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A PRECONDITIONER FOR THE FETI-DP METHOD FOR MORTAR-TYPE CROUZEIX-RAVIART ELEMENT DISCRETIZATION

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Abstract. In this paper, we consider mortar-type Crouzeix-Raviart element discretizations for second order elliptic problems with discontinuous coefficients. A preconditioner for the FETI-DP method is proposed. We prove that the condition number of the preconditioned operator is bounded by $(1+\log(H/h))^2$, where H and h are mesh sizes. Finally, numerical tests are presented to verify the theoretical results.

Keywords: FETI-DP; Crouzeix-Raviart element; nonstandard mortar condition; preconditioner

MSC 2010: 65N30, 65N38

1. INTRODUCTION

Models with discontinuous coefficients play an important role in scientific computing, for instance, in the simulation of fluid flow in porous media, where the permeability of the media may have large jumps across subdomain interfaces. Large jumps in the coefficients may result in bad convergence for the iterative methods.

Nonconforming discretizations are important for multi-physics simulations, contact-impact problems, generations of grids and partitions aligned with jumps in diffusion coefficients, *hp*-adaptive methods and special discretizations in the neighborhood of singularities. In order to make sure that the overall discretization makes sense, an

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optimal coupling between the grids is required. Of the many methods for coupling different discretization schemes, we consider the mortar method, which was first introduced by Bernardi, Maday, and Patera in [1]. In the standard mortar method, we need to know the function on the interface. For P_1 conforming elements, it is enough to know the node values along the interface. However, when it comes to Crouzeix-Raviart elements, the function on the interface depends on the node values corresponding to interface nodes and some subdomain interior nodes lying closest to the interface. In this paper, we adopt the nonstandard mortar condition for Crouzeix-Raviart element discretizations introduced by Xu Xuejun et al. [18], which is only associated with the node values on the interface in the calculation of the mortar condition.

The domain decomposition methods and especially FETI-DP methods form a class of fast and efficient iterative solvers for algebraic systems of equations arising from the finite element discretizations of PDEs, cf. [4], [5], [7], [8], [9], [17]. Recently, FETI-DP methods have been applied to mortar-type finite element methods, cf. [10], [13]. There has been also some work about FETI-DP methods for Crouzeix-Raviart element discretizations, cf. [16].

To our best knowledge, there is no work in the literature on the preconditioners for FETI-DP methods for solving systems of equations discretized by mortar-type Crouzeix-Raviart elements of second order problems with discontinuous coefficients. In this paper, we are interested in the application of the nonstandard mortar condition on nonmatching grids, where in each subgrid, Crouzeix-Raviart elements are used for the discretization. We extend the results of Kim and Lee [6] to mortar-type Crouzeix-Raviart element discretizations.

The paper is organized as follows: We describe mortar-type Crouzeix-Raviart element discretizations for second order elliptic problems with discontinuous coefficients in Section 2. In Section 3, the FETI-DP operator is introduced, then a parallel preconditioner for the FETI-DP operator is proposed in Section 4. Sections 5–6 are devoted to establishing the condition number bounds of the preconditioned problem. In Section 7, we compare the proposed preconditioner with that of Dryja and Widlund [3] and numerical tests are in accord with our theory.

2. Discrete problem

We consider a polygonal domain Ω in the plane, partitioned into a set of nonoverlapping polygonal subdomains $\{\Omega_k\}_{k=1,\ldots,N}$, such that $\overline{\Omega} = \bigcup\limits_{k=1}^{N} \overline{\Omega}_k$ and $\Omega_k \cap \Omega_l = \emptyset$, $k \neq l$, and $\{\Omega_k\}_{k=1,\ldots,N}$ form a coarse triangulation of $\overline{\Omega}$, which is shape regular. The second order elliptic problem is considered as follows: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u, v) = \sum_{k=1}^{N} \int_{\Omega_k} \varrho_k \nabla u \nabla v \,\mathrm{d}x \tag{2.2}$$

with $\varrho_k (k = 1, \ldots, N)$ being positive constant coefficients.

Let $T_h(\Omega_k) = \{\tau\}$ be a quasi-uniform triangulation consisting of triangles τ in each subdomain Ω_k. Let $h_k = \max\limits_{\tau \in T_h(\Omega_k)} \operatorname{diam}(\tau)$ be the mesh size. Let $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ denote the interface between two neighboring subdomains Ω_k and Ω_l, and $\Gamma = \bigcup\limits_{k=1}^{N} \partial\Omega_k \setminus \partial\Omega$ be the interface skeleton. We call the midpoints of all edges of an element in the triangulation Crouzeix-Raviart nodes or CR nodes. Denote the sets of CR nodes and P1 conforming nodes that are contained in Ω, $\partial\Omega$, Ω_k, $\partial\Omega_k$, and Γ_{kl} by Ω_h^{CR}, $\partial\Omega_h^{CR}$, $\Omega_{k,h}^{CR}$, $\partial\Omega_{k,h}^{CR}$, $\Gamma_{kl,h}^{CR}$, and Ω_h, $\partial\Omega_h$, $\Omega_{k,h}$, $\partial\Omega_{k,h}$, $\Gamma_{kl,h}$, respectively.

Let $W^h(\Omega_k)$ be the CR element space defined on the triangulation $T_h(\Omega_k)$, consisting of functions that are piecewise linear in each triangle $\tau \in T_h(\Omega_k)$, continuous in $\Omega_{k,h}^{CR}$, and equal to zero in $\partial\Omega_h^{CR} \cap \partial\Omega_{k,h}^{CR}$. The local subspace is equipped with the local broken H^1-seminorm and -norm: $|u|^2_{H_h^1(\Omega_k)} = \sum\limits_{\tau \in T_h(\Omega_k)} |u|^2_{H^1(\tau)}$, $\|u\|^2_{H_h^1(\Omega_k)} = |u|^2_{L^2(\Omega_k)} + |u|^2_{H_h^1(\Omega_k)}$. Let

$$W^h(\Omega) = \prod_{k=1}^{N} W^h(\Omega_k)$$

be the global space defined on the domain Ω and equipped with the broken H^1-seminorm and norm: $|u|^2_{H_H^1(\Omega)} = \sum\limits_{k=1}^{N} |u|^2_{H_h^1(\Omega_k)}$, $\|u\|^2_{H_H^1(\Omega)} = \sum\limits_{k=1}^{N} \|u\|^2_{H_h^1(\Omega_k)}$.

