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MINIMIZING LAPLACIAN SPECTRAL RADIUS OF  
UNICYCLIC GRAPHS WITH FIXED GIRTH

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*Abstract.* In this paper we consider the following problem: Over the class of all simple connected unicyclic graphs on  $n$  vertices with girth  $g$  ( $n, g$  being fixed), which graph minimizes the Laplacian spectral radius? Let  $U_{n,g}$  be the lollipop graph obtained by appending a pendent vertex of a path on  $n - g$  ( $n > g$ ) vertices to a vertex of a cycle on  $g \geq 3$  vertices. We prove that the graph  $U_{n,g}$  uniquely minimizes the Laplacian spectral radius for  $n \geq 2g - 1$  when  $g$  is even and for  $n \geq 3g - 1$  when  $g$  is odd.

*Keywords:* Laplacian matrix; Laplacian spectral radius; girth; unicyclic graph

*MSC 2010:* 05C50

## 1. INTRODUCTION

Let  $G = (V, E)$  be a simple graph with vertex set  $V = \{v_1, v_2, \dots, v_n\}$  and edge set  $E$ . The *adjacency matrix* of  $G$  is defined as  $A(G) = (a_{ij})$ , where  $a_{ij}$  is equal to 1 if  $\{v_i, v_j\} \in E$  and 0 otherwise. Let  $D(G)$  be the *diagonal matrix* of  $G$  whose  $i$ -th diagonal entry is the degree of the vertex  $v_i$  of  $G$ . The *Laplacian matrix*  $L(G)$  of  $G$  is defined by  $L(G) = D(G) - A(G)$ . Clearly  $L(G)$  is real symmetric. It is known that  $L(G)$  is a positive semi-definite matrix. So all its eigenvalues are real and non-negative. Since the sum of the entries in each row of  $L(G)$  is zero, the all one vector  $\mathbf{e} = [1, \dots, 1]^T$  is an eigenvector of  $L(G)$  corresponding to the smallest eigenvalue zero. Here  $X^T$  denotes the transpose of a given matrix  $X$ . For more about the Laplacian matrix and its eigenvalues we refer the reader to [10], [11], [12] and the references therein.

The largest eigenvalue of  $L(G)$  is called the *Laplacian spectral radius* of  $G$ , we denote it by  $\lambda(G)$ . Among all trees on  $n$  vertices, the Laplacian spectral radius is uniquely minimized by the path, and uniquely maximized by the star. The tree that

uniquely maximizes the Laplacian spectral radius over all trees on  $n$  vertices with fixed diameter or with fixed number of pendant vertices is characterized in [9] or [6], respectively. Over all unicyclic graphs on  $n$  vertices, the cycle has the minimum Laplacian spectral radius and the graph obtained by joining  $n - 3$  pendant vertices to a vertex of a 3-cycle has the maximum Laplacian spectral radius.

The second smallest eigenvalue of  $L(G)$  is called the *algebraic connectivity* of  $G$  [3]. The Laplacian spectral radius of  $G$  is related to the algebraic connectivity of the complement graph of  $G$ . The study of the algebraic connectivity and the Laplacian spectral radius of graphs has received a good deal of attention in the recent past. As far as the class of connected unicyclic graphs on  $n$  vertices with fixed girth is concerned, the problem of minimizing and maximizing the algebraic connectivity has been studied in [1] and [2], respectively, and that of maximizing the Laplacian spectral radius is solved by Guo ([7], Corollary 4.1). However, the question of minimizing the Laplacian spectral radius has not been studied so far and we deal with this problem in the present paper.

Recall that the *girth* of a graph  $G$  is the length of a shortest cycle (if any) of  $G$ . A graph  $G$  is *unicyclic* if it has exactly one cycle, that is, the number of edges and the number of vertices of  $G$  are the same. Consider a cycle on  $g \geq 3$  vertices and append a pendent vertex of the path on  $n - g$  ( $n > g$ ) vertices to a vertex of the cycle. The new graph thus obtained is a unicyclic graph on  $n$  vertices with girth  $g$ , called a *lollipop graph*. We denote it by  $U_{n,g}$  (see Figure 2). In this paper, we prove

**Theorem 1.1.** *Let  $G$  be a unicyclic graph on  $n$  vertices with girth  $g$  which is not isomorphic to  $U_{n,g}$ . Then the following assertions hold.*

- (1) *If  $g$  is even and  $n \geq 2g - 1$ , then  $\lambda(U_{n,g}) < \lambda(G)$ .*
- (2) *If  $g$  is odd and  $n \geq 3g - 1$ , then  $\lambda(U_{n,g}) < \lambda(G)$ .*

In Section 2, we recall some basic definitions and results from literature which are needed in the subsequent sections. In Section 3, we study the Laplacian spectral radius of the graph  $U_{n,g}$ . Finally, we prove Theorem 1.1 in Section 4.

## 2. PRELIMINARIES

All graphs considered in this paper are finite, simple and connected. For any graph  $G$ , define  $B(G) = D(G) + A(G)$ . We denote by  $\mu(G)$  the largest eigenvalue of  $B(G)$ . By Rayleigh-Ritz theorem ([8], p. 176), we have

$$\lambda(G) = \max_{X \in W} X^T L(G) X,$$

and

$$\mu(G) = \max_{X \in W} X^T B(G) X,$$

where  $W = \{X \in \mathbb{R}^n; X^T X = 1\}$ . Since  $G$  is connected,  $B(G)$  is a nonnegative irreducible matrix. So  $\mu(G)$  is simple and there exists a positive eigenvector of  $B(G)$  corresponding to  $\mu(G)$ . This is a consequence of the Perron-Frobenius theory. If  $X$  is a unit eigenvector of  $B(G)$  corresponding to  $\mu(G)$ , then we have

$$\mu(G) = X^T B(G) X = \sum_{\{v_i, v_j\} \in E} (x_i + x_j)^2, \quad \text{where } X^T = [x_1, x_2, \dots, x_n].$$

Recall that  $G$  is *bipartite* if its vertex set  $V$  is a disjoint union of two sets  $V_1$  and  $V_2$  such that every edge in  $G$  joins a vertex of  $V_1$  to a vertex of  $V_2$ . The next proposition and the following lemmas relates  $L(G)$  and  $B(G)$  for a bipartite graph  $G$ .

