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MAXIMAL DISTRIBUTIONAL CHAOS OF WEIGHTED SHIFT OPERATORS ON KÖTHE SEQUENCE SPACES

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Abstract. During the last ten some years, many research works were devoted to the chaotic behavior of the weighted shift operator on the Köthe sequence space. In this note, a sufficient condition ensuring that the weighted shift operator $B_w^n\colon \lambda_p(A) \to \lambda_p(A)$ defined on the Köthe sequence space $\lambda_p(A)$ exhibits distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$ and any $n \in \mathbb{N}$ is obtained. Under this assumption, the principal measure of B_w^n is equal to 1. In particular, every Devaney chaotic shift operator exhibits distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$.

Keywords: weighted shift operator; principal measure; distributional chaos

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1. Introduction and preliminaries

The complexity of a topological dynamical system is a central topic of research since the term of chaos was introduced by Li and Yorke [4] in 1975, known as Li-Yorke chaos today. In their study, Li and Yorke suggested that 'divergent pairs' to consider are the pairs (x, y) which are *proximal but not asymptotic*, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0. \tag{1.1}$$

In their context, (X, f) is said to be *chaotic in the sense of Li-Yorke*, if there exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$,

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(x, y) is proximal but not asymptotic. During the last years many researchers paid attention to the 'chaotic behavior' of backward shift operators on Köthe sequence space (more generally, on Banach or Fréchet spaces, see, e.g., [5], [6], [7], [14], [15], [16]). In [2], Li-Yorke chaos was studied by Duan et al. for linear operators. In 2011, Bermúdez et al. [1] gave some equivalent conditions for Li-Yorke chaotic operators and obtained a few sufficient criteria for distributionally chaotic operators. Recently, we [14] proved that for a bounded operator defined on a Banach space, Li-Yorke chaos, Li-Yorke sensitivity, spatio-temporal chaos, and distributional chaos in a sequence are equivalent, and they are all strictly stronger than sensitivity. In [16], we investigated chaos generated by a class of non-constantly weighted shift operators.

A very important generalization of Li-Yorke chaos is that proposed by Schweizer and Smítal in [10], mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to the compact interval case [10] or hyperbolic symbolic spaces [8]. It is also remarkable that this equivalence does not transfer to higher dimensions, e.g., positive topological entropy does not imply distributional chaos (DC1) in the case of triangular maps of the unit square [12] (the same happens when the dimension is zero [9]).

Let f be a continuous self-map on a metric space (X, d). For any pair of points $x, y \in X$, define the *lower* and *upper distributional functions* $\mathbb{R} \to [0, 1]$ generated by f as

$$F_{x,y}(t, f) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}(d(f^j(x), f^j(y))), \tag{1.2}$$

and

$$F^*_{x,y}(t, f) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}(d(f^j(x), f^j(y))), \tag{1.3}$$

respectively, where $\chi_{[0,t)}(x)$ is the characteristic function of the set $[0, t)$. Both functions F_{xy} and F^*_{xy} are nondecreasing. Now, we are ready to state the definition of distributional chaos. A dynamical system (X, f) is *distributionally ε-chaotic* for a given $\varepsilon > 0$ if there exists an uncountable subset $D \subset X$ such that for any pair of distinct points $x, y \in D$, one has $F^*_{x,y}(t, f) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, f) = 0$. The set D is a *distributionally ε-chaotic set* and the pair (x, y) a *distributionally ε-chaotic pair.* If (X, f) is distributionally ε-chaotic for any given $0 < \varepsilon < \operatorname{diam} X$, then (X, f) is said to exhibit *maximal distributional chaos.*

Note that distributional ε-chaos is very complex. If a system exhibits such kinds of chaos, then there exists a constant $\delta > 0$ such that for any pair of distinct points of the distributionally chaotic set, during almost every time step, their iterates are arbitrarily close when looking from one time perspective and almost every iterate of these points is separated by δ when the time perspective is changed. Distributional ε-chaos implies chaos in the sense of Li and Yorke, as it requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic.

In [11], the following measure for a dynamical (X, f) was introduced:

$$\mu_p(f) = \sup_{x,y \in X} \frac{1}{\operatorname{diam} X} \int_0^\infty F^*_{x,y}(t, f) - F_{x,y}(t, f)\,\mathrm{d}t. \tag{1.4}$$

It is called the *principal measure* of system (X, f). It is a hard task to calculate this supremum; however, positiveness of $\mu_p(f)$ may be used as a preliminary test for chaos. This approach is nicer than that of topological entropy where the (usually easier to calculate) upper bound does not help to answer whether the system is chaotic or not. In [15], we proved that the principal measure of an annihilator operator of the unforced quantum harmonic oscillator is 1. Moreover, we obtained that the operator exhibits maximal distributional chaos.

