sigma(\mathcal{B}_\lambda) = \bigcup\limits_\lambda \sigma(\mathcal{B}') = \sigma(\mathcal{V}')$, and so $\sigma(\mathcal{V}) = \sigma(\mathcal{V}')$. Moreover, it follows from the assumptions that

$$\begin{aligned}\varphi^{-1}(\sigma(\mathcal{V})) = \varphi^{-1}\Big(\bigcup_\lambda \sigma(\mathcal{B}_\lambda)\Big) &= \bigcup_\lambda \varphi^{-1}(\sigma(\mathcal{B}_\lambda)) \\ &= \bigcup_\lambda \breve{\varphi}^{-1}(\mathcal{B}'_\lambda) = \breve{\varphi}^{-1}\Big(\bigcup_\lambda \mathcal{B}_\lambda\Big) = \breve{\varphi}^{-1}(\mathcal{V}'),\end{aligned}$$

which proves that φ is semicompatible with Γ. □

E x a m p l e 3.3. If Γ is not a reversible topology on $\mathcal{A}(X)$, then assertion (b) in Theorem 3.1 is not necessarily true. In fact, define $X = \{a,b,c,d\}$ and on $\mathcal{A}(X)$ consider the topology Γ generated by the basis

$$\mathfrak{B}_\Gamma = \{\{\{a\}\}, \{\{a\},\{b\}\}, \{\{a,b\}\}, \{\{b,d\}\}, \{\{a\},\{a,c,d\}\}, \mathcal{A}(X)\}.$$

Since $\sigma\Big(\bigcup\limits_\lambda \mathcal{B}_\lambda\Big) = \bigcup\limits_\lambda \sigma(\mathcal{B}_\lambda)$ (by Lemma 2.1), the set $\sigma(\Gamma)$ is the collection of all reunions of subcollections of

$$\sigma(\mathfrak{B}_\Gamma) \cup \{\emptyset\} = \{\{a\}, \{a,b\}, \{b,d\}, \{a,c,d\}, X\} \cup \{\emptyset\}.$$

Thus, since $\sigma(\Gamma)$ is a subbasis for $\mathcal{R}(\Gamma)$, we have $\{b\} = \{a,b\} \cap \{b,d\} \in \mathcal{R}(\Gamma)$, although $\{b\} \notin \sigma(\Gamma)$, which shows that Γ is not a reversible topology on $\mathcal{A}(X)$.

Consider the set $Y = \{1,2\}$ with the topology $\tau = \{\emptyset, \{1\}, Y\}$ and let $\varphi\colon Y \multimap X$ be the multimap given by $\varphi(1) = \{a\}$ and $\varphi(2) = \{b\}$.

It is easy to check that the multimap φ and the topology Γ generated by the basis $\mathfrak{B}_\Gamma$ satisfy the assumption of the previous lemma and that, moreover, for every $\mathcal{B} \in \mathfrak{B}_\Gamma$ there exists $\mathcal{B}' \in \mathfrak{B}_\Gamma$ such that $\sigma(\mathcal{B}) = \sigma(\mathcal{B}')$ and $\varphi^{-1}(\sigma(\mathcal{B})) = \breve{\varphi}^{-1}(\mathcal{B}')$. Thus, we conclude by the previous lemma that φ is semicompatible (but not compatible, as we can verify easily) with Γ. Now, since the pre-image by $\breve{\varphi}$ of any element of $\mathfrak{B}_\Gamma$ is either $\emptyset$ or $\{1\}$ or X, it follows that $\breve{\varphi}\colon Y \to (\mathcal{A}(X), \Gamma)$ is continuous. However, since $\{b\} \in \mathcal{R}(\Gamma)$ and $\varphi^{-1}(\{b\}) = \{2\} \notin \tau$, it follows that $\varphi\colon Y \multimap (X, \mathcal{R}(\Gamma))$ is not upper semicontinuous.

References

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