**Lemma 2.1** ([5], p. 220). *Let  $G$  be a bipartite graph. Then  $B(G)$  and  $L(G)$  are unitarily similar. In particular,  $\lambda(G)$  is simple.*

By Lemma 2.1 and the Perron-Frobenius theory, the following results are immediate.

**Lemma 2.2.** *Let  $G$  be a bipartite graph and  $G'$  a graph obtained from  $G$  by adjoining a new vertex to a vertex of  $G$ . Then  $\lambda(G') > \lambda(G)$ .*

**Lemma 2.3** ([5], p. 233). *Let  $G$  be a graph and  $G'$  a graph obtained from  $G$  by joining two non-adjacent vertices of  $G$  with an edge. Then  $\lambda(G') \geq \lambda(G)$ .*

**Lemma 2.4** ([4], p. 224). *Let  $G$  be a graph on  $n \geq 2$  vertices. Then  $\lambda(G) \geq \Delta(G) + 1$ , where  $\Delta(G)$  is the maximum vertex degree of  $G$ . Further, equality holds if and only if  $\Delta(G) = n - 1$ .*

Lemma 2.3 says that introducing a new edge in a graph  $G$  cannot decrease the Laplacian spectral radius. The next result mentions a case in which the Laplacian spectral radius remains the same even after introducing a new edge in the given graph.

**Proposition 2.5** ([7], p. 712). *Let  $G$  be a graph on  $n \geq 2$  vertices and  $v$  a vertex of  $G$ . Let  $G_s^k$  be the graph obtained from  $G$  by attaching  $s \geq 2$  new paths  $P_i: vv_{ik}v_{i(k-1)} \dots v_{i1}$  ( $1 \leq i \leq s$ ) at  $v$  each of length  $k \geq 1$ . Let  $G_{s,t}^k$  be the graph obtained from  $G_s^k$  by adding  $t$  ( $1 \leq t \leq \frac{1}{2}s(s-1)$ ) edges among the vertices  $v_{11}, v_{21}, \dots, v_{s1}$ . Then  $\lambda(G_s^k) = \lambda(G_{s,t}^k)$ .*

Let  $G$  be a graph on  $n \geq 2$  vertices and  $v$  a vertex of  $G$ . For  $l \geq k \geq 1$ , we construct a new graph  $G_{k,l}$  from  $G$  by attaching two new paths  $P: vv_1v_2 \dots v_k$  and  $Q: vu_1u_2 \dots u_l$  of lengths  $k$  and  $l$ , respectively, at  $v$ . Let  $\tilde{G}_{k,l}$  be the graph obtained from  $G_{k,l}$  by removing the edge  $\{v_{k-1}, v_k\}$  and adding the edge  $\{u_l, v_k\}$  (see Figure 1). We say that  $\tilde{G}_{k,l}$  is obtained from  $G_{k,l}$  by *grafting* an edge. The next result compares the Laplacian spectral radius of  $G_{k,l}$  and  $\tilde{G}_{k,l} \simeq G_{k-1,l+1}$ .

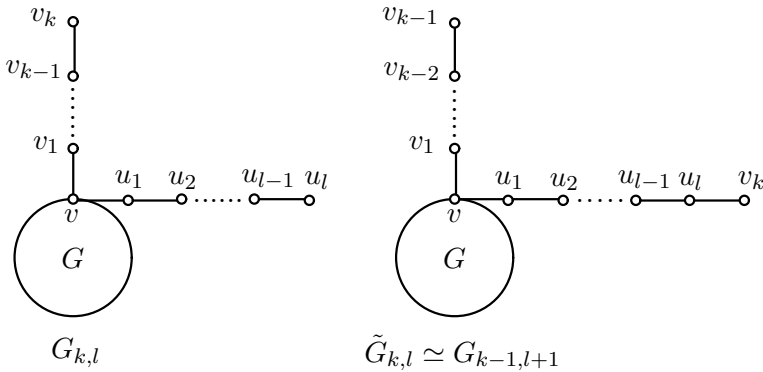


Figure 1. Grafting an edge

**Proposition 2.6** ([6], p. 65, 68). *Let  $G$  be a graph on  $n \geq 2$  vertices and  $v$  a vertex of  $G$ . For  $l \geq k \geq 1$ , let  $G_{k,l}$  be the graph defined as above. Then*

$$\lambda(G_{k-1,l+1}) \leq \lambda(G_{k,l}),$$

*with equality if and only if there exists an eigenvector of  $G_{k,l}$  corresponding to  $\lambda(G_{k,l})$  whose  $v$ -th component is zero. In particular, if  $G$  is bipartite, then  $\lambda(G_{k-1,l+1}) < \lambda(G_{k,l})$ .*

**Lemma 2.7.** *Let  $f_1(y) = y - 1$  and define  $f_i(y) = y - 2 - 1/f_{i-1}(y)$  for  $i \geq 2$ . Then for  $i, j \geq 1$ ,*

- (i)  $f_i(y) > y/(y - 2)$  for  $y \geq 4.383$ ;
- (ii)  $f_i(y) > f_{i+1}(y) > 1$  for  $y \geq 4$ ;
- (iii)  $f_i(y)f_{i+1}(y) > f_j(y)$  for  $y \geq 4.383$ .

**Proof.** The proofs of (i) and (ii) are similar to that of Lemma 3.2 in [6]. We prove (iii) here. We have

$$f_i(y)f_{i+1}(y) = f_i(y)\left(y - 2 - \frac{1}{f_i(y)}\right) = (y - 2)f_i(y) - 1.$$

Since  $y \geq 4.383$ , (i) implies that

$$f_i(y)f_{i+1}(y) > (y-2)\frac{y}{y-2} - 1 = y-1 = f_1(y) \geq f_j(y).$$

Here the last inequality follows from (ii).  $\square$

For an eigenvector  $X$  of a graph  $G$  corresponding to  $\mu(G)$  (or  $\lambda(G)$ ), we associate with  $X$  a labeling of  $G$  in which a vertex  $v_i$  is labeled  $x_{v_i}$  or simply  $x_i$ .

