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BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL OPERATOR ON BLOCK SPACES WITH VARIABLE EXPONENT

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Abstract. The family of block spaces with variable exponents is introduced. We obtain some fundamental properties of the family of block spaces with variable exponents. They are Banach lattices and they are generalizations of the Lebesgue spaces with variable exponents. Moreover, the block space with variable exponents is a pre-dual of the corresponding Morrey space with variable exponents.

The main result of this paper is on the boundedness of the Hardy-Littlewood maximal operator on the block space with variable exponents. We find that the Hardy-Littlewood maximal operator is bounded on the block space with variable exponents whenever the Hardy-Littlewood maximal operator is bounded on the corresponding Lebesgue space with variable exponents.

Keywords: block space; variable exponent analysis; Hardy-Littlewood maximal operator

MSC 2010: 42B25, 46E30

1. Introduction

In this paper, we present and study the block space with variable exponents $\mathfrak{B}_{\omega,L^{p(x)}}$. Recently, there are a number of researches on the Lebesgue space with variable exponents $L^{p(x)}$ and the Morrey space with variable exponents $M_{\omega,L^{p(x)}}$. As in the classical case, the block space is a pre-dual of the Morrey space [2]. Thus, the family of block spaces is an important variety of Lebesgue spaces and Morrey spaces. Therefore, in this paper, we extend the study of variable exponent analysis to block spaces.

The classical block space is introduced and studied in [2]. A well-known result on the block space is that it is a pre-dual of the Morrey space. One of the main result

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of this paper is an extension of this duality to the variable exponent setting. We find that the block space with variable exponents is a pre-dual of the Morrey space with variable exponents.

The other main result of this paper is related to the boundedness of the Hardy-Littlewood maximal operator. We find that if $p(x)$ is an exponent function such that the Hardy-Littlewood maximal operator is bounded on $L^{p(x)}$, then it is also bounded on $\mathfrak{B}_{\omega,L^{p(x)}}$.

We define the block spaces with variable exponents in Section 2. We also show that they are pre-duals of Morrey spaces with variable exponents in that section. The boundedness of the Hardy-Littlewood maximal operator on block spaces with variable exponents is established in Section 3.

2. Block spaces with variable exponent

The main theme of this section is the duality between block spaces with variable exponents and Morrey spaces with variable exponents. We begin with some notions and notation from variable exponent analysis.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the class of Lebesgue measurable functions on $\mathbb{R}^n$. For any Lebesgue measurable set E, the characteristic function of E is denoted by χ_E. For any $x \in \mathbb{R}^n$ and $r > 0$, write $B(x,r) = \{y \in \mathbb{R}^n\colon |x-y| < r\}$ and $\mathbb{B} = \{B(x,r)\colon x \in \mathbb{R}^n,\ r > 0\}$.

We recall the definition of Lebesgue spaces with variable exponents from [19].

Definition 2.1. Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(x)}$ consists of all Lebesgue measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^{p(x)}} = \inf\{\lambda > 0\colon \varrho(f(x)/\lambda) \leqslant 1\} < \infty$$

where
$$\varrho(f(x)) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x.$$

We call $p(x)$ the exponent function of $L^{p(x)}$.

Theorem 2.1. *If $p(x)\colon \mathbb{R}^n \to (1,\infty)$ is a Lebesgue measurable function with $\operatorname{ess\,sup} p(x) < \infty$, then the dual space of $L^{p(x)}$ is $L^{p'(x)}(\mathbb{R}^n)$ where p' satisfies $1/p(x) + 1/p'(x) = 1$.*

We call $p'(x)$ the conjugate function of $p(x)$. The reader is referred to [19, Theorem 2.6] for the proof of the above theorem.