As we triangulate each subdomain independently of its neighboring subdomains, we note that each interface $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ inherits two independent triangulations. One of the sides of Γ_{kl} is defined as a mortar (master) side, denoted by γ_{kl}, and the other as a nonmortar (slave) side, denoted by δ_{lk}. We always choose the mortar side to be the side whose subdomain has larger ϱ value. Let γ_{kl} be the mortar side associated with Ω_k and δ_{lk} be the nonmortar side associated with Ω_l, obviously then $\varrho_l \preceq \varrho_k$. We further assume in this case that $h_k \preceq h_l$. We have two sets of CR nodes belonging to Γ_{kl}, the midpoints of elements belonging to $T_h(\gamma_{kl}) = T_h^k(\Gamma_{kl})$, the h_k-triangulation of Γ_{kl} inherited from $T_h(\Omega_k)$, and to $T_h(\delta_{lk}) = T_h^l(\Gamma_{kl})$, the h_l-triangulation of Γ_{kl} inherited from $T_h(\Omega_l)$, denoted by $\gamma_{kl,h}^{CR}$ and $\delta_{lk,h}^{CR}$, respectively. Since the triangulations on Ω_k and Ω_l do not match on their common interface Γ_{kl}, the functions in $W^h(\Omega)$ are discontinuous on the set $\gamma_{kl,h}^{CR}$ or $\delta_{lk,h}^{CR}$. Here, we adopt the nonstandard mortar condition introduced by Xu Xuejun et al. in [18], i.e., a function $u=\{u_k\}_{k=1}^N \in W^h(\Omega)$ satisfies

$$Q_m J_m u_k = Q_m u_l, \tag{2.3}$$

where J_m is an interpolation operator defined below (see Figure 1) and Q_m: $L^2(\Gamma_{kl}) \to V^h(\delta_{lk})$ is a L^2-orthogonal projection operator defined as

$$(Q_m u, \psi)_{L^2(\delta_{lk})} = (u, \psi)_{L^2(\delta_{lk})} \quad \forall\, \psi \in V^h(\delta_{lk}), \tag{2.4}$$

where $V^h(\delta_{lk}) \subset L^2(\Gamma_{kl})$ is the test space of functions that are piecewise constant on elements of the nonmortar triangulation of Γ_{kl}. Let

$$V^h = \prod_{\delta_{lk}\subset\Gamma} V^h(\delta_{lk})$$

be the auxiliary interface space, which will be used as the Lagrange multipliers space further.

Let e denote a triangle edge. Let $Z_h(\gamma_{kl}) = \prod_{e\in T_h(\gamma_{kl})} P_1(e)$ be the space of piecewise linear functions defined on the triangulation $T_h(\gamma_{kl})$, and let $T_{h/2}(\gamma_{kl})$ be the triangulation obtained by dividing the edges of $T_h(\gamma_{kl})$ into equal segments. Let $Y_{h/2}(\gamma_{kl})$ be the conforming space of piecewise linear continuous functions on the triangulation $T_{h/2}(\gamma_{kl})$. The midpoint, left and right endpoint of each edge $e \in T_h(\gamma_{kl})$ are denoted by x_m^e, x_l^e, and x_r^e, respectively. The length of e is denoted by $|e|$.

Definition 2.1. For $u \in W^h(\gamma_{kl}) = W^h(\Omega)|\gamma_{kl}$, $I_m u \in Y_{h/2}(\gamma_{kl})$ is defined as

$$I_m u(x) = \begin{cases} u(x), & x \in \gamma_{kl,h}^{CR}, \\ \dfrac{|e_r|}{|e_l|+|e_r|} u(x_m^{e_l}) + \dfrac{|e_l|}{|e_l|+|e_r|} u(x_m^{e_r}), & x \in \gamma_{kl,h}, \\ u(x_m^{e_e}) + \dfrac{|e_e|}{|e_e|+|e_e'|}(u(x_m^{e_e}) - u(x_m^{e_e'})), & x \in \partial\gamma_{kl,h}, \end{cases}$$

where e_l and e_r are the left and right neighboring edge of $x \in \gamma_{kl,h}$, respectively, e_e represents a triangle edge of $T_h(\gamma_{kl})$ touching $\partial\gamma_{kl}$, and e_e' is the corresponding neighboring edge.

Definition 2.2. For $u \in W^h(\gamma_{kl})$, $J_m u \in Z_h(\gamma_{kl})$ is a piecewise linear function on the edges $\{e\}$ of γ_{kl}, defined by its values at the two endpoints $x_l^e, x_r^e \in \overline{\gamma}_{kl,h}$ of each edge e. If e is an interior edge of γ_{kl}, then

$$J_m u(x) = \begin{cases} u(x_m^e) + \frac{1}{2}\{I_m u(x_l^e) - I_m u(x_r^e)\}, & x = x_l^e, \\ u(x_m^e) + \frac{1}{2}\{I_m u(x_r^e) - I_m u(x_l^e)\}, & x = x_r^e. \end{cases}$$

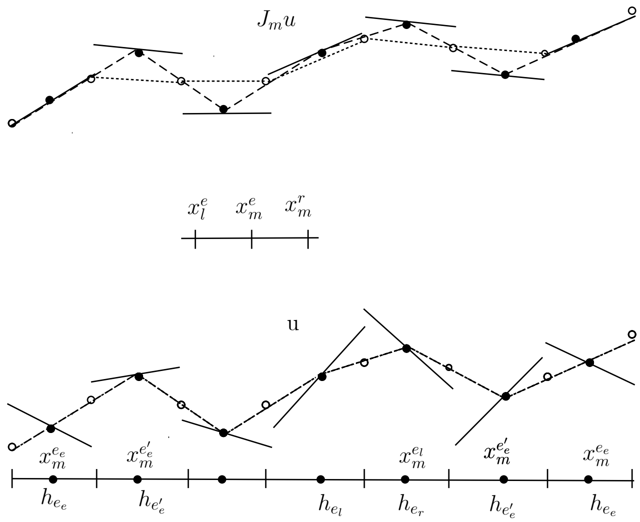

Figure 1. Here $u \in W^h(\gamma_{kl})$ is a piecewise linear function corresponding to solid lines in the lower figure. The dashed lines in both upper and lower figures correspond to $I_m u$. The solid lines in the upper figure correspond to $J_m u$.

It is easy to see that if e is a boundary edge of γ_{kl}, then $J_m u(x) = I_m u(x)$ for $x = x_l^e, x_r^e$.

Let $\widetilde{W}^h(\Omega) = \{u \in W^h(\Omega)\colon \int_{\delta_{lk}} (u_k - u_l)\,\mathrm{d}x = 0 \text{ for all } \delta_{lk} \subset \Gamma\}$ and let $\widehat{W}^h(\Omega) \subset \widetilde{W}^h(\Omega)$ be the subspace of functions which satisfy the nonstandard mortar condition (2.3) for all $\delta_{lk} \subset \Gamma$.

A discrete formulation of the problem (2.1) is then: Find $u_h^* \in \widehat{W}^h(\Omega)$ such that

$$a_h(u_h^*, v_h) = f(v_h) \quad \forall\, v_h \in \widehat{W}^h(\Omega), \tag{2.5}$$

where

$$a_h(u,v) = \sum_{k=1}^N a_{k,h}(u,v), \quad a_{k,h}(u,v) = \sum_{\tau \in T_h(\Omega_k)} \int_\tau \varrho_k \nabla u \nabla v \,\mathrm{d}x.$$

The problem (2.5) has a unique solution, cf. [2].