**Lemma 2.8.** *Let  $v$  be a vertex of a bipartite graph  $H$  and  $G$  a graph obtained from  $H$  by attaching a path  $P: v = v_0v_1 \dots v_k$  at  $v$ . Let  $X$  be a positive eigenvector of  $B(G)$  corresponding to  $\mu = \mu(G)$ . Then  $x_{v_i} = f_{k-i}(\mu)x_{v_{i+1}}$  for  $0 \leq i \leq k-1$ , where  $f_i(y)$  is the function defined in Lemma 2.7. Further, if  $\mu \geq 4$ , then  $x_{v_i} > x_{v_{i+1}}$  for  $0 \leq i \leq k-1$ .*

*Proof.* From  $B(G)X = \mu X$ , we have  $x_{v_{k-1}} = (\mu-1)x_{v_k}$  and  $x_{v_{i-2}} + x_{v_i} = (\mu-2)x_{v_{i-1}}$  for  $2 \leq i \leq k$ . Using these two equations it follows that  $x_{v_i} = f_{k-i}(\mu)x_{v_{i+1}}$  for  $0 \leq i \leq k-1$ . If  $\mu \geq 4$ , then  $f_{k-i}(\mu) > 1$  by Lemma 2.7(ii). So  $x_{v_i} > x_{v_{i+1}}$  for  $0 \leq i \leq k-1$ .  $\square$

Let  $G$  be a graph with vertex set  $V = \{v_1, \dots, v_n\}$  and Laplacian matrix  $L = L(G) = (l_{ij})$ . Let  $\tau$  be an automorphism of  $G$ . Since  $\tau$  is a permutation of  $V$ , it induces a permutation matrix  $P = (p_{ij})$ , where  $p_{ij}$  is defined by:

$$p_{ij} = \begin{cases} 1 & \text{if } \tau(v_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

If  $\tau(v_t) = v_i$  and  $\tau(v_j) = v_s$ , then the  $ij$ -th entry of  $PL$  is  $l_{tj}$  and that of  $LP$  is  $l_{is}$ . We have that  $v_t \neq v_j$  if and only if  $v_i \neq v_s$ , and that  $v_t$  and  $v_j$  are adjacent if and only if  $v_i$  and  $v_s$  are adjacent. This, together with the fact that  $\tau$  preserves the degree of a vertex, implies that  $l_{tj} = l_{is}$ . Thus  $PL = LP$ . The same argument implies that  $PB = BP$ , where  $B = B(G)$ .

**Lemma 2.9.** *Let  $G$  be a graph and  $\tau$  an automorphism of  $G$ . If  $\lambda = \lambda(G)$  is a simple eigenvalue of  $L = L(G)$  and  $X$  is an eigenvector corresponding to  $\lambda$ , then  $|x_{v_k}| = |x_{\tau(v_k)}|$  for every  $v_k \in V$ .*

*Proof.* We have  $LPX = PLX = P\lambda X = \lambda PX$ . So  $PX$  is also an eigenvector of  $L$  corresponding to  $\lambda$ . Since  $\lambda$  has algebraic multiplicity one,  $X$  and  $PX$  are linearly dependent. So  $PX = \alpha X$  for some real number  $\alpha$ . Since  $P$  has finite order (as a group element in  $GL(n, \mathbb{R})$ ), it follows that  $\alpha$  is a  $k$ -th root of unity for some positive integer  $k$ . So  $\alpha = \pm 1$  and  $PX = \pm X$ . Then  $\sum_{j=1}^n p_{ij}x_{v_j} = \pm x_{v_i}$ . If  $\tau(v_k) = v_i$ , then  $x_{v_k} = \pm x_{v_i}$  and so  $|x_{v_k}| = |x_{v_i}| = |x_{\tau(v_k)}|$ .  $\square$

**Lemma 2.10.** *Let  $G$  be a bipartite graph and  $\tau$  an automorphism of  $G$ . Let  $X$  be a positive eigenvector of  $B(G)$  corresponding to  $\mu(G)$ . Then  $x_{v_k} = x_{\tau(v_k)}$  for  $v_k \in V$ .*

*Proof.* Since  $G$  is bipartite,  $\mu(G)$  is a simple eigenvalue of  $B(G)$ . Now, the proof is similar to that of Lemma 2.9.  $\square$

### 3. THE GRAPH $U_{n,g}$

In Section 1, we defined the graph  $U_{n,g}$  for  $n > g$ . We take the vertex set  $V$  of  $U_{n,g}$  as  $V = \{1, 2, \dots, n\}$  and the edges of  $U_{n,g}$  as shown in Figure 2. Note that, by Lemma 2.4,  $\lambda(U_{n,g}) \geq 4$  with equality if and only if  $n = 4$ .

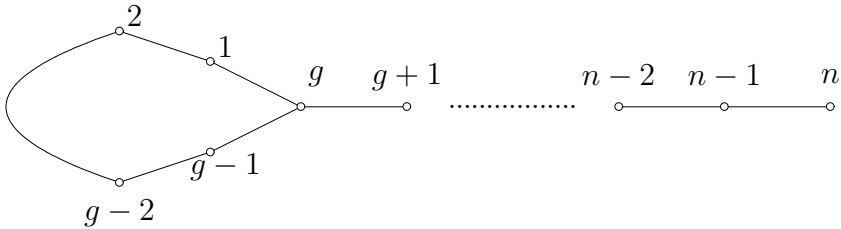


Figure 2. The graph  $U_{n,g}$

We first consider the graph  $U_{n,g}$  with  $g$  even. So  $U_{n,g}$  is a bipartite graph and hence  $\lambda(U_{n,g}) = \mu(U_{n,g})$  by Lemma 2.1.