Definition 2.2. Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ and $\omega(x,r)\colon \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. A $b \in \mathcal{M}(\mathbb{R}^n)$ is an $(\omega, L^{p(x)})$-block if it is supported in a ball $B(x_0,r)$, $x_0 \in \mathbb{R}^n$, $r > 0$, and

$$\|b\|_{L^{p(x)}} \leqslant \frac{1}{\omega(x_0,r)}. \tag{2.1}$$

Define $\mathfrak{B}_{\omega,L^{p(x)}}$ by

$$\mathfrak{B}_{\omega,L^{p(x)}} = \biggl\{\sum_{k=1}^{\infty} \lambda_k b_k\colon \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is an } (\omega, L^{p(x)})\text{-block}\biggr\}. \tag{2.2}$$

The space $\mathfrak{B}_{\omega,L^{p(x)}}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}} = \inf\biggl\{\sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k\biggr\}. \tag{2.3}$$

We call $\mathfrak{B}_{\omega,L^{p(x)}}$ the block space with variable exponent.

The family of block spaces for the Lebesgue spaces is introduced and studied in [2].

Whenever $p(x)$ is a constant function, then $\mathfrak{B}_{\omega,L^{p(x)}}$ coincides with the classical block space introduced in [2].

Furthermore, according to (2.3), for any $(\omega, L^{p(x)})$-block b, we have

$$\|b\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant 1. \tag{2.4}$$

This observation is applied frequently in the proofs of the subsequent results for $\mathfrak{B}_{\omega,L^{p(x)}}$.

Notice that in [22], [26], the term "block space" was used to represent another family of function spaces.

We now establish a fundamental result for $\mathfrak{B}_{\omega,L^{p(x)}}$.

Proposition 2.1. *If $p(x)\colon \mathbb{R}^n \to (1,\infty)$ is a Lebesgue measurable function, then $\mathfrak{B}_{\omega,L^{p(x)}}$ is a Banach lattice.*

Proof. Obviously, $\|\cdot\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$ satisfies the triangle inequality. Let $f_i \in \mathfrak{B}_{\omega,L^{p(x)}}$, $i \in \mathbb{N}$ satisfy

$$\sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{B}_{\omega,L^{p(x)}}} < \infty.$$

According to the definition of $\mathfrak{B}_{\omega,L^{p(x)}}$, for any $\varepsilon > 0$ we have

$$f_i = \sum_{k=1}^{\infty} \lambda_{k,i} b_{k,i}$$

where $b_{k,i},\ i,k \in \mathbb{N}$ are $(\omega, L^{p(x)})$-blocks and

$$\sum_{k=1}^{\infty} |\lambda_{k,i}| \leqslant (1+\varepsilon)\|f_i\|_{\mathfrak{B}_{\omega,L^{p(x)}}}.$$

Therefore,

$$\sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \lambda_{k,i} b_{k,i}$$

and $\lambda_{k,i},\ i,k \in \mathbb{N}$ satisfy

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |\lambda_{k,i}| \leqslant (1+\varepsilon)\sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{B}_{\omega,L^{p(x)}}} < \infty.$$

That is, $\sum\limits_{i=1}^{\infty} f_i$ converges in $\mathfrak{B}_{\omega,L^{p(x)}}$. Moreover, as $\varepsilon > 0$ is arbitrary, we also have

$$\Big\|\sum_{i=1}^{\infty} f_i\Big\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \sum_{i=1}^{\infty} \|f_i\|_{\mathfrak{B}_{\omega,L^{p(x)}}}.$$

Hence, $\mathfrak{B}_{\omega,L^{p(x)}}$ is a Banach space.

Next, assume that $|g| \leqslant |f|$ where $f \in \mathfrak{B}_{\omega,L^{p(x)}}$ and $f, g \in \mathcal{M}$. Since $f \in \mathfrak{B}_{\omega,L^{p(x)}}$, for any $\varepsilon > 0$, we have a family of $(\omega, L^{p(x)})$-blocks $\{b_i\}_{i=1}^{\infty}$ and a family of scalars $\{\lambda_i\}_{i=1}^{\infty}$ such that

$$f = \sum_{i=1}^{\infty} \lambda_i b_i$$

and $\sum\limits_{i=1}^{\infty} |\lambda_i| \leqslant (1+\varepsilon)\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$. Therefore

$$g = \sum_{i=1}^{\infty} \lambda_i c_i$$

where

$$c_i(x) = \begin{cases} \dfrac{g(x)}{f(x)}\, b_i(x), & f(x) \neq 0, \\ 0, & f(x) = 0. \end{cases}$$

It is easy to see that $\{c_i\}_{i=1}^{\infty}$ are $(\omega, L^{p(x)})$-blocks because $|g| \leqslant |f|$. Thus, $g \in \mathfrak{B}_{\omega,L^{p(x)}}$. Moreover, as ε is arbitrary, we also have $\|g\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$. □

The subsequent results show that the family of block spaces with variable exponents is an extension of $L^{p(x)}$.