3. FETI-DP method

In this section, we introduce FETI-DP methods for solving problem (2.5) by using the framework given in [21].

The bilinear form $b(\cdot,\cdot)\colon \widetilde{W}^h(\Omega) \times V^h \to R$ is defined as

$$b(u,\psi) = \sum_{\delta_{lk} \subset \Gamma} \int_{\delta_{lk}} (J_m u_k - u_l)\psi_{lk} \,\mathrm{d}s,$$

for $u = (u_k)_{k=1}^N \in W^h(\Omega)$ and $\psi = (\psi_{lk})_{\delta_{lk}\subset\Gamma} \in V^h$ with $u_k \in W^h(\Omega_k)$ and $\psi_{lk} \in V^h(\delta_{lk})$. So we can rewrite the nonstandard mortar condition (2.3) as

$$b(u,\psi) = 0. \tag{3.1}$$

This brings us to another equivalent definition of $\widehat{W}^h(\Omega)$:

$$\widehat{W}^h(\Omega) = \{u \in \widetilde{W}^h(\Omega)\colon\ b(u,\psi) = 0 \quad \forall\, \psi \in V^h\}.$$

Note that

$$\widehat{W}^h(\Omega) \subset \widetilde{W}^h(\Omega) \subset W^h(\Omega).$$

Problem (2.5) can be reformulated as a saddle point problem: Find a pair $(w_h^*, \psi^*) \in \widetilde{W}^h(\Omega) \times V^h$ such that

$$\begin{aligned} a(w_h^*, v) + b(v, \psi^*) &= f(v) \quad \forall\, v \in \widetilde{W}^h(\Omega),\\ b(w_h^*, \varphi) &= 0 \quad \forall\, \varphi \in V^h. \end{aligned} \tag{3.2}$$

We see that w_h^* is the solution of (2.5), cf. [13]. For simplicity, we use the same notation to represent a function in a space and its vector representation with respect to the nodal basis of that space.

We next introduce a local decomposition of any function $u_k \in W^h(\Omega_k)$ as $u_k = P_k u_k + H_k^+ u_k$, where $P_k u_k \in W_0^h(\Omega_k)$ is defined by

$$a_{k,h}(P_k u_k, v) = a_{k,h}(u_k, v) \quad \forall\, v \in W_0^h(\Omega_k). \tag{3.3}$$

Here $W_0^h(\Omega_k) = \{u \in W^h(\Omega_k)\colon\ u(m) = 0$ for all $m \in \partial\Omega_{k,h}^{CR}\} \subset W^h(\Omega_k)$ and $H_k^+ u_k \in W^h(\Omega_k)$ is the discrete harmonic part of u_k, i.e., $H_k^+ u_k = u_k - P_k u_k$, which is defined as the solution of the following problem: Find $H_k^+ u_k \in W^h(\Omega_k)$ such that

$$\begin{cases} a_{k,h}(H_k^+ u_k, v) = 0 & \forall\, v \in W_0^h(\Omega_k),\\ H_k^+ u_k(m) = u_k(m) & \forall\, m \in \partial\Omega_{k,h}^{CR}. \end{cases} \tag{3.4}$$

We introduce

$$W_k = H_k^+ W^h(\Omega_k) \tag{3.5}$$

as the local space of discrete harmonic functions. Consequently, $Hu = (H_k^+ u_k)_{k=1}^N$ for $u = (u_k)_{k=1}^N \in W^h(\Omega)$.

Let the global spaces of discrete harmonic functions corresponding to the spaces $W^h(\Omega)$, $\widetilde{W}^h(\Omega)$, and $\widehat{W}^h(\Omega)$ be defined as follows:

$$W = HW^h(\Omega) = \prod_{k=1}^{N} W_k, \tag{3.6}$$

$$\widetilde{W} = H\widetilde{W}^h(\Omega) = \Big\{u \in W\colon \int_{\delta_{lk}} (u_k - u_l)\,\mathrm{d}s = 0\ \forall\, \delta_{lk} \subset \Gamma\Big\}, \tag{3.7}$$

$$\widehat{W} = H\widehat{W}^h(\Omega) = \{u \in \widetilde{W}\colon\ b(u,\psi) = 0\ \forall\, \psi \in V^h\}. \tag{3.8}$$

The solution of problem (2.5) can be decomposed as

$$u_h^* = (u_{h,k}^*)_{k=1}^N = u_I^* + w_h^*,$$

where $u_I^* = (P_k u_{h,k}^*)_{k=1}^N$ and $P_k u_{h,k}^*$ $(k = 1,\dots,N)$ can be computed by solving N independent local subproblems. The discrete harmonic part of u_h^*, i.e., $w_h^* = (H_k^+ u_{h,k}^*)_{k=1}^N \in \widehat{W}$, is the unique solution of the following problem: Find $w_h^* \in \widehat{W}$ such that

$$a_h(w_h^*, v) = f(v) \quad \forall\, v \in \widehat{W}. \tag{3.9}$$

In the FETI-DP method, we denote the basis for $V^h(\delta_{lk})$ by $\{\xi_k^{\delta_{lk}}\}_{k=1}^{N_{\delta_{lk}}}$, the basis for $W_l(\delta_{lk}) = W_l|_{\delta_{lk}}$ by $\{\varphi_k^{\delta_{lk}}\}_{k=1}^{N_{\delta_{lk}}}$, and the basis for $(J_m W_k)(\gamma_{kl}) = (J_m W_k)|_{\gamma_{kl}}$ by $\{\varphi_k^{\gamma_{kl}}\}_{k=1}^{N_{\gamma_{kl}}}$, where $N_{\gamma_{kl}}$ and $N_{\delta_{lk}}$ are the number of elements of $T_h(\gamma_{kl})$ and $T_h(\delta_{lk})$, respectively. We define matrices $B_{\delta_{lk}}^l$ and $B_{\gamma_{kl}}^k$ with entries

$$(B_{\delta_{lk}}^l)_{ij} = \int_{\delta_{lk}} \xi_i^{\delta_{lk}} \varphi_j^{\delta_{lk}}\,\mathrm{d}s, \quad i = 1,\dots,N_{\delta_{lk}},\ j = 1,\dots,N_{\delta_{lk}},$$
$$(B_{\gamma_{kl}}^k)_{ij} = -\int_{\delta_{lk}} \xi_i^{\delta_{lk}} \varphi_j^{\gamma_{kl}}\,\mathrm{d}s, \quad i = 1,\dots,N_{\delta_{lk}},\ j = 1,\dots,N_{\gamma_{kl}}.$$

Then we write (3.1) as

$$B_{\delta_{lk}}^l u_{l,\delta_{lk}} + B_{\gamma_{kl}}^k u_{k,\gamma_{kl}} = 0,$$

where $u_{l,\delta_{lk}} = u_l|_{\delta_{lk}}$ and $u_{k,\gamma_{kl}} = u_k|_{\gamma_{kl}}$.