**Proposition 3.1.** *Let  $G = U_{n,g}$  with  $g$  even and let  $X = [x_1, \dots, x_n]^T$  be a positive eigenvector of  $B = B(G)$  corresponding to  $\mu = \mu(G)$ . Then the following assertions hold:*

- (i)  $x_j = x_{g-j}$  for  $j \in \{1, \dots, \frac{1}{2}g\}$ .
- (ii)  $x_j > x_{j+1}$  for  $j \in \{g, \dots, n-1\}$ .
- (iii)  $x_g > x_1$  and  $x_j > x_{j+1}$  for  $j \in \{1, \dots, \frac{1}{2}g-1\}$ .
- (iv) If  $\mu \geq 4.5$ , then  $x_j > 2x_{j+1}$  for  $j \in \{g, \dots, n-1\}$ ,  $x_g < 2x_1$  and  $x_j < 2x_{j+1}$  for  $j \in \{1, \dots, \frac{1}{2}g-1\}$ .
- (v) If  $\mu \geq 4.5$ , then  $x_i > x_{g+i}$ , where  $1 \leq i \leq \min\{\frac{1}{2}g, n-g\}$ .

*Proof.* (i) The map  $\tau$  fixing the vertices  $g, g+1, \dots, n$  and taking  $j$  to  $g-j$  for  $j \in \{1, \dots, g-1\}$  is an automorphism of  $G$ . Since  $G$  is a bipartite graph, the result follows from Lemma 2.10.

(ii) This follows from Lemma 2.8, since  $\mu = \lambda(G) \geq 4$ .

(iii) From  $BX = \mu X$  at the vertex  $\frac{1}{2}g$ , we have  $x_{g/2-1} + 2x_{g/2} + x_{g/2+1} = \mu x_{g/2}$ . Since  $x_{g/2-1} = x_{g/2+1}$  by (i) and  $\mu > 4$ , we have  $x_{g/2} = (2/(\mu - 2))x_{g/2-1} < x_{g/2-1}$ . Assume that  $x_{j+1} < x_j$  for  $j \in \{2, \dots, \frac{1}{2}g - 1\}$ . Now  $x_{j-1} = (\mu - 2)x_j - x_{j+1} > (\mu - 3)x_j > x_j$ . So  $x_j > x_{j+1}$  for  $j \in \{1, \dots, \frac{1}{2}g\}$ . A similar proof holds for  $x_g > x_1$ .

(iv) For  $j = n - 1$  we have  $x_{n-1} = (\mu - 1)x_n > 2x_n$ . Assume that  $x_j > 2x_{j+1}$  for  $j \in \{g + 1, \dots, n - 1\}$ . Now

$$x_{j-1} = (\mu - 2)x_j - x_{j+1} \geq 2.5x_j - x_{j+1} > 2.5x_j - 0.5x_j = 2x_j.$$

So  $x_j > 2x_{j+1}$  for  $j \in \{g, \dots, n - 1\}$ . From  $BX = \mu X$  at the vertex  $g$  and using  $x_1 = x_{g-1}$  by (i), we have

$$2x_1 = (\mu - 3)x_g - x_{g+1} \geq 1.5x_g - x_{g+1} > 1.5x_g - 0.5x_g = x_g.$$

So  $x_g < 2x_1$ . Now at the vertex 1,

$$x_2 = (\mu - 2)x_1 - x_g \geq 2.5x_1 - x_g > 2.5x_1 - 2x_1 = 0.5x_1.$$

So  $x_1 < 2x_2$ . Assume that  $x_{j-1} < 2x_j$  for  $j \in \{2, \dots, 0.5g - 1\}$ . Then

$$x_{j+1} = (\mu - 2)x_j - x_{j-1} \geq 2.5x_j - x_{j-1} > 2.5x_j - 2x_j = 0.5x_j.$$

So  $x_j < 2x_{j+1}$  for  $j \in \{1, \dots, 0.5g - 1\}$ .

(v) By (iv),  $2x_1 > x_g > 2x_{g+1}$ . So  $x_1 > x_{g+1}$ . Inductively, for  $1 \leq i \leq \min\{0.5g, n - g\}$ ,  $2x_i > x_{i-1} > x_{g+i-1} > 2x_{g+i}$  gives that  $x_i > x_{g+i}$ .  $\square$

**Lemma 3.2.** *Let  $g$  be even,  $k = 0.5g$  and  $n = g + k$ . Then  $\mu(U_{n,g}) \geq 4.5$ .*

*Proof.* Define the numbers  $a_g = 2^k$ ,  $a_i = a_{g-i} = a_{g+i} = 2^{k-i}$  for  $i \in \{1, 2, \dots, k\}$  and  $a = \left(\sum_{j=1}^n a_j^2\right)^{1/2}$ . We have

$$\begin{aligned} a^2 &= a_g^2 + 3(a_{g+1}^2 + \dots + a_{g+k-1}^2) + 2a_n^2 \\ &= 2^{2k} + 3(2^{2k-2} + \dots + 2^2) + 2 = 2(2^{2k} - 1). \end{aligned}$$

Let  $X = (x_1, \dots, x_n)$  be the unit vector, where  $x_j = a_j/a$  for  $j \in \{1, 2, \dots, n\}$ . Then

$$\begin{aligned} \sum_{\{i,j\} \in E} (x_i + x_j)^2 &= 2 \sum_{j=1}^n x_j^2 + x_g^2 - x_n^2 + 6 \sum_{j=g}^{n-1} x_j x_{j+1} \\ &= 2 + \frac{1}{a^2}(2^{2k} - 1) + \frac{6}{a^2}(2^{2k-1} + 2^{2k-3} + \dots + 2^3 + 2) \\ &= 2 + \frac{1}{a^2}(2^{2k} - 1) + \frac{4}{a^2}(2^{2k} - 1) \\ &= 4.5. \end{aligned}$$

So  $\mu(U_{n,g}) \geq X^T B(U_{n,g}) X = \sum_{\{i,j\} \in E} (x_i + x_j)^2 = 4.5$ .  $\square$



**Corollary 3.3.** *If  $g$  is even and  $n \geq 1.5g$ , then  $\lambda(U_{n,g}) \geq 4.5$ .*

*Proof.* This follows from Lemmas 2.2 and 3.2. □

We now consider the graph  $U_{n,g}$  with  $g$  odd. Let  $\overline{U}_{n,g}$  be the graph obtained from  $U_{n,g}$  by deleting the edge  $\{(g-1)/2, (g+1)/2\}$ . Then  $\overline{U}_{n,g}$  is a tree, and so  $\lambda(\overline{U}_{n,g}) = \mu(\overline{U}_{n,g})$ .