Proposition 2.2. *Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ and $\omega(x,r)\equiv 1$. If* $\operatorname{ess\,sup} p(x) < \infty$ *then we have* $\mathfrak{B}_{\omega,L^{p(x)}} = L^{p(x)}$.

Proof. Let $f \in \mathfrak{B}_{\omega,L^{p(x)}}$. For any $\varepsilon > 0$ we have $f = \sum\limits_{k=1}^{\infty} \lambda_k b_k$ where $\{b_k\}_{k=1}^{\infty}$ are $(\omega, L^{p(x)})$-blocks and $\sum\limits_{k=1}^{\infty} |\lambda_k| \leqslant (1+\varepsilon)\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$. We find that

$$\|f\|_{L^{p(x)}} \leqslant \sum_{k=1}^{\infty} |\lambda_k| \|b_k\|_{L^{p(x)}} \leqslant \sum_{k=1}^{\infty} |\lambda_k| \leqslant (1+\varepsilon)\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}. \tag{2.5}$$

As $\varepsilon > 0$ is arbitrary, we find that $\|f\|_{L^{p(x)}} \leqslant \|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$. That is, $\mathfrak{B}_{\omega,L^{p(x)}} \hookrightarrow L^{p(x)}$.

For the reverse embedding, notice that for any $f \in L^{p(x)}$ and $R > r > 0$,

$$\frac{1}{\|f\chi_{B(0,R)\setminus B(0,r)}\|_{L^{p(x)}}} f\chi_{B(0,R)\setminus B(0,r)}$$

is an $(\omega, L^{p(x)})$-block. Thus, by (2.4), we have

$$\|f\chi_{B(0,R)\setminus B(0,r)}\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|f\chi_{B(0,R)\setminus B(0,r)}\|_{L^{p(x)}}. \tag{2.6}$$

Next, we show that the sequence $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(x)}$. As $\|f\|_{L^{p(x)}} < \infty$, $|f(x)/c|^{p(x)}$ is an integrable function for some $c > 0$. For any $\varepsilon > 0$ there exists a $J \in \mathbb{N}$ such that $\int_{|x|>2^j} |f(x)/c|^{p(x)}\,\mathrm{d}x < \varepsilon$, for all $j > J$.

In view of [19, (2.28)], we have $\varrho(f_j) \to 0$ if and only if $\|f_j\|_{L^{p(x)}} \to 0$ when $\operatorname{ess\,sup} p(x) < \infty$ where $f_j = f\chi_{\mathbb{R}^n\setminus B(0,2^j)}$, $j \in \mathbb{N}$.

Since $L^{p(x)}$ is a Banach function space [10, Theorem 3.2.13], $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(x)}$. So, (2.6) ensures that it is also a Cauchy sequence in $\mathfrak{B}_{\omega,L^{p(x)}}$. Inequalities (2.5) guarantee that the limit functions of $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ in $L^{p(x)}$ and in $\mathfrak{B}_{\omega,L^{p(x)}}$ are f.

Similarly to the above argument, for any $R > 0$,

$$F = \frac{1}{\|f\chi_{B(0,R)}\|_{L^{p(x)}}} f\chi_{B(0,R)}$$

is an $(\omega, L^{p(x)})$-block. Therefore, (2.4) yields $\|F\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant 1$. That is,

$$\|f\chi_{B(0,R)}\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|f\chi_{B(0,R)}\|_{L^{p(x)}}. \tag{2.7}$$

Since $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ converges to f both in $L^{p(x)}$ and in $\mathfrak{B}_{\omega,L^{p(x)}}$ when $j \to \infty$, by taking limit on both sides of the above inequality, we obtain $\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|f\|_{L^{p(x)}}$. Thus, $L^{p(x)} \hookrightarrow \mathfrak{B}_{\omega,L^{p(x)}}$. $\square$

Next, we establish the duality relation between the block spaces and the Morrey spaces in the variable exponent setting.