Now we define a zero extension operator $E_{\delta_{lk}}\colon V^h(\delta_{lk}) \to V^h$, and a restriction operator $R_{\delta_{lk}}^l\colon W_l \to W_l(\delta_{lk})$, $R_{\gamma_{lj}}^l\colon W_l \to W_l(\gamma_{lj})$. Let

$$B_l = \sum_{\delta_{lk} \subset \partial\Omega_l} E_{\delta_{lk}} B_{\delta_{lk}}^l R_{\delta_{lk}}^l + \sum_{\gamma_{lj} \subset \partial\Omega_l} E_{\delta_{jl}} B_{\gamma_{lj}}^j R_{\gamma_{lj}}^j.$$

Then the nonstandard mortar condition (3.1) becomes

$$\sum_{l=1}^{N} B_l u_l = 0.$$

Define

$$\widetilde{W}_\Delta(\delta_{lk}) = \left\{ w_{\delta_{lk}} \in \widetilde{W}(\delta_{lk}) = \widetilde{W}|_{\delta_{lk}} : \int_{\delta_{lk}} w_{\delta_{lk}} \,\mathrm{d}s = 0 \right\},$$

and let

$$\widetilde{W}_\Delta = \prod_{\delta_{lk} \subset \Gamma} \widetilde{W}_\Delta(\delta_{lk}).$$

For $w_{\delta_{lk}} \in \widetilde{W}_\Delta(\delta_{lk})$, we define $\widetilde{w}_{\delta_{lk}} \in \widetilde{W}_l = \widetilde{W}|_{\partial\Omega_l}$ to be the zero extension of $w_{\delta_{lk}}$ on $\partial\Omega_l$. Let $\widetilde{w}_l = \sum_{\delta_{lk} \subset \partial\Omega_l} \widetilde{w}_{\delta_{lk}}$ and $\widetilde{w} = (\widetilde{w}_1, \ldots, \widetilde{w}_N)$, and we have $\widetilde{w} \in \widetilde{W}$. Hence, for $w \in \widetilde{W}_\Delta$, we define a norm by

$$\|w\|_{\widetilde{W}_\Delta} = \|\widetilde{w}\|_{\widetilde{W}}. \tag{3.10}$$

We decompose $\widetilde{W}$ into two subspaces $\widetilde{W}_\Delta$ and $\widehat{W}_\Pi$ such that

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta, \tag{3.11}$$

where $\widehat{W}_\Pi$ is called the primal subspace, and $\widetilde{W}_\Delta$ is called the dual subspace.

For any $u \in \widetilde{W}$, we regroup its unknowns so that we can write it in its vector representation as (cf. (3.11)) $u = (u_\Pi, u_\Delta)^t$, where $u_\Pi \in \widehat{W}_\Pi$ and $u_\Delta \in \widetilde{W}_\Delta$. Similarly, we partition the vector $w_k \in \widetilde{W}_k$ as $w_k = (w_{k,\Pi}, w_{k,\Delta})^t$. Let L_Δ^k represent a matrix such that $L_\Delta^k w_\Delta$ restricts the values of the degrees of freedom of $w_\Delta \in \widetilde{W}_\Delta$ to the respective degrees of freedom of $\partial\Omega_k$, i.e., for any $w \in \widetilde{W}$, we can write $w = (w_1, \ldots, w_N)$ with $w_k = (w_{k,\Pi}, L_\Delta^k w_\Delta)^t$.

Recall that $S^{(k)}$ is the Schur complement matrix obtained from the bilinear form $a_{k,h}(\cdot,\cdot)$ and let $g^{(k)}$ be the Schur complement forcing vector obtained from $\int_{\Omega_k} f v_k \,\mathrm{d}x$. The matrix $S^{(k)}$ and vector $g^{(k)}$ are ordered in the following way:

$$S^{(k)} = \begin{pmatrix} S_{\Pi\Pi}^{(k)} & S_{\Pi\Delta}^{(k)} \\ S_{\Delta\Pi}^{(k)} & S_{\Delta\Delta}^{(k)} \end{pmatrix}, \quad g^{(k)} = \begin{pmatrix} g_\Pi^{(k)} \\ g_\Delta^{(k)} \end{pmatrix}.$$

Then the problem (3.2) becomes the following: Find $(w_\Pi, w_\Delta, \psi) \in \widehat{W}_\Pi \times \widetilde{W}_\Delta \times V^h$ such that

$$S_{\Pi\Pi} w_\Pi + S_{\Pi\Delta} w_\Delta + B_\Pi^t \psi = g_\Pi, \tag{3.12}$$

$$S_{\Delta\Pi} w_\Pi + S_{\Delta\Delta} w_\Delta + B_\Delta^t \psi = g_\Delta, \tag{3.13}$$

$$B_\Pi w_\Pi + B_\Delta w_\Delta = 0, \tag{3.14}$$

where

$$S_{\Pi\Pi} = \mathrm{diag}_k(S_{\Pi\Pi}^{(k)}),$$
$$S_{\Pi\Delta} = \begin{pmatrix} (S_{\Pi\Delta}^{(1)})^t L_\Delta^1 \\ \vdots \\ (S_{\Pi\Delta}^{(N)})^t L_\Delta^N \end{pmatrix},$$
$$S_{\Delta\Pi} = S_{\Pi\Delta}^t,$$
$$S_{\Delta\Delta} = \sum_{k=1}^N (L_\Delta^k)^t S_{\Delta\Delta}^{(k)} L_\Delta^k,$$
$$B_\Pi = (B_{1,\Pi}, \ldots, B_{N,\Pi}), \quad B_\Delta = \sum_{k=1}^N B_{k,\Delta} L_\Delta^k,$$
$$g_\Pi = \begin{pmatrix} g_\Pi^{(1)} \\ \vdots \\ g_\Pi^{(N)} \end{pmatrix}, \quad g_\Delta = \sum_{k=1}^N (L_\Delta^k)^t g_\Delta^{(k)}, \quad w_\Pi = \begin{pmatrix} w_{1,\Pi} \\ \vdots \\ w_{N,\Pi} \end{pmatrix}.$$

Since $S_{\Pi\Pi}$ is invertible, we solve (3.12) for w_Π to get

$$w_\Pi = S_{\Pi\Pi}^{-1}(g_\Pi - S_{\Pi\Delta} w_\Delta - B_\Pi^t \psi).$$

After substituting w_Π into (3.13) and (3.14), we obtain

$$\begin{aligned} &B_\Pi S_{\Pi\Pi}^{-1} B_\Pi^t \psi + (B_\Pi S_{\Pi\Pi}^{-1} S_{\Pi\Delta} - B_\Delta) w_\Delta = B_\Pi S_{\Pi\Pi}^{-1} g_\Pi, \\ &(S_{\Delta\Pi} S_{\Pi\Pi}^{-1} B_\Pi^t - B_\Delta^t)\psi - (S_{\Delta\Delta} - S_{\Delta\Pi} S_{\Pi\Pi}^{-1} S_{\Pi\Delta}) w_\Delta = -(g_\Delta - S_{\Delta\Pi} S_{\Pi\Pi}^{-1} g_\Pi). \end{aligned}$$