**Proposition 3.4.** *Let  $G = \overline{U}_{n,g}$  and let  $X = [x_1, \dots, x_n]^T$  be a positive eigenvector of  $B(G)$  corresponding to  $\mu = \mu(G)$ . Then the following assertions hold:*

- (i)  $x_j = x_{g-j}$  for  $j \in \{1, \dots, (g-1)/2\}$ .
- (ii)  $x_j > x_{j+1}$  for  $j \in \{g, \dots, n-1\}$ .
- (iii)  $x_g > x_1$  and  $x_j > x_{j+1}$  for  $j \in \{1, \dots, (g-3)/2\}$ .
- (iv) If  $\mu \geq 4.383$ , then  $x_i > x_{g+2i}$ , where  $1 \leq i \leq \min\{(g-1)/2, \lfloor (n-g)/2 \rfloor\}$ .

*Proof.* The proofs of (i), (ii) and (iii) are similar to those of Proposition 3.1 (i), (ii) and (iii). We now prove (iv). Along the path  $g(g+1)(g+2) \dots n$  in  $\overline{U}_{n,g}$ , by Lemma 2.8 we have

$$\begin{aligned} x_g &= f_{n-g}(\mu)x_{g+1} \\ &= f_{n-g}(\mu)f_{n-g-1}(\mu)x_{g+2} \\ &\quad \vdots \\ &= f_{n-g}(\mu)f_{n-g-1}(\mu) \dots f_2(\mu)f_1(\mu)x_n. \end{aligned}$$

Similarly, along the path  $g12 \dots (g-1)/2$  in  $\overline{U}_{n,g}$  we have

$$x_g = f_{(g-1)/2}(\mu)x_1 = f_{(g-1)/2}(\mu)f_{(g-3)/2}(\mu)x_2 = \dots = f_{(g-1)/2}(\mu) \dots f_1(\mu)x_{(g-1)/2}.$$

Thus, for  $1 \leq i \leq \min\{(g-1)/2, \lfloor (n-g)/2 \rfloor\}$ ,

$$f_{(g-1)/2}(\mu) \dots f_{(g-2i+1)/2}(\mu)x_i = x_g = f_{n-g}(\mu) \dots f_{n-g-2i+1}(\mu)x_{g+2i}.$$

Since  $\mu \geq 4.383$ ,

$$f_{n-g}(\mu) \dots f_{n-g-2i+1}(\mu) > f_{(g-1)/2}(\mu) \dots f_{(g-2i+1)/2}(\mu)$$

by Lemma 2.7 (iii). So  $x_i > x_{g+2i}$ . This completes the proof. □

**Lemma 3.5.** *If  $g \geq 5$  is odd and  $n \geq g + 2$ , then  $\mu(\overline{U}_{n,g}) = \lambda(U_{n,g}) > 4.383$ .*

*Proof.* Since  $g$  is odd,  $\lambda(U_{n,g}) = \lambda(\overline{U}_{n,g})$  by Proposition 2.5. Now consider the star graph  $S$  on 4 vertices. Form a new graph  $S_1$  by appending one vertex to each of the pendant vertices of  $S$ . Using MATLAB, we have  $\lambda(S_1) \approx 4.4142 > 4.383$ . So for odd  $g \geq 5$  and  $n \geq g + 2$ ,  $\mu(\overline{U}_{n,g}) = \lambda(\overline{U}_{n,g}) \geq \lambda(S_1) > 4.383$  (see Lemma 2.2).  $\square$

#### 4. PROOF OF THEOREM 1.1

For a unicyclic graph  $G$  on  $n$  vertices with girth  $g \geq 3$  ( $n > g$ ), we take the vertices on the cycle of  $G$  as  $1, 2, \dots, g$  and denote by  $l_i$  the number of vertices on the tree attached to the vertex  $i$  for  $1 \leq i \leq g$ . We define  $C_G$  to be the set of all vertices  $i$  on the cycle of  $G$  for which  $l_i \geq 1$ . Then  $l_1 + \dots + l_g = n - g$  and  $1 \leq |C_G| \leq g$ .

For two distinct vertices  $u$  and  $v$  of  $G$ , we denote by  $d(u, v)$  the *distance* between  $u$  and  $v$  (that is, the length of a shortest path between  $u$  and  $v$ ). The following lemma is useful for us.

**Lemma 4.1.** *Let  $G$  be a unicyclic graph on  $n$  vertices with girth  $g$ . Suppose that  $|C_G| = r \geq 2$ . Then the following assertions hold:*

- (a) *If  $n \geq 2g - 1$ , then there exist two vertices  $i, j \in C_G$  with  $l_i \geq d(i, j)$ .*
- (b) *If  $n \geq 3g - 1$ , then there exist two vertices  $i, j \in C_G$  with  $l_i \geq 2d(i, j)$ .*

*Proof.* Let  $i_1, i_2, \dots, i_r$  be the vertices in  $C_G$  with  $i_1 < i_2 < \dots < i_r$ . Set  $d_j = d(i_j, i_{j+1})$  for  $1 \leq j \leq r - 1$  and  $d_r = d(i_r, i_1)$ .

To prove (a), it is enough to show that  $d_j \leq \max\{l_j, l_{j+1}\}$  for some  $1 \leq j \leq r - 1$  or  $d_r \leq \max\{l_r, l_1\}$ . Suppose that this is not true. Then  $l_j \leq d_j - 1$  for  $1 \leq j \leq r$ . Now  $g - 1 \leq l_1 + \dots + l_r \leq d_1 + \dots + d_r - r \leq g - r$ . This implies that  $r \leq 1$ , a contradiction.

To prove (b), we show that  $2d_j \leq \max\{l_j, l_{j+1}\}$  for some  $1 \leq j \leq r - 1$  or  $2d_r \leq \max\{l_r, l_1\}$ . If this is not true, then  $l_j \leq 2d_j - 1$  for  $1 \leq j \leq r$ . Now  $2g - 1 \leq l_1 + \dots + l_r \leq 2(d_1 + \dots + d_r) - r \leq 2g - r$ . This gives  $r \leq 1$ , a contradiction.  $\square$

**4.1. The case  $g$  even.** Let  $G$  be a unicyclic graph on  $n$  vertices with even girth  $g$ . By a repeated use of graph operations consisting of grafting of edges we can transform the graph  $G$  into a new graph  $G_1$  that has a path  $P_i$  (on  $l_i$  vertices) appended to the vertex  $i$  for each  $i \in C_G$  and satisfies  $\lambda(G_1) \leq \lambda(G)$ . This is possible by Proposition 2.6.