We state the definition for the Morrey space with variable exponents.

Definition 2.3. Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ and $\omega(x,r)\colon \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The Morrey space with variable exponent $M_{\omega,L^{p(x)}}$ consists of the Lebesgue measurable functions f satisfying

$$\|f\|_{M_{\omega,L^{p(x)}}} = \sup_{x_0\in\mathbb{R}^n, r>0} \frac{1}{\omega(x_0,r)} \|f\chi_{B(x_0,r)}\|_{L^{p(x)}} < \infty.$$

The following theorem is the main result of this section. It asserts that the dual space of $\mathfrak{B}_{\omega,L^{p(x)}}$ is $M_{\omega,L^{p'(x)}}$. We denote the dual space of $\mathfrak{B}_{\omega,L^{p(x)}}$ by $\mathfrak{B}^*_{\omega,L^{p(x)}}$.

Theorem 2.2. *Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ and $\omega(x,r)\colon \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. If* $\operatorname{ess\,sup} p(x) < \infty$, *then we have*

$$\mathfrak{B}^*_{\omega,L^{p(x)}} = M_{\omega,L^{p'(x)}}.$$

Proof. Let b be an $(\omega, L^{p(x)})$-block supported in $B(x_0,r)$. For any $f \in M_{\omega,L^{p'(x)}}$, the Hölder inequality for $L^{p(x)}$ [19, Theorem 2.1] ensures that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)b(x)\,\mathrm{d}x\right| &\leqslant C\|\chi_{B(x_0,r)}f\|_{L^{p'(x)}}\|\chi_{B(x_0,r)}b\|_{L^{p(x)}}\\ &\leqslant C\frac{1}{\omega(x_0,r)}\|\chi_{B(x_0,r)}f\|_{L^{p'(x)}}\end{aligned}$$

for some $C>0$.

Consequently, for any $g = \sum\limits_{k\in\mathbb{N}} \lambda_k b_k \in \mathfrak{B}_{\omega,L^{p(x)}}$ we obtain

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x\right| \leqslant \sum_{k\in\mathbb{N}} |\lambda_k| \left|\int_{\mathbb{R}^n} f(x)b_k(x)\,\mathrm{d}x\right| \leqslant C\|g\|_{\mathfrak{B}_{\omega,L^{p(x)}}}\|f\|_{M_{\omega,L^{p'(x)}}}$$

for some $C>0$. Thus, $M_{\omega,L^{p'(x)}} \hookrightarrow \mathfrak{B}^*_{\omega,L^{p(x)}}$.

Next, we prove the reverse embedding.

For any $r>0$ and $L \in \mathfrak{B}^*_{\omega,L^{p(x)}}$, define $X = \{g\chi_{B(0,r)}\colon g \in L^{p(x)}\}$. Obviously, X is a subspace of $L^{p(x)}$.

Define the linear functional $l(h)$ for $h \in X$ by

$$l(h) = L(\chi_{B(0,r)}g)$$

where $h = \chi_{B(0,r)}g$ and $g \in L^{p(x)}$.

In view of (2.7) and $L \in \mathfrak{B}^*_{\omega,L^{p(x)}}$, we find that

$$|l(h)| = |L(g\chi_{B(0,r)})| \leqslant C\|g\chi_{B(0,r)}\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant A\|g\chi_{B(0,r)}\|_{L^{p(x)}} = A\|h\|_{L^{p(x)}}$$

for some $A > 0$. That is, l is bounded on X. According to the Hahn-Banach theorem, the linear functional l can be extended to be a member of $(L^{p(x)})^*$. Therefore, Theorem 2.1 ensures that there exists a $f_r \in L^{p'(x)}$ such that

$$l(g) = \int_{\mathbb{R}^n} f_r(x)g(x)\,\mathrm{d}x, \quad \forall g \in L^{p(x)}.$$

Moreover, we can assume that $\operatorname{supp} f_r \subseteq B(0,r)$.