Let

$$\begin{aligned} F_{I_{\Pi\Pi}} &= B_\Pi S_{\Pi\Pi}^{-1} B_\Pi^t, \\ F_{I_{\Pi\Delta}} &= B_\Pi S_{\Pi\Pi}^{-1} S_{\Pi\Delta} - B_\Delta, \\ F_{I_{\Delta\Pi}} &= S_{\Delta\Pi} S_{\Pi\Pi}^{-1} B_\Pi^t - B_\Delta^t, \\ F_{I_{\Delta\Delta}} &= S_{\Delta\Delta} - S_{\Delta\Pi} S_{\Pi\Pi}^{-1} S_{\Pi\Delta}, \\ d_\Pi &= B_\Pi S_{\Pi\Pi}^{-1} g_\Pi, \\ d_\Delta &= g_\Delta - S_{\Delta\Pi} S_{\Pi\Pi}^{-1} g_\Pi. \end{aligned} \tag{3.15}$$

Then (ψ, w_Δ) satisfies

$$\begin{pmatrix} F_{I_{\Pi\Pi}} & F_{I_{\Pi\Delta}} \\ F_{I_{\Delta\Pi}} & -F_{I_{\Delta\Delta}} \end{pmatrix} \begin{pmatrix} \psi \\ w_\Delta \end{pmatrix} = \begin{pmatrix} d_\Pi \\ -d_\Delta \end{pmatrix}.$$

By eliminating w_Δ in the above equation, we obtain

$$(F_{I_{\Pi\Pi}} + F_{I_{\Pi\Delta}} F_{I_{\Delta\Delta}}^{-1} F_{I_{\Delta\Pi}})\psi = d_\Pi - F_{I_{\Pi\Delta}} F_{I_{\Delta\Delta}}^{-1} d_\Delta. \tag{3.16}$$

Here $F_{DP} = F_{I_{\Pi\Pi}} + F_{I_{\Pi\Delta}} F_{I_{\Delta\Delta}}^{-1} F_{I_{\Delta\Pi}}$ is called the FETI-DP operator.

4. Preconditioner

In this section, we propose a preconditioner for the system (3.16); cf. [21]. Let the matrix $S_\Delta \colon \widetilde{W}_\Delta \to \widetilde{W}_\Delta$ be the submatrix of S obtained by restricting the matrix S to the subspace $\widetilde{W}_\Delta$, where $S = \mathrm{diag}_k(S^{(k)})$. We note that $S_\Delta = \mathrm{diag}_k(S_\Delta^{(k)})$, with $S_\Delta^{(k)}$ being the restriction of the local Schur complement matrix $S^{(k)}$ to the space $\widetilde{W}_\Delta(\delta_{kl})$.

We equip $w \in \widetilde{W}_\Delta$ with the norm

$$\|w\|_{S_\Delta}^2 = \langle S_\Delta w, w\rangle = \|\widetilde{w}\|_S^2,$$

where $\widetilde{w} \in \widetilde{W}$ is the extension of $w \in \widetilde{W}_\Delta$ by zero onto the trace spaces associated with mortars.

Then we can define

$$B_{\delta_{lk}} = \sum_{\delta_{lk} \subset \Gamma} E_{\delta_{lk}} B_{\delta_{lk}}^l R_{\delta_{lk}}^l, \quad B_{\gamma_{kl}} = \sum_{\gamma_{kl} \subset \Gamma} E_{\delta_{lk}} B_{\gamma_{kl}}^l R_{\gamma_{kl}}^l.$$

The matrices $B_{\delta_{lk},\Delta}$ and $B_{\delta_{lk},\Pi}$ with subscripts Δ and Π are the submatrices of the matrix $B_{\delta_{lk}}$ corresponding to the splitting (3.11). Then we define $B_\Delta = \mathrm{diag}_{\delta_{lk}\subset\Gamma}(B_{\delta_{lk},\Delta})$. Note that for any $(\psi_{lk}, w_{\delta_{lk}}) \in V^h(\delta_{lk}) \times \widetilde{W}_\Delta(\delta_{lk})$, we have

$$\langle w_{\delta_{lk}}, (B_{\delta_{lk},\Delta})^t \psi_{lk}\rangle = \langle B_{\delta_{lk},\Delta} w_{\delta_{lk}}, \psi_{lk}\rangle = \int_{\delta_{lk}} \psi_{lk} w_{\delta_{lk}} \,\mathrm{d}s. \tag{4.1}$$

Here B_Δ, B_Δ^t are block diagonal matrices with invertible blocks, cf. [11]. Finally, we introduce the inverse of the preconditioner as $\widehat{F}_{DP} = B_\Delta S_\Delta^{-1} B_\Delta^t$, which is nonsingular, and thus we choose

$$\widehat{F}_{DP}^{-1} = B_\Delta^{-t} S_\Delta B_\Delta^{-1} \tag{4.2}$$

as the preconditioner for problem (3.16).

In this section, we state and prove a few technical lemmas necessary for the proof of Theorem 6.1 in Section 6.

Let $Y_{h/2}(\Omega_k)$ be the conforming space of piecewise linear continuous functions on the triangulation $T_{h/2}(\Omega_k)$, which is constructed by joining the midpoints of the edges of the elements of $T_h(\Omega_k)$. For each open edge $\varepsilon \subset \partial\Omega_k$, we define $W^h_\varepsilon(\Omega_k)$ as a subspace of $W^h(\Omega_k)$ formed by all functions which are equal to zero in $\partial\Omega^{CR}_{k,h} \setminus \varepsilon^{CR}_h$.

Definition 5.1 ([19])**.** For a given $u \in W^h_\varepsilon(\Omega_k)$, we introduce $M^\varepsilon_k u \in Y_{h/2}(\Omega_k)$ by defining the values of $M^\varepsilon_k u$ at the nodes of the triangulation $T_{h/2}(\Omega_k)$:

- ▷ For $p \in \Omega^{CR}_{k,h} \cup \partial\Omega^{CR}_{k,h}$, let $M^\varepsilon_k u(p) = u(p)$.
- ▷ For $p \in \Omega_{k,h}$, let $M^\varepsilon_k u(p) = N(p)^{-1} \sum_{\tau^h_j} u|_{\tau^h_j}(p)$, where the sum is taken over all triangles τ^h_j with a common vertex p and $N(p)$ is the number of elements with p as an vertex.
- ▷ For $p \in \partial\Omega_{k,h} \setminus \varepsilon_h$, let $M^\varepsilon_k u(p) = 0$.
- ▷ For $p \in \varepsilon_h$, let $M^\varepsilon_k u(p) = |pp_r| |p_l p_r|^{-1} u(p_l) + |p_l p| |p_l p_r|^{-1} u(p_r)$, where p_l, p_r are the left and right neighboring CR nodes of p, respectively, and $|ab|$ is the length of the segment with a, b as its ends.