**Proposition 4.2.** *Let  $G$  be a unicyclic graph on  $n$  vertices with even girth  $g$  which is not isomorphic to  $U_{n,g}$ . If  $|C_G| = 1$ , then  $\lambda(U_{n,g}) < \lambda(G)$ .*

*Proof.* Since  $G$  is not isomorphic to  $U_{n,g}$ , we use the operation grafting of edges at least once to get the graph  $G_1$ . Since  $G$  is bipartite,  $\lambda(G_1) < \lambda(G)$  by Proposition 2.6. Now  $|C_G| = 1$  implies that  $G_1$  is isomorphic to  $U_{n,g}$ . So  $\lambda(U_{n,g}) < \lambda(G)$ .  $\square$

**Proposition 4.3.** *Let  $G$  be a unicyclic graph on  $n$  vertices with even girth  $g$ . Suppose that  $|C_G| = r \geq 2$  and that we can arrange the vertices in  $C_G$  in some ordering, say  $i_1, i_2, \dots, i_r$ , such that  $l_{i_1} + \dots + l_{i_{j-1}} \geq d(i_1, i_j)$  for  $2 \leq j \leq r$ . If  $\lambda(U_{n,g}) \geq 4.5$ , then  $\lambda(U_{n,g}) < \lambda(G)$ .*

*Proof.* Consider the graph  $G_1$  obtained from  $G$  as above. For  $1 \leq j \leq r$ , let  $i_j 1$  be the vertex on the path  $P_{i_j}$  adjacent to the vertex  $i_j$ , and let  $i_j l_{i_j}$  be the pendant vertex of  $P_{i_j}$ . That is, the path  $P_{i_j}$  is  $(i_j 1)(i_j 2) \dots (i_j l_{i_j})$ , where the vertex  $i_j 1$  is adjacent to  $i_j$ . Note that  $i_j 1$  and  $i_j l_{i_j}$  are the same vertices if  $l_{i_j} = 1$ . With the sequence of vertices  $i_1, i_2, \dots, i_r$ , we perform the following graph operations on  $G_1$ :

Delete the edge  $\{i_2, i_2 1\}$  and add a new edge  $\{i_1 l_{i_1}, i_2 1\}$ ,  
delete the edge  $\{i_3, i_3 1\}$  and add a new edge  $\{i_2 l_{i_2}, i_3 1\}$  and so on.

Let  $G_2$  be the new graph thus obtained from  $G_1$ . Then  $G_2$  is isomorphic to  $U_{n,g}$ . Both  $G_1$  and  $G_2$  are bipartite as  $g$  is even. By Lemma 2.1,  $\mu(G_t) = \lambda(G_t)$  for  $t = 1, 2$ . Let  $X$  be the positive unit eigenvector of  $B(G_2)$  corresponding to  $\mu(G_2)$ . Then

$$\begin{aligned} \lambda(G_1) - \lambda(G_2) &= \mu(G_1) - \mu(G_2) \\ &\geq X^T B(G_1) X - \mu(G_2) \\ &= X^T B(G_1) X - X^T B(G_2) X \\ &= [(x_{i_2} + x_{i_2 1})^2 - (x_{i_1 l_{i_1}} + x_{i_2 1})^2] \\ &\quad + [(x_{i_3} + x_{i_3 1})^2 - (x_{i_2 l_{i_2}} + x_{i_3 1})^2] \\ &\quad + \dots + [(x_{i_r} + x_{i_r 1})^2 - (x_{i_{r-1} l_{i_{r-1}}} + x_{i_r 1})^2]. \end{aligned}$$

Since  $\lambda(U_{n,g}) \geq 4.5$  and  $l_{i_1} + \dots + l_{i_{j-1}} \geq d(i_1, i_j)$  for  $2 \leq j \leq r$ , Proposition 3.1(v) implies that  $x_{i_2} > x_{i_1 l_{i_1}}, x_{i_3} > x_{i_2 l_{i_2}}, \dots, x_{i_r} > x_{i_{r-1} l_{i_{r-1}}}$ . So  $\lambda(G_1) - \lambda(G_2) > 0$ . Hence  $\lambda(G) \geq \lambda(G_1) > \lambda(G_2) = \lambda(U_{n,g})$ .  $\square$

As an immediate consequence of Propositions 4.2 and 4.3, we have

**Corollary 4.4.** *Let  $G$  be a unicyclic graph on  $n$  vertices with even girth  $g$  which is not isomorphic to  $U_{n,g}$ . If  $l_k \geq \frac{1}{2}g$  for some vertex  $k$  in  $C_G$ , then  $\lambda(U_{n,g}) < \lambda(G)$ .*

*Proof.* Since  $l_k \geq \frac{1}{2}g$ , we have  $\lambda(U_{n,g}) \geq 4.5$  by Corollary 3.3. Now the result follows from Proposition 4.2 for  $|C_G| = 1$  and from Proposition 4.3 for  $|C_G| \geq 2$ . In the latter case, we can take any ordering of the vertices in  $C_G$  starting with the vertex  $k$ . Note that the distance between two vertices in  $C_G$  is at most  $\frac{1}{2}g$ .  $\square$

**Proposition 4.5.** *Let  $G$  be a unicyclic graph on  $n$  vertices with even girth  $g$ . Suppose that  $|C_G| = r \geq 2$  and that  $n \geq 2g - 1$ . Then  $\lambda(U_{n,g}) < \lambda(G)$ .*

*Proof.* We have  $\lambda(U_{n,g}) \geq 4.5$  by Corollary 3.3. So, by Proposition 4.3, it is enough to show that we can arrange the vertices in  $C_G$  in an ordering  $i_1, i_2, \dots, i_r$  such that  $l_{i_1} + \dots + l_{i_{j-1}} \geq d(i_1, i_j)$  for  $2 \leq j \leq r$ . We shall prove this by induction on  $r$ . By Lemma 4.1(a), there exist two distinct vertices  $i_1$  and  $i_2$  in  $C_G$  such that  $l_{i_2} \geq d(i_1, i_2)$ . If  $r = 2$ , then we take the ordering  $i_2, i_1$ .