Our framework will be linear and continuous operators $T\colon E \to E$ on separable Fréchet spaces E, i.e., vector spaces E which have an increasing sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of seminorms that define a metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}, \qquad x, y \in E, \tag{1.5}$$

under which E is complete and separable.

The *unilateral backward shift* B on a sequence space is defined by

$$B(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots). \tag{1.6}$$

In 2009, F. Martínez-Giménez et al. [5] gave several conditions ensuring distributional ε-chaos for backward shifts. Motivated by their study, in this paper a sufficient condition ensuring that the weighted shift operator exhibits distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$ is obtained (see Theorem 2.1). This not only shows that for the backward shift operators on Köthe sequence space $\lambda_p(A)$, Devaney chaos implies distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$, but also is a stronger formulation of [5, Theorem 6] (see Corollary 2.1 and Remark 2.1).

2. The principal measure and distributional chaos of backward shift operators

Throughout this paper, our notation for Köthe sequence spaces and Fréchet spaces is standard and we refer to [3], [7].

An infinite non-negative matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ is called a *Köthe matrix* if for each $j \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ with $a_{j,k} > 0$ and $0 \leqslant a_{j,k} \leqslant a_{j,k+1}$ for all $j, k \in \mathbb{N}$. For $1 \leqslant p < \infty$, we consider the (separable) Fréchet spaces

$$\lambda_p(A) = \left\{x \in \mathbb{K}^{\mathbb{N}} \colon \ \|x\|_k := \left(\sum_{j=1}^{\infty} |x_j a_{j,k}|^p\right)^{1/p} < \infty, \ \forall k \in \mathbb{N}\right\}, \tag{2.1}$$

and for $p = 0$

$$\lambda_0(A) = \{x \in \mathbb{K}^{\mathbb{N}} \colon \ \lim_{j\to\infty} x_j a_{j,k} = 0, \ \|x\|_k := \sup_{j\in\mathbb{N}} |x_j a_{j,k}|, \ \forall k \in \mathbb{N}\}, \tag{2.2}$$

which are the corresponding *Köthe sequence spaces.*

In order to apply notions from topological dynamics, the backward shift must be continuous and well defined on the Köthe sequence space $\lambda_p(A)$. This is equivalent to the following condition on the matrix A:

$$\forall n \in \mathbb{N}, \quad \exists m > n \colon \sup_{j\in\mathbb{N}} \left|\frac{a_{j,n}}{a_{j+1,m}}\right| < \infty, \tag{2.3}$$

where in the case $a_{j+1,m} = 0$, we have $a_{j,n} = 0$ and we consider $\frac{0}{0}$ as 1.

Given a sequence $\{w_i\}_{i\geqslant 2}$ of strictly positive scalars we may consider its associated *weighted backward shift* $B_w \colon \lambda_p(A) \to \lambda_p(A)$

$$B_w(x_1, x_2, \ldots) := (w_2 x_2, w_3 x_3, \ldots). \tag{2.4}$$

Set

$$\mathcal{O}_1 = 1, \quad \mathcal{O}_i = \frac{1}{w_2 \ldots w_i}, \quad \mathcal{O}_i^k = w_i \ldots w_{i-k+1}, \quad i > 1, \ k < i. \tag{2.5}$$

Accordingly, the operator B_w is continuous if and only if

$$\forall n \in \mathbb{N}, \quad \exists m > n \colon \sup_{j\in\mathbb{N}} \left|w_{j+1} \frac{a_{j,n}}{a_{j+1,m}}\right| < \infty. \tag{2.6}$$

By the definition of B_w, it is easy to see that for any $k \in \mathbb{N}$ and any $c = (c_1, c_2, \ldots) \in \lambda_p(A)$,

$$B_w^k(c) = (\mathcal{O}_{k+1}^k c_{k+1}, \mathcal{O}_{k+2}^k c_{k+2}, \ldots). \tag{2.7}$$

We recall that the *upper density* $\mathscr{D}(A)$ of a set $A \subset \mathbb{N}$ is defined by:

$$\mathscr{D}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}. \tag{2.8}$$

Theorem 2.1. *Let A be a Köthe matrix satisfying (2.6) and $1 \leqslant p < \infty$ (or $p = 0$). If there exists an increasing sequence $E \subset \mathbb{N}$ such that for any $n \in \mathbb{N}$ it holds that $\sum\limits_{j\in E} |\mathcal{O}_j a_{j,n}|^p < \infty$ (or $\lim\limits_{E\ni j\to\infty} |\mathcal{O}_j a_{j,n}| = 0$) and $\mathscr{D}(E) = 1$, then for any $n \in \mathbb{N}$,*

(i) *$B_w^n\colon\ \lambda_p(A) \to \lambda_p(A)$ exhibits distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$.*

(ii) *$\mu_p(B_w^n) = 1$.*

Proof. (i) *Case 1:* $1 \leqslant p < \infty$.