Note that $M^\varepsilon_k u$ is piecewise linear between the CR nodes of ε^{CR}_h, and $M^\varepsilon_k u$ is equal to zero on $\partial\Omega_k \setminus \varepsilon$. The mapping $M^\varepsilon_k u$ has the following properties (cf. [10]):

$$|u|_{H^1_h(\Omega_k)} \asymp |M^\varepsilon_k u|_{H^1(\Omega_k)}, \quad \|u\|_{L^2(\Omega_k)} \asymp \|M^\varepsilon_k u\|_{L^2(\Omega_k)},$$
$$\|M^\varepsilon_k u - u\|_{L^2(\Omega_k)} \preceq h_k |u|_{H^1_h(\Omega_k)}, \quad \|M^\varepsilon_k u - u\|_{L^2(\varepsilon)} \preceq h_k^{1/2} |u|_{H^1_h(\Omega_k)}.$$

We also need a special mortar operator $\Pi_{\delta_{lk}}\colon L^2(\delta_{lk}) \to W^{h_l}_0(\delta_{lk})$ defined over trace spaces, where $W^{h_l}_0(\delta_{lk})$ is a set of continuous functions that are equal to zero at the ends of δ_{lk} and are piecewise linear over all segments that have their ends in $\delta^{CR}_{lk,h}$.

Definition 5.2 ([10])**.** The operator $\Pi_{\delta_{lk}}\colon L^2(\delta_{lk}) \to W^{h_l}_0(\delta_{lk})$ is defined as

$$\Pi_{\delta_{lk}} u(x) = Q_m u(x) \quad \forall\, x \in \delta^{CR}_{lk,h}. \tag{5.1}$$

We have the stability property of the above operator in the L^2 and $H^{1/2}_{00}$ norms, cf. [10], i.e.,

$$\|\Pi_{\delta_{lk}} u\|_{L^2(\delta_{lk})} \preceq \|u\|_{L^2(\delta_{lk})} \quad \forall\, u \in L^2(\delta_{lk}), \tag{5.2}$$

$$\|\Pi_{\delta_{lk}} u\|_{H^{1/2}_{00}(\delta_{lk})} \preceq \|u\|_{H^{1/2}_{00}(\delta_{lk})} \quad \forall\, u \in H^{1/2}_{00}(\delta_{lk}). \tag{5.3}$$

Lemma 5.1 ([12]). *Let $u \in W_l$ and $u = 0$ in $\partial\Omega_{l,h}^{CR} \setminus \delta_{lk,h}^{CR}$ for a slave edge $\delta_{lk} \subset \partial\Omega_l$. Then we have*

$$|u|_{H_h^1(\Omega_l)} \preceq \|M_l^{\delta_{lk}} u\|_{H_{00}^{1/2}(\delta_{lk})}.$$

Lemma 5.2 ([14], [21]). *For any $\psi \in V^h$, we have*

$$\langle \widehat{F}_{DP}\psi, \psi\rangle^{1/2} = \sup_{w\in\widetilde{W}_\Delta\setminus\{0\}} \sum_{\delta_{lk}\subset\Gamma} \frac{\int_{\delta_{lk}} w_{\delta_{lk}} \psi_{lk}\, \mathrm{d}s}{\|w\|_{S_\Delta}} = \sup_{w\in\widetilde{W}_\Delta\setminus\{0\}} \frac{b(\widetilde{w}, \psi)}{\|\widetilde{w}\|_S},$$

where $\psi_{lk} \in V^h(\delta_{lk})$ and $w_{\delta_{lk}} \in \widetilde{W}_\Delta(\delta_{lk})$, cf. (4.1), *and $\widetilde{w} \in \widetilde{W}$ is the extension of $w \in \widetilde{W}_\Delta$ by zero.*

Lemma 5.3 ([20]). *For any $\psi \in V^h$ we have*

$$\langle F_{DP}\psi, \psi\rangle^{1/2} = \sup_{w\in\widetilde{W}\setminus\{0\}} \frac{b(w, \psi)}{\|w\|_S}.$$

Lemma 5.4 ([10]). *Let a slave side $\delta_{lk} \subset \partial\Omega_l$. Then for any $u \in W_l$ we have*

$$|u^{\delta_{lk}}|^2_{H_h^1(\Omega_l)} \preceq (1 + \log(H_l/h))^2 (H_l^{-2}\|u\|^2_{L^2(\Omega_l)} + |u|^2_{H_h^1(\Omega_l)}),$$

where $u^{\delta_{lk}}$ is a discrete harmonic function taking the same values as u at CR nodes on $\delta_{lk,h}^{CR}$ and is equal to zero at the remaining CR nodes on $\partial\Omega_l$, and $H_l = \operatorname{diam}\Omega_l$.

Definition 5.3 ([15]). Given $u \in W^h(\Omega_k)$, we define an operator $\widehat{O}_k\colon\ W^h(\Omega_k) \to \widehat{W}$ as follows:

- ▷ For $p \in \gamma_{kl,h}^{CR}$, let $\widehat{O}_k u(p) = u(p)$.
- ▷ For $p \in \delta_{lk,h}^{CR}$, let $\widehat{O}_k u(p) = Q_m J_m u(p)$.
- ▷ For $p \in \delta_{kj,h}^{CR}$, for any slave side $\delta_{kj} \subset \partial\Omega_k$, or if the edge midpoint p is on $\partial\Omega$ or on any remaining master or slave side, let $\widehat{O}_k u(p) = 0$.

Definition 5.4 ([15]). Given $u \in W^h(\Omega_k)$, which can be decomposed into $u = H_k^+ u + P_k u$, we define $O_k u$ as follows, based on Definition 5.3:

$$O_k u = \widehat{O}_k u + \widetilde{P}_k u. \tag{5.4}$$

Here $\widetilde{P}_k u = (0, \ldots, P_k u, \ldots, 0)$, i.e., it is the $P_k u$ from (3.3) extended by zero onto the remaining subdomains.

We define a projection operator $O\colon \widetilde{W} \to \widetilde{W}_\Delta$ by

$$(Ou)|_{\delta_{lk}} = Q_m(J_m u_k - u_l) \quad \text{on } \delta_{lk}, \tag{5.5}$$

where u_k and u_l are the restriction of $u \in \widetilde{W}$ to the mortar side γ_{kl} and to the slave side δ_{lk} of an interface Γ_{kl}, respectively.