So assume that  $r > 2$ . Now consider the graph  $G_1$  obtained from  $G$  as above. Let  $K$  be the graph obtained from  $G_1$  by disconnecting the path  $P_{i_1}$  from the vertex  $i_1$  and appending it to the pendant vertex of the path  $P_{i_2}$ . We have  $C_G = C_{G_1}$  and  $C_K = C_G - \{i_1\}$ . Set  $l'_{i_2} = l_{i_1} + l_{i_2}$ . Since  $|C_K| < r$ , applying the induction hypothesis to the graph  $K$  we can get an ordering  $k_2, \dots, k_t (= i_2), \dots, k_r$  of the vertices in  $C_K$  such that

$$\begin{aligned} l_{k_2} + \dots + l_{k_{j-1}} &\geq d(k_2, k_j) \quad \text{for } 3 \leq j \leq t, \\ l_{k_2} + \dots + l_{k_{t-1}} + l'_{k_t} &\geq d(k_2, k_{t+1}), \end{aligned}$$

and

$$l_{k_2} + \dots + l_{k_{t-1}} + l'_{k_t} + l_{k_{t+1}} + \dots + l_{k_{j-1}} \geq d(k_2, k_j) \quad \text{for } t+2 \leq j \leq r.$$

Now the facts  $l_{i_2} \geq d(i_1, i_2)$  and  $l'_{i_2} = l_{i_1} + l_{i_2}$  imply that the ordering  $k_2, \dots, k_t (= i_2), i_1, k_{t+1}, \dots, k_r$  of the vertices in  $C_G$  satisfies our requirement.  $\square$

Now Theorem 1.1(1) follows from Propositions 4.2 and 4.5.

Consider the case  $g = 4$ . There is only one unicyclic graph on five vertices. For  $n \geq 6$ , we have  $\lambda(U_{n,4}) \geq 4.5$  by Corollary 3.3. By Theorem 1.1(1),  $U_{n,4}$  uniquely minimizes the Laplacian spectral radius when  $n \geq 7$ . There are four non-isomorphic unicyclic graphs on six vertices and exactly one of them does not satisfy the conditions in Propositions 4.2 or 4.3. This graph  $G$  is obtained from a cycle of

length four by appending one pendant vertex to each of its two opposite vertices. Using MATLAB we can see that  $\lambda(G) \approx 4.73205 > 4.5615 \approx \lambda(U_{6,4})$ .

Now, consider the case  $g = 6$ . There is only one unicyclic graph when  $n = 7$  and five non-isomorphic unicyclic graphs when  $n = 8$ . In the latter case, we can verify that  $U_{8,6}$  (with  $\lambda(U_{8,6}) \approx 4.4989$ ) uniquely minimizes the Laplacian spectral radius. For  $n \geq 9$ , we have  $\lambda(U_{n,6}) \geq 4.5$  by Corollary 3.3. By Theorem 1.1(1),  $U_{n,6}$  uniquely minimizes the Laplacian spectral radius for  $n \geq 11$ . There are three graphs for  $n = 10$  (and also for  $n = 9$ ) with  $|C_G| \geq 2$  for which the hypothesis in Proposition 4.3 is not satisfied. In both cases, we can check that  $U_{n,6}$  uniquely minimizes the Laplacian spectral radius. Thus we have

**Proposition 4.6.** *For  $g \in \{4, 6\}$ , the graph  $U_{n,g}$  uniquely minimizes the Laplacian spectral radius over all unicyclic graphs on  $n$  vertices with girth  $g$ .*

Having seen that  $U_{n,g}$  uniquely minimizes the Laplacian spectral radius when  $g = 4$  and  $6$ , it is natural to expect that the same might be true for all even  $g$ . However, this is false when  $n$  is not large comparing to  $g$ . We give an example below.

**Example 4.7.** Consider a cycle on 10 vertices. Let  $v$  and  $w$  be two opposite vertices on this cycle, that is,  $d(v, w) = 5$ . Add one pendant vertex to each of the vertices  $v$  and  $w$ . The new graph is a unicyclic graph on 12 vertices with girth 10, denote it by  $C_{10}^{1,1}$ . We have  $4.4383 \approx \lambda(C_{10}^{1,1}) < \lambda(U_{12,10}) \approx 4.4763$ .

**4.2. The case  $g$  odd.** In this subsection we prove Theorem 1.1 in the case  $g$  odd.

**Proposition 4.8.** *Let  $H$  be a unicyclic graph on  $n$  vertices with odd girth  $g$  which is not isomorphic to  $U_{n,g}$ . If  $|C_H| = 1$ , then  $\lambda(U_{n,g}) < \lambda(H)$ .*

*Proof.* Let  $C_H = \{j\}$  and let  $H_1$  be the bipartite graph obtained from  $H$  by deleting the edge opposite to the vertex  $j$ . Since  $H$  is not isomorphic to  $U_{n,g}$ , we can use the operation grafting of edges on the graph  $H_1$  (at least once) to get a new graph  $H_2$  which is isomorphic to  $\overline{U}_{n,g}$ . By Proposition 2.6,  $\lambda(H_2) < \lambda(H_1)$ . Now Proposition 2.5 and Lemma 2.3 imply that  $\lambda(U_{n,g}) = \lambda(\overline{U}_{n,g}) = \lambda(H_2) < \lambda(H_1) \leq \lambda(H)$ .  $\square$