There exists an increasing sequence of positive integers $\{m_k\}_{k\in\mathbb{N}}$ such that

$$1 = \mathscr{D}(E) = \lim_{k\to\infty} \frac{|E \cap [1,m_k]|}{m_k}. \tag{2.9}$$

Set $\mathscr{A} = \{k \in \mathbb{N}\colon\ a_{j,k} = 0,\ \forall j \in \mathbb{N}\}$. Without loss of generality, we may assume that $\mathscr{A} = \emptyset$. Then there exists $j_0 \in \mathbb{N}$ such that $a_{j_0,1} > 0$. According to the definition of Köthe matrix, it is easy to see that for each $k \in \mathbb{N}$,

$$a_{j_0,k} \geqslant a_{j_0,1} > 0, \tag{2.10}$$

and

$$\operatorname{diam} \lambda_p(A) = 1. \tag{2.11}$$

As $\sum\limits_{j\in E} |\mathcal{O}_j a_{j,n}|^p < \infty$, it follows that

$$\nu = (\nu_1, \nu_2, \ldots) = \sum_{j\in E} \mathcal{O}_j e_j \in \lambda_p(A), \tag{2.12}$$

where $e_j = (x_1, x_2, x_3, \ldots)$ such that

$$x_i = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Then there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\sum_{j=n_k}^{\infty} |\nu_j a_{j,k}|^p < \frac{1}{2^k}, \quad k \in \mathbb{N}. \tag{2.13}$$

Let us define $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots)$ by the formula:

$$\tilde{\nu}_j = \begin{cases} k\nu_j, & n_k \leqslant j < n_{k+1}, \\ \nu_j, & 1 \leqslant j < n_1. \end{cases} \tag{2.14}$$

Because

$$\begin{aligned} \sum_{j=n_k}^{\infty} |\tilde{\nu}_j a_{j,k}|^p &= \sum_{l \geqslant k} \sum_{j=n_l}^{n_{l+1}-1} |\tilde{\nu}_j a_{j,k}|^p = \sum_{l \geqslant k} \sum_{j=n_l}^{n_{l+1}-1} l^p |\nu_j a_{j,k}|^p \\ &\leqslant \sum_{l \geqslant k} \sum_{j=n_l}^{n_{l+1}-1} l^p |\nu_j a_{j,l}|^p \leqslant \sum_{l \geqslant k} \frac{l^p}{2^l} < \infty, \end{aligned}$$

we have $\tilde{\nu} \in \lambda_p(A)$.

Choose

$$\tilde{\nu}[k] = B_w^k(\tilde{\nu}) = (\mathcal{O}_{k+1}^k \tilde{\nu}_{k+1}, \mathcal{O}_{k+2}^k \tilde{\nu}_{k+2}, \ldots), \tag{2.15}$$

and

$$\tilde{\nu}[k,n] = (\underbrace{0, 0, \ldots, 0}_{n}, \mathcal{O}_{k+n+1}^k \tilde{\nu}_{k+n+1}, \mathcal{O}_{k+n+2}^k \tilde{\nu}_{k+n+2}, \ldots). \tag{2.16}$$

Clearly,

$$\lim_{n \to \infty} d(0, \tilde{\nu}[k,n]) = 0. \tag{2.17}$$

Then for $k, n, l \in \mathbb{N}$, there exists some $\zeta(k,n,l) \in \mathbb{N}$ such that for any $m \geqslant \zeta(k,n,l)$ and any $1 \leqslant i \leqslant n$,

$$d(0, \tilde{\nu}[k+i, m]) < \frac{1}{l}. \tag{2.18}$$

Let us choose $L_1 = 2$ and $\mathscr{L}_1 = L_1 + 2^{L_1}$. Define inductively the numbers $L_2, \mathscr{L}_2, L_3, \mathscr{L}_3, \ldots$ as follows:

$$\begin{aligned} L_{k+1} &= \max\{\zeta(\mathscr{L}_k, 2^{\mathscr{L}_k}, k+1), 2^{\mathscr{L}_k}, n_{k+1}\}, \\ \mathscr{L}_{k+1} &= \min\{m_i \colon\ m_i \geqslant \mathscr{L}_k + L_{k+1} + 2^{\mathscr{L}_k + L_{k+1}},\ i \in \mathbb{N}\}. \end{aligned} \tag{2.19}$$