Lemma 5.5. *For all $u \in \widetilde{W}$ we have*

$$\|Ou\|_{S_\Delta} \preceq (1 + \log(H/h))\|u\|_S,$$

where $H = \max\limits_k H_k$ and $h = \min\limits_k h_k$.

Proof. Let $Ou = (O_1 u, \cdots, O_N u) \in \widetilde{W}$. Then we have $\|Ou\|_S^2 = \sum\limits_{k=1}^N \|O_k u\|_{S^{(k)}}^2$. Next we will estimate the term $\|O_k u\|_{S^{(k)}}^2$.

Note that $O_k u$ can be nonzero only over Ω_k and a neighboring subdomain Ω_l that shares a common edge Γ_{kl} with Ω_k, such that the slave side δ_{lk} is associated with Ω_l. Let N_k denote the set of indices of such subdomains. Thus

$$\|O_k u\|_{S^{(k)}}^2 = \varrho_k |O_k u|_{H_h^1(\Omega_k)}^2 + \sum_{l \in N_k} \varrho_l |O_k u|_{H_h^1(\Omega_l)}^2.$$

When we consider Ω_k, we have $O_k u = u - \sum\limits_{\delta_{kl} \subset \partial\Omega_k} u^{\delta_{kl}}$ with $u^{\delta_{kl}}$ being a discrete harmonic function which equals to u at the CR nodes of $\delta_{kl,h}^{CR}$ and to zero at all remaining nodes of $\partial\Omega_{k,h}^{CR}$. Thus, by Lemma 5.4,

$$\begin{aligned} |O_k u|_{H_h^1(\Omega_k)}^2 &\preceq |u|_{H_h^1(\Omega_k)}^2 + \sum_{\delta_{kl} \subset \partial\Omega_k} |u^{\delta_{kl}}|_{H_h^1(\Omega_k)}^2 \\ &\preceq (1 + \log(H_k/h_k))^2 (H_k^{-2} \|u\|_{L^2(\Omega_k)}^2 + |u|_{H_h^1(\Omega_k)}^2). \end{aligned}$$

Next we consider those Ω_l with $l \in N_k$. Consequently, the slave side δ_{lk} of the edge Γ_{kl} is an edge of Ω_l, and the mortar side γ_{kl} is an edge of Ω_k. Then by Lemma 5.1, we get

$$|O_k u|_{H_h^1(\Omega_l)} \preceq \|M_l^\varepsilon O_k u\|_{H_{00}^{1/2}(\delta_{lk})}.$$

From Definitions 5.1, 5.3, and 5.4, we have

$$M_l^\varepsilon O_k u(p) = O_k u(p) = Q_m O_k u(p) = Q_m J_m u(p) \quad \forall\, p \in \delta_{lk,h}^{CR}. \tag{5.6}$$

Thus we get

$$\|M_l^\varepsilon O_k u\|_{H_{00}^{1/2}(\delta_{lk})} \leqslant \|M_l^\varepsilon O_k u - \Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}\|_{H_{00}^{1/2}(\delta_{lk})} + \|\Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}\|_{H_{00}^{1/2}(\delta_{lk})}.$$

The second term above can be estimated by (5.2), the trace theorem and Lemma 5.4 as follows:

$$
\begin{aligned}
\|\Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}\|^2_{H_{00}^{1/2}(\delta_{lk})} &\preceq \|M_k^\varepsilon u^{\gamma_{kl}}\|^2_{H_{00}^{1/2}(\delta_{lk})} \\
&\preceq |M_k^\varepsilon u^{\gamma_{kl}}|^2_{H^1(\Omega_k)} \preceq |u^{\gamma_{kl}}|^2_{H_h^1(\Omega_k)} \\
&\preceq (1+\log(H_k/h_k))^2 (H_k^{-2}\|u\|^2_{L^2(\Omega_k)} + |u|^2_{H_h^1(\Omega_k)}).
\end{aligned}
$$

The first term is bounded using an inverse inequality, (5.6) and Definition 5.2 as follows:

$$
\begin{aligned}
\|M_l^\varepsilon O_k u - \Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}\|^2_{H_{00}^{1/2}(\delta_{lk})} &\preceq h_l^{-1}\|M_l^\varepsilon O_k u - \Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}\|^2_{L^2(\delta_{lk})} \\
&\preceq \sum_{p\in\delta_{lk,h}^{CR}} |M_l^\varepsilon O_k u(p) - \Pi_{\delta_{lk}} M_k^\varepsilon u^{\gamma_{kl}}(p)|^2 \\
&= \sum_{p\in\delta_{lk,h}^{CR}} |Q_m(J_m u - M_k^\varepsilon u^{\gamma_{kl}})(p)|^2 \\
&\preceq h_l^{-1}\|Q_m(J_m u - M_k^\varepsilon u^{\gamma_{kl}})\|^2_{L^2(\delta_{lk})} \\
&\preceq h_l^{-1}\|J_m u - M_k^\varepsilon u^{\gamma_{kl}}\|_{L^2(\delta_{lk})}.
\end{aligned}
$$

The fact that $J_m u = J_m u^{\gamma_{kl}}$ yields

$$
\|J_m u - M_k^\varepsilon u^{\gamma_{kl}}\|_{L^2(\delta_{lk})} \leqslant \|M_k^\varepsilon u^{\gamma_{kl}} - u^{\gamma_{kl}}\|_{L^2(\delta_{lk})} + \|u^{\gamma_{kl}} - J_m u^{\gamma_{kl}}\|_{L^2(\delta_{lk})}.
$$

Applying the property of $M_k^\varepsilon u$ to the first term and Lemma 3.2 from [18] to the second term and then using the assumption that $h_k \preceq h_l$ and Lemma 5.4, we get

$$
\begin{aligned}
h_l^{-1}\|M_k^\varepsilon u^{\gamma_{kl}} - J_m u\|^2_{L^2(\delta_{lk})} &\preceq \frac{h_k}{h_l}|u^{\gamma_{kl}}|^2_{H_h^1(\Omega_k)} \\
&\preceq (1+\log(H_k/h_k))^2 (H_k^{-2}\|u\|^2_{L^2(\Omega_k)} + |u|^2_{H_h^1(\Omega_k)}) \\
&\preceq (1+\log(H_k/h_k))^2 |u|^2_{H_h^1(\Omega_k)}.
\end{aligned}
$$

Finally, summing over all edges in $\partial\Omega_k$ and then over all subdomain ends the proof. $\square$

6. Condition number estimate

In this section, we give the condition number estimate of the preconditioned operator, which forms the main theorem of this paper.

Theorem 6.1. *For any $\psi \in V^h$, it holds that*

$$\langle \widehat{F}_{DP}\psi, \psi\rangle \preceq \langle F_{DP}\psi, \psi\rangle \preceq (1+\log(H/h))^2 \langle \widehat{F}_{DP}\psi, \psi\rangle,$$

where $H = \max_k H_k$ and $h = \min_k h_k$.