Consider the case  $g = 3$ . Let  $H$  be a unicyclic graph on  $n$  vertices with girth three which is not isomorphic to  $U_{n,3}$ . If  $|C_H| = 1$ , then  $\lambda(U_{n,3}) < \lambda(H)$  by Proposition 4.8. Assume that  $|C_H| \geq 2$ . For  $n = 5$ , we can see using MATLAB that  $U_{5,3}$  uniquely minimizes the Laplacian spectral radius. So assume that  $n \geq 6$ . Choose an edge  $\{u, v\}$  on the cycle of  $H$  such that  $H_1 = H - \{u, v\}$  is not a path. Then  $H_1$  is

a tree. By a finite sequence of grafting of edges (at least once) we can transform  $H_1$  to a new tree  $H_2$  such that  $H_2$  is isomorphic to  $\overline{U}_{n,3}$  and that  $\lambda(H_2) < \lambda(H_1)$  (see Proposition 2.6). Now, by Proposition 2.5 and Lemma 2.3,  $\lambda(U_{n,3}) = \lambda(\overline{U}_{n,3}) = \lambda(H_2) < \lambda(H_1) \leq \lambda(H)$ . Thus we have

**Proposition 4.9.** *The graph  $U_{n,3}$  uniquely minimizes the Laplacian spectral radius over all unicyclic graphs on  $n$  vertices with girth three.*

We give an example where  $U_{n,g}$  does not minimize the Laplacian spectral radius among all unicyclic graphs on  $n$  vertices with odd girth  $g$ .

**Example 4.10.** Consider a cycle on 7 vertices. Let  $v$  and  $w$  be two vertices on this cycle with  $d(v, w) = 3$ . Add one pendant vertex to each of the vertices  $v$  and  $w$ . The new graph is a unicyclic graph on 9 vertices with girth 7, denote it by  $C_7^{1,1}$ . We have  $\lambda(C_7^{1,1}) \approx 4.4142 < 4.4605 \approx \lambda(U_{9,7})$ .

Now, let  $H$  be a unicyclic graph on  $n$  vertices with odd girth  $g \geq 5$ . By a sequence of grafting of edges we can transform  $H$  into a new graph  $H_1$  that has a path  $P_i$  (on  $l_i$  vertices) appended to the vertex  $i$  for each  $i \in C_H$  and satisfies  $\lambda(H_1) \leq \lambda(H)$  (see Proposition 2.6).

The proof of the next result is similar to that of Proposition 4.3 with some modifications. We write the proof with necessary changes for the sake of completeness.

**Proposition 4.11.** *Let  $H$  be a unicyclic graph on  $n$  vertices with odd girth  $g \geq 5$ . Suppose that  $|C_H| = r \geq 2$  and that we can arrange the vertices in  $C_H$  in an ordering, say  $i_1, i_2, \dots, i_r$ , such that  $l_{i_1} + \dots + l_{i_{j-1}} \geq 2d(i_1, i_j)$  for  $2 \leq j \leq r$ . Then  $\lambda(U_{n,g}) < \lambda(H)$ .*

*Proof.* Consider the graph  $H_1$  obtained from  $H$  as above. For  $1 \leq j \leq r$ , let  $i_{j,1}$  be the vertex on the path  $P_{i_j}$  adjacent to the vertex  $i_j$ , and let  $i_j l_{i_j}$  be the pendant vertex of  $P_{i_j}$ . Let  $H_2$  be the graph obtained from  $H_1$  by deleting the edge opposite to the vertex  $i_1$ . With the given ordering of vertices  $i_1, i_2, \dots, i_r$ , perform the same sequence of graph operations on  $H_2$  as we have done on the graph  $G_1$  in the proof of Proposition 4.3 to get a new graph  $H_3$ . Then  $H_3$  is isomorphic to  $\overline{U}_{n,g}$ . Since  $H_2$  and  $H_3$  are bipartite,  $\mu(H_t) = \lambda(H_t)$  for  $t = 2, 3$ . Let  $X$  be the positive unit eigenvector of  $B(H_3)$  corresponding to  $\mu(H_3)$ . Then  $\lambda(H_2) - \lambda(H_3) = \mu(H_2) - \mu(H_3) \geq X^T B(H_2)X - \mu(H_3) = X^T B(H_2)X - X^T B(H_3)X = [(x_{i_2} + x_{i_{2,1}})^2 - (x_{i_1 l_{i_1}} + x_{i_{2,1}})^2] + [(x_{i_3} + x_{i_{3,1}})^2 - (x_{i_2 l_{i_2}} + x_{i_{3,1}})^2] + \dots + [(x_{i_r} + x_{i_{r,1}})^2 - (x_{i_{r-1} l_{i_{r-1}}} + x_{i_{r,1}})^2]$ . Since  $\mu(\overline{U}_{n,g}) > 4.383$  (Lemma 3.5) and  $l_{i_1} + \dots + l_{i_{j-1}} \geq 2d(i_1, i_j)$  for  $2 \leq j \leq r$ , Proposition 3.4(iv) implies that  $\lambda(H_2) - \lambda(H_3) > 0$ . So  $\lambda(U_{n,g}) = \lambda(\overline{U}_{n,g}) = \lambda(H_3) < \lambda(H_2) \leq \lambda(H)$ .  $\square$

As a consequence of Propositions 4.8 and 4.11, we have

**Corollary 4.12.** *Let  $H$  be a unicyclic graph on  $n$  vertices with odd girth  $g \geq 5$  which is not isomorphic to  $U_{n,g}$ . If  $l_k \geq g - 1$  for some vertex  $k$  in  $C_H$ , then  $\lambda(U_{n,g}) < \lambda(H)$ .*

*Proof.* This follows from Proposition 4.8 for  $|C_H| = 1$  and from Proposition 4.11 for  $|C_H| \geq 2$ . In the latter case, we can take any ordering of the vertices in  $C_H$  starting with the vertex  $k$ .  $\square$

**Proposition 4.13.** *Let  $H$  be a unicyclic graph on  $n$  vertices with odd girth  $g \geq 5$ . Suppose that  $|C_H| = r \geq 2$  and that  $n \geq 3g - 1$ . Then  $\lambda(U_{n,g}) < \lambda(H)$ .*

*Proof.* The proof of this result is similar to that of Proposition 4.5 (In the proof, one has to replace Proposition 4.3 by Proposition 4.11, Lemma 4.1 (a) by Lemma 4.1 (b),  $G$  by  $H$  and  $G_1$  by  $H_1$ ).  $\square$

Now Theorem 1.1(2) follows from Propositions 4.8, 4.9 and 4.13.

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