Let $r, s > 0$. For any $B \in \mathbb{B}$ with $B \subseteq B(0,r) \cap B(0,s)$,

$$\int_B f_r(x)\,\mathrm{d}x = l(\chi_B) = \int_B f_s(x)\,\mathrm{d}x.$$

That is, $f_r = f_s$ almost everywhere on $B(0,r) \cap B(0,s)$. Therefore, there is a unique Lebesgue measurable function f such that $f(x) = f_r(x)$ on $B(0,r)$ for all r.

Next, we show that $f \in M_{\omega,L^{p'(x)}}$. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, let $s > 0$ be chosen so that $B(x_0,r) \subseteq B(0,s)$. For any $h \in L^{p(x)}$ and $B(x_0,r) \in \mathbb{B}$,

$$H = \frac{\chi_{B(x_0,r)}h}{\|\chi_{B(x_0,r)}h\|_{L^{p(x)}}\omega(x_0,r)} \tag{2.8}$$

is an $(\omega, L^{p(x)})$-block. Therefore, according to (2.4),

$$\|H\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant 1.$$

That is, $\|\chi_{B(x_0,r)}h\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|\chi_{B(x_0,r)}h\|_{L^{p(x)}}\omega(x_0,r)$.

As the function given in (2.8) is an $(\omega, L^{p(x)})$-block, we have

$$\begin{aligned}
\frac{1}{\omega(x_0,r)}\|\chi_{B(x_0,r)}f\|_{L^{p'(x)}} &= \frac{1}{\omega(x_0,r)} \sup_{\|h\|_{L^{p(x)}}=1}\left|\int_{B(x_0,r)} f(x)h(x)\,\mathrm{d}x\right| \\
&\leqslant \sup_{\|h\|_{L^{p(x)}}=1}\left|\int_{B(0,s)} f_s(x)\frac{\chi_{B(x_0,r)}h}{\omega(x_0,r)}\,\mathrm{d}x\right| \\
&\leqslant \|L\|_{\mathfrak{B}^*_{\omega,L^{p(x)}}} \sup_{\|h\|_{L^{p(x)}}=1}\left\|\frac{h\chi_{B(x_0,r)}}{\omega(x_0,r)}\right\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant \|L\|_{\mathfrak{B}^*_{\omega,L^{p(x)}}}.
\end{aligned}$$

The functionals $L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x$ and L coincide on the set of $(\omega, L^{p(x)})$-blocks and the set of finite linear combinations of $(\omega, L^{p(x)})$-blocks is dense in $\mathfrak{B}_{\omega,L^{p(x)}}$, therefore $L_f = L$ and $\mathfrak{B}^*_{\omega,L^{p(x)}} \hookrightarrow M_{\omega,L^{p'(x)}}$. $\square$

The preceding theorem is a generalization of the well known result on the "classical" Morrey spaces and block spaces, see [2, Theorem 1].

As an application of Theorem 2.2, we obtain the following remarkable property for Morrey spaces with variable exponents.

Corollary 2.1. *Let $p(x)\colon \mathbb{R}^n \to (1,\infty)$ and $\omega(x,r)\colon \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The unit ball of the Morrey space with variable exponent $M_{\omega,L^{p(x)}}$ is weak-star compact.*

The above corollary follows from Alaoglu's theorem and Theorem 2.1.

3. Boundedness of maximal operator

In this section, we present an important boundedness result of the Hardy-Littlewood maximal operator on $\mathfrak{B}_{\omega,L^{p(x)}}$. Recall that for any locally integrable function f, the Hardy-Littlewood maximal operator $\mathrm{M}(f)$ is defined by

$$\mathrm{M}(f) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(x)|\,\mathrm{d}x$$

where the supremum is taking over all balls B containing x.