Take $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2, \ldots)$ by the formula:

$$\overline{\nu}_j = \begin{cases} \tilde{\nu}_j, & j \in E \cap \Big([L_1, \mathscr{L}_1] \cup \Big(\bigcup_{n=1}^{\infty} [\mathscr{L}_n + 2L_{n+1}, \mathscr{L}_{n+1}]\Big)\Big), \\ 0, & \text{otherwise}. \end{cases} \tag{2.20}$$

Clearly, $\overline{\nu} \in \lambda_p(A)$.

Now, we assert that $\Gamma := \{\alpha\overline{\nu}\colon \alpha \in (0,1)\}$ is a distributionally ε-scrambled set with any $0 < \varepsilon < \operatorname{diam} \lambda_p(A) = 1$.

In fact, for any pair $x, y \in \Gamma$ with $x \neq y$, it is clear that there exists $\lambda \in (0,1)$ such that $x - y = \lambda\overline{\nu}$.

According to the choice of L_{n+1} and the construction of $\overline{\nu}$, it follows that for any $\mathscr{L}_n + 1 \leqslant m \leqslant \mathscr{L}_n + L_{n+1}$,

$$d(B_w^m(x), B_w^m(y)) = d(0, B_w^m(\lambda\overline{\nu})) \leqslant d(0, B_w^m(\overline{\nu})) \leqslant d(0, B_w^m(\tilde{\nu})) < \frac{1}{n+1}. \tag{2.21}$$

Then for any $t > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$ and any $\mathscr{L}_n + 1 \leqslant m \leqslant \mathscr{L}_n + L_{n+1}$, $d(B_w^m(x), B_w^m(y)) < t$. Thus

$$\begin{aligned} F^*_{x,y}(t, B_w) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,t)}(d(B_w^j(x), B_w^j(y))) \\ &\geqslant \limsup_{n\to\infty} \frac{1}{\mathscr{L}_n + L_{n+1}} \sum_{j=1}^{\mathscr{L}_n + L_{n+1}} \chi_{[0,t)}(d(B_w^j(x), B_w^j(y))) \\ &\geqslant \limsup_{n\to\infty} \frac{1}{\mathscr{L}_n + L_{n+1}} \cdot L_{n+1} \quad (L_{n+1} \geqslant 2^{\mathscr{L}_n}) \\ &\geqslant \limsup_{n\to\infty} \frac{2^{\mathscr{L}_n}}{\mathscr{L}_n + 2^{\mathscr{L}_n}} = 1. \end{aligned} \tag{2.22}$$

For any fixed $0 < \varepsilon < 1$, there exists some $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{M}{1+M} \geqslant \varepsilon. \tag{2.23}$$

Without loss of generality, let us assume that for any $j > j_0$, $B_w^{j-j_0}(\overline{\nu}) = (\xi_1^{(j)}, \xi_2^{(j)}, \ldots)$. Note the fact that for any $n \geqslant j_0$ and any $j \in E \cap [\mathscr{L}_n + 2L_{n+1}, \mathscr{L}_{n+1}]$,

$$|\xi_{j_0}^{(j)}| \geqslant |(n+1)\mathcal{O}_j^{j-j_0}\mathcal{O}_j| = \left|\frac{n+1}{w_2 \ldots w_{j_0}}\right|. \tag{2.24}$$

Combining this with (2.10), we have that for any $k \in \mathbb{N}$,

$$\begin{aligned} \|B_w^{j-j_0}(\lambda\overline{\nu})\|_k &= \left(\sum_{i=1}^{\infty} |\lambda\xi_i^{(j)} a_{i,k}|^p\right)^{1/p} \\ &\geqslant |\lambda\xi_{j_0}^{(j)} a_{j_0,k}| \geqslant \left|\lambda\frac{n+1}{w_2 \ldots w_{j_0}} a_{j_0,1}\right| \to \infty \quad (n \to \infty). \end{aligned} \tag{2.25}$$

This implies that there exists some $N' > j_0$ such that for any $n \geqslant N'$ and any $j \in E \cap [\mathscr{L}_n + 2L_{n+1}, \mathscr{L}_{n+1}]$, $\|B_w^j(\lambda\overline{\nu})\|_k \geqslant M$ holds for any $k \in \mathbb{N}$. This leads with (2.23) to that

$$d(0, B_w^j(\lambda\overline{\nu})) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|B_w^j(\lambda\overline{\nu})\|_k}{1 + \|B_w^j(\lambda\overline{\nu})\|_k} \geqslant \varepsilon. \tag{2.26}$$