We show that whenever the Hardy-Littlewood maximal operator is bounded on $L^{p(x)}$, it is also bounded on $\mathfrak{B}_{\omega,L^{p(x)}}$.

Let $\mathcal{B}$ denote the set of all $p(x)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(x)}$. Some important subsets of $\mathcal{B}$ are given in [4], [6], [9], [21], [20], [24], [23], [25].

For the boundedness of the Hardy-Littlewood maximal operator on a Morrey space with variable exponents, the reader is referred to [11]. In addition, the extension of the boundedness of the maximal operator on a vector-valued Morrey space with variable exponents is obtained in [17], [12].

Proposition 3.1. *If $p(\cdot) \in \mathcal{B}$, then we have constants $C_1, C_0 > 0$ such that for any $B \in \mathbb{B}$,*

$$|B| \leqslant C_0 \|\chi_B\|_{L^{p(x)}} \|\chi_B\|_{L^{p'(x)}} \leqslant C_1 |B|. \tag{3.1}$$

P r o o f. We only need to prove the second inequality because the first inequality follows from a general result for Banach function spaces, see [1, Chapter 1, Definitions 2.1 and 2.3].

For any $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n$ and $r > 0$, we define the projection operator $P_B(g)$ by

$$(P_B g)(y) = \left(\frac{1}{|B|}\int_B g(x)\,\mathrm{d}x\right)\chi_B(y).$$

The projection operator P_B is uniformly dominated by the maximal operator M. More precisely, there exists a constant $C > 0$ such that for any $B = B(x_0, r)$, $|P_B(f)| \leqslant C\mathrm{M}(f)$. Consequently, $\sup\limits_B \|P_B\|_{L^{p(x)}\to L^{p(x)}} < C\|\mathrm{M}\|_{L^{p(x)}\to L^{p(x)}}$ where $\|\cdot\|_{L^{p(x)}\to L^{p(x)}}$ is the operator norm of a mapping on $L^{p(x)}$.

The uniform boundedness of P_B and [1, Chapter 1, Lemma 2.8] yield

$$\|\chi_B\|_{L^{p(x)}}\|\chi_B\|_{L^{p'(x)}} = \sup\left\{\left|\int_B g(x)\,\mathrm{d}x\right|\|\chi_B\|_{L^{p(x)}} \colon \|g\|_{L^{p(x)}} \leqslant 1\right\} \leqslant C|B|.$$

□

The above result can be extended to some other families of function spaces. For instance, (3.1) is valid for rearrangement-invariant quasi-Banach function spaces, the reader is referred to [14, Lemma 5]. For the general Banach function space, the reader may consult [17].

We are now ready to obtain the boundedness of the Hardy-Littlewood maximal operator on $\mathfrak{B}_{\omega,L^{p(x)}}$.

Theorem 3.1. *Let $p(x) \in \mathcal{B}$ and let $\omega\colon \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If the Hardy-Littlewood maximal operator is bounded on $L^{p(x)}$ and there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, ω fulfils*

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p'(x)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p'(x)}}}\,\omega(x, 2^{j+1}r) < C\omega(x,r), \tag{3.2}$$

then the Hardy-Littlewood maximal operator is bounded on $\mathfrak{B}_{\omega,L^{p(x)}}$.

Proof. Let $x_0 \in \mathbb{R}^n$, $r > 0$. Suppose that b is an $(\omega, L^{p(x)})$-block with support $B(x_0, r)$. For any $k \in \mathbb{N}$, let $B_k = B(x_0, 2^k r)$. Write $m_k = \chi_{B_{k+1}\setminus B_k}\mathrm{M}(b)$, $k \in \mathbb{N}\setminus\{0\}$ and $m_0 = \chi_{B_0}\mathrm{M}(b)$. We have $\operatorname{supp} m_k \subseteq B_{k+1}\setminus B_k$ and $\mathrm{M}(b) = \sum\limits_{k=0}^{\infty} m_k$.

In view of $p(\cdot) \in \mathcal{B}$, we have

$$\|m_0\|_{L^{p(x)}} \leqslant C\|\mathrm{M}(b)\|_{L^{p(x)}} \leqslant \frac{C}{\omega(x_0, r)}$$

for a constant $C > 0$ independent of x_0 and r. Consequently, m_0 is a constant-multiple of an $(\omega, L^{p(x)})$-block.

The definition of the Hardy-Littlewood maximal operator and the Hölder inequality for $L^{p(x)}$ (see [19, Theroem 2.1]) assert that

$$\begin{aligned} m_k = \chi_{B_{k+1}\setminus B_k}|\mathrm{M}(b)| &\leqslant \frac{\chi_{B_{k+1}\setminus B_k}}{2^{kn}r^n}\int_{B(x_0,r)}|b(x)|\,\mathrm{d}x \\ &\leqslant C\chi_{B_{k+1}\setminus B_k}\frac{1}{2^{kn}r^n}\|b\|_{L^{p(x)}}\|\chi_{B(x_0,r)}\|_{L^{p'(x)}} \end{aligned}$$

for a $C>0$ independent of k.

Proposition 3.1 ensures that

$$\begin{aligned} \|m_k\|_{L^{p(x)}} &\leqslant \frac{\|\chi_{B_{k+1}\setminus B_k}\|_{L^{p(x)}}}{2^{kn}r^n}\|b\|_{L^{p(x)}}\|\chi_{B(x_0,r)}\|_{L^{p'(x)}} \\ &\leqslant C\frac{\|\chi_{B(x_0,r)}\|_{L^{p'(x)}}}{\|\chi_{B_{k+1}}\|_{L^{p'(x)}}}\frac{\omega(x_0,2^{k+1}r)}{\omega(x_0,r)}\frac{1}{\omega(x_0,2^{k+1}r)}. \end{aligned}$$

Define $m_k=\sigma_k b_k$ where

$$\sigma_k = \frac{\|\chi_{B(x_0,r)}\|_{L^{p'(x)}}}{\|\chi_{B_{k+1}}\|_{L^{p'(x)}}}\frac{\omega(x_0,2^{k+1}r)}{\omega(x_0,r)}.$$

Consequently, b_k is a constant-multiple of an $(\omega, L^{p(x)})$-block and this constant does not depend on k. Inequality (3.2) yields $\sum_{k=0}^{\infty}\sigma_k < C$ for some $C>0$. Hence, $\mathrm{M}(b)\in\mathfrak{B}_{\omega,L^{p(x)}}$ and there exists a constant $C_0>0$ such that for any $(\omega, L^{p(x)})$-block b,

$$\|\mathrm{M}(b)\|_{\mathfrak{B}_{\omega,L^{p(x)}}} < C_0.$$

Now, we consider $f\in\mathfrak{B}_{\omega,L^{p(x)}}$. The definition of the block space ensures that there exist a family of $(\omega, L^{p(x)})$-blocks $\{c_k\}_{k=1}^{\infty}$ and a sequence $\Lambda=\{\lambda_k\}_{k=1}^{\infty}\in l^1$ such that $f=\sum_{k=1}^{\infty}\lambda_k c_k$ with $\|\Lambda\|_{l^1}\leqslant 2\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}$. Finally, we have

$$\begin{aligned} \|\mathrm{M}(f)\|_{\mathfrak{B}_{\omega,L^{p(x)}}} &\leqslant \sum_{k=1}^{\infty}|\lambda_k|\|\mathrm{M}(c_k)\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \\ &\leqslant C_0\sum_{k=1}^{\infty}|\lambda_k|\|c_k\|_{\mathfrak{B}_{\omega,L^{p(x)}}} \leqslant 2C_0\|f\|_{\mathfrak{B}_{\omega,L^{p(x)}}}. \end{aligned}$$

□

Condition similar to (3.2) is introduced in [16, Theorem 5.2] for the investigation of the boundedness of the maximal operator on a vector-valued Morrey space associated with a rearrangement-invariant Banach function space. It is also used in [13] to show the Fefferman-Stein vector-valued maximal inequalities for weighted Morrey spaces.

The reader may have a wrong impression that condition (3.2) is too complicated to apply. In fact, the subsequent result shows that (3.2) is satisfied by a number of Lebesgue measurable functions ω.