Then

$$\begin{aligned} 1 &\geqslant \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[\varepsilon,\infty)}(d(B_w^j(x), B_w^j(y))) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[\varepsilon,\infty)}(d(0, B_w^j(\lambda\overline{\nu}))) \\ &\geqslant \limsup_{n\to\infty} \frac{1}{\mathscr{L}_{n+1}} \sum_{j=1}^{\mathscr{L}_{n+1}} \chi_{[\varepsilon,\infty)}(d(0, B_w^j(\lambda\overline{\nu}))) \\ &\geqslant \limsup_{n\to\infty} \frac{|E \cap [\mathscr{L}_n + 2L_{n+1}, \mathscr{L}_{n+1}]|}{\mathscr{L}_{n+1}} \\ &\geqslant \limsup_{n\to\infty} \frac{|E \cap [1, \mathscr{L}_{n+1}]| - (\mathscr{L}_n + 2L_{n+1})}{\mathscr{L}_{n+1}} \\ &= \lim_{n\to\infty} \frac{|E \cap [1, \mathscr{L}_{n+1}]|}{\mathscr{L}_{n+1}} - \lim_{n\to\infty} \frac{\mathscr{L}_n + 2L_{n+1}}{\mathscr{L}_{n+1}} = \mathscr{D}(E) - 0 = 1. \end{aligned} \tag{2.27}$$

Thus

$$\begin{aligned} F_{x,y}(\varepsilon, B_w) &= \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[0,\varepsilon)}(d(B_w^j(x), B_w^j(y))) \\ &= 1 - \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{[\varepsilon,\infty)}(d(B_w^j(x), B_w^j(y))) = 0. \end{aligned} \tag{2.28}$$

Hence B_w is distributionally ε-chaotic as Γ is uncountable. Combining this with [13, Lemma 3.1], it follows that B_w^n is distributionally ε-chaotic for any $n \in \mathbb{N}$.

Case 2: $p = 0$.

Similarly to the proof of Case 1, this holds trivially.

(ii) Since B_w^n is distributionally ε-chaotic for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$, we have

$$\begin{aligned} 1 \geqslant \mu_p(B_w^n) &= \sup_{x,y\in\lambda_p(A)} \frac{1}{\operatorname{diam} \lambda_p(A)} \int_0^{\infty} F_{x,y}^*(t, B_w^n) - F_{x,y}(t, B_w^n)\,\mathrm{d}t \\ &\geqslant \sup_{0<\varepsilon<\operatorname{diam} \lambda_p(A)} \frac{1}{\operatorname{diam} \lambda_p(A)} \int_0^{\varepsilon} 1\,\mathrm{d}t = 1. \end{aligned} \tag{2.29}$$

That is $\mu_p(B_w^n) = 1$. $\square$

Corollary 2.1. *If there exists a decreasing sequence of sets* $\mathbb{N} \supset S_1 \supset S_2 \supset \ldots$ *such that for any* $n \in \mathbb{N}$ *it holds that* $\sum_{j \in S_n} |\mathcal{O}_j a_{j,n}|^p < \infty$ *and* $\mathscr{D}(S_n) = 1$, *then* $B_w \colon \lambda_p(A) \to \lambda_p(A)$ *is distributionally* ε*-chaotic for any* $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$ *and* $\mu_p(B_w) = 1$.

P r o o f. According to the proof of [5, Theorem 6], it follows that there exists $D \subset \mathbb{N}$ such that $\sum_{j \in D} |\mathcal{O}_j a_{j,n}|^p < \infty$ and $\mathscr{D}(D) = 1$. Combining this with Theorem 2.1, the proof is completed. □

Remark 2.1. The stronger condition given in [5, Corollary 3.4] characterizing chaos in the sense of Devaney was

$$\sum_{j=1}^{\infty} |\mathcal{O}_j a_{j,n}|^p < \infty. \tag{2.30}$$

This condition implies distributional ε-chaos for any $0 < \varepsilon < \operatorname{diam} \lambda_p(A)$, since the hypothese of Theorem 2.1 are satisfied for $E = \mathbb{N}$. Applying this conclusion, it follows that the derivative operator D and the Bessel operator $\Delta_\mu = z^{-2\mu-1} D z^{2\mu+1} D$ ($\mu > -1/2$) (see [5], [6] for more details) are both distributionally ε-chaotic for any $0 < \varepsilon < 1$ and $\mu_p(D) = \mu_p(\Delta_\mu) = 1$.

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