Definition 3.1. For any $p(x) \in \mathcal{B}$, let $\kappa_{p(x)}$ denote the supremum of those $q > 1$ that $p(x)/q \in \mathcal{B}$. Let $e_{p(\cdot)}$ be the conjugate of $\kappa_{p'(\cdot)}$.

The above indices can be viewed as a generalization of Boyd's indices to $L^{p(x)}$, see [15].

We have the following proposition from [12]. For completeness, we provide the proof of the subsequent result from [12].

Proposition 3.2. *Let $p \in \mathcal{B}$ and $\operatorname{ess\,sup} p(x) < \infty$. For any $1 < q < \kappa_{p(\cdot)}$ and $1 < s < \kappa_{p'(\cdot)}$, there exist constants $C_1, C_2 > 0$ such that for any $x_0 \in \mathbb{R}^n$ and $r > 0$ we have*

$$C_2 2^{jn(1-1/s)} \leqslant \frac{\|\chi_{B(x_0,2^j r)}\|_{L^{p(x)}}}{\|\chi_{B(x_0,r)}\|_{L^{p(x)}}} \leqslant C_1 2^{jn/q}, \quad \forall j \in \mathbb{N}. \tag{3.3}$$

Proof. For any $B = B(x_0, r) \in \mathbb{B}$ and $j \in \mathbb{N}$ we have a constant $C > 0$ such that

$$C 2^{-jn} \leqslant \mathrm{M}(\chi_B)(x)$$

when $x \in B(x_0, 2^j r)$, $j \in \mathbb{N}$. Let $q < \kappa_{p(\cdot)}$. Since $\alpha p(\cdot) \in \mathcal{B}$ for any $\alpha > 1$ and $p(\cdot) \in \mathcal{B}$ we have $p(\cdot)/q \in \mathcal{B}$. Subsequently,

$$2^{-jn} \|\chi_{B(x_0,2^j r)}\|_{L^{p(\cdot)/q}} \leqslant C \|\mathrm{M}(\chi_B)\|_{L^{p(\cdot)/q}} \leqslant C \|\chi_B\|_{L^{p(\cdot)/q}}.$$

Since, for any $B \in \mathbb{B}$ and $q > 0$, $\|\chi_B\|_{L^{p(\cdot)/q}} = \|\chi_B\|^q_{L^{p(\cdot)}}$, we obtain the second inequality of (3.3).

According to [7, Theorem 8.1], $p'(\cdot) \in \mathcal{B}$. Thus, for any $s < \kappa_{p'(\cdot)}$, we also have

$$\frac{\|\chi_{B(x_0,2^j r)}\|_{L^{p'(x)}}}{\|\chi_{B(x_0,r)}\|_{L^{p'(x)}}} \leqslant C_1 2^{jn/s}, \quad \forall j \in \mathbb{N}.$$

Therefore, Proposition 3.1 yields the first inequality in (3.3). □

Proposition 3.3. *Let $p \in \mathcal{B}$ with $\operatorname{ess\,sup} p(x) < \infty$ and $0 \leqslant \lambda < n/e_{p'(\cdot)}$. If ω satisfies*

$$\omega(x, 2^j r) \leqslant C 2^{j\lambda} \omega(x, r), \quad \forall x \in \mathbb{R}^n, \ r > 0 \ \text{ and } \ j \in \mathbb{N} \tag{3.4}$$

for some $C > 0$, then ω fulfils (3.2).

P r o o f. By using Proposition 3.1 and the inequality on the right-hand side of (3.3), we find that for any $1 < q < \kappa_{p(\cdot)}$,

$$\frac{\|\chi_{B(x,r)}\|_{L^{p'(x)}}}{\|\chi_{B(x,2^j r)}\|_{L^{p'(x)}}} \leqslant C 2^{-jn(1-1/q)}. \tag{3.5}$$

As $e_{p'(\cdot)}$ is the conjugate of $\kappa_{p(\cdot)}$, (3.4) and (3.5) guarantee that ω satisfies (3.2). $\square$

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