Proof. We use the algebraic arguments from [21] in the proof of this theorem.

Lower bound: For any nonzero $w \in \widetilde{W}_\Delta$, define $\widetilde{w} \in \mathrm{Ext}(\widetilde{W}_\Delta) \subset \widetilde{W}$ as the extension of w by zero. Then we have $\|w\|_{S_\Delta} = \|\widetilde{w}\|_S$. Thus by Lemma 5.2 and Lemma 5.3, we have

$$\begin{aligned}\langle \widehat{F}_{DP}\psi, \psi\rangle^{1/2} &= \sup_{w \in \widetilde{W}_\Delta \setminus \{0\}} \frac{b(\widetilde{w}, \psi)}{\|w\|_{S_\Delta}} = \sup_{\widetilde{w} \in \mathrm{Ext}(\widetilde{W}_\Delta)\setminus\{0\}} \frac{b(\widetilde{w}, \psi)}{\|\widetilde{w}\|_S} \\ &\leqslant \sup_{w \in \widetilde{W}\setminus\{0\}} \frac{b(w,\psi)}{\|w\|_S} = \langle F_{DP}\psi, \psi\rangle^{1/2}.\end{aligned}$$

Upper bound: For any $w \in \widetilde{W}$, by (5.5), we have

$$\begin{aligned}b(w,\psi) &= \sum_{\delta_{lk}\subset\Gamma} \int_{\delta_{lk}} (J_m w_k - w_l)\psi_{lk}\,\mathrm{d}s \\ &= \sum_{\delta_{lk}\subset\Gamma} \int_{\delta_{lk}} Q_m(J_m w_k - w_l)\psi_{lk}\,\mathrm{d}s = \sum_{\delta_{lk}\subset\Gamma} \int_{\delta_{lk}} (Ow)|_{\delta_{lk}}\psi_{lk}\,\mathrm{d}s.\end{aligned}$$

Hence, by Lemmas 5.2, 5.3, and 5.5 we conclude that

$$\begin{aligned}\langle F_{DP}\psi, \psi\rangle^{1/2} &= \sup_{w \in \widetilde{W}\setminus\{0\}} \frac{b(w,\psi)}{\|w\|_S} \\ &= \sup_{w \in \widetilde{W}\setminus\{0\}} \sum_{\delta_{lk}\subset\Gamma} \frac{\int_{\delta_{lk}} (Ow)|_{\delta_{lk}}\psi_{lk}\,\mathrm{d}s}{\|w\|_S} \\ &\preceq \langle \widehat{F}_{DP}\psi, \psi\rangle^{1/2} \sup_{w \in \widetilde{W}\setminus\{0\}} \frac{\|Ow\|_S}{\|w\|_S} \\ &\preceq \Big(1 + \log\frac{H}{h}\Big)\langle \widehat{F}_{DP}\psi, \psi\rangle^{1/2}.\end{aligned}$$

$\square$

7. Numerical results

We consider the second order elliptic problem

$$-\nabla \cdot (\varrho(x,y)\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{7.1}$$

where $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$.

We compare our preconditioner (4.2), denoted by $\widehat{F}_{DP}^{-1}$, with the preconditioner developed by Dryja and Widlund (cf. (3.13) in [3]), denoted by $\widehat{F}_{DW}^{-1}$, for the cases where $\varrho(x,y) = 1$ and mesh sizes are comparable, and where $\varrho(x,y)$ is highly discontinuous across subdomain interfaces and mesh sizes are not comparable.

First, we compare the above two preconditioners for the same problem with nonmatching grids. We take $\varrho(x,y) = 1$ and the exact solution $u(x,y) = y(1-y)\sin \pi x$. The CG iteration continues until the relative residual norm is less than 10^{-6}. The number of nodes on edges including endpoints and the number of subdomains are denoted by n and N, respectively. Here, we use the same n for all subdomains and divide Ω into rectangular subdomains, where each subdomain is denoted by Ω_{ij}, see Figure 2. For the case of nonmatching grids across subdomain interfaces, the triangulations in each subdomain are generated as follows: we choose n random quasi-uniform nodes on each horizontal and vertical edge in each subdomain to generate nonuniform structured grids with comparable mesh sizes between neighboring subdomains.

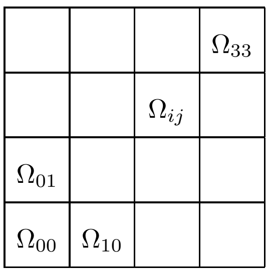

Figure 2. Partition of subdomains with $N = 4 \times 4$.

In Table 1, we divide Ω into 4×4 subdomains, increase the number of nodal points n and compute the number of CG iterations and condition numbers for the above two preconditioners. In Table 2, we fix $n - 1 = 4$ for the cases $N = 8 \times 8$, 16×16, and 32×32, where Ω is divided into subdomains in the same way as in the case $N = 4 \times 4$. We can see from Tables 1 and 2 that both preconditioners seem to give the $\log^2$-growth of the condition number bound and that the number of CG iterations of the preconditioner $\widehat{F}_{DW}^{-1}$ is smaller than that of $\widehat{F}_{DP}^{-1}$.

$n-1$	$\widehat{F}_{DP}^{-1}$		$\widehat{F}_{DW}^{-1}$	
	Iter	Cond	Iter	Cond
4	15	4.52	7	1.94
8	19	6.68	8	2.68
16	21	9.02	10	3.69
32	22	11.6	10	4.80
64	22	14.8	11	6.14

Table 1. Comparison between $\widehat{F}_{DP}^{-1}$ and $\widehat{F}_{DW}^{-1}$ on nonmatching grids when n increases with $N = 4 \times 4$: Iter (number of CG iterations), Cond (condition number of the preconditioned FETI-DP operator).

$N \times N$	$\widehat{F}_{DP}^{-1}$		$\widehat{F}_{DW}^{-1}$	
	Iter	Cond	Iter	Cond
4×4	14	5.36	7	1.94
8×8	16	5.64	8	2.13
16×16	16	5.82	8	2.11
32×32	16	5.96	8	2.10

Table 2. Comparison between $\widehat{F}_{DP}^{-1}$ and $\widehat{F}_{DW}^{-1}$ on nonmatching grids when N increases with $n-1=4$: Iter, Cond.

Next, we consider problem (7.1) for highly discontinuous $\varrho(x,y)$ across subdomain interfaces and noncomparable mesh sizes between neighboring subdomains. The cases of $N = 2 \times 2, 4 \times 4, 8 \times 8$ subdomains are considered. For each subdomain Ω_{ij}, $\varrho(x,y) = \varrho_{ij}$ is chosen as follows:

$$\varrho_{ij} = \begin{cases} 1 & \text{if both } i \text{ and } j \text{ are even,} \\ 250 & \text{if } i \text{ is odd and } j \text{ is even,} \\ 5000 & \text{if } i \text{ is even and } j \text{ is odd,} \\ 10 & \text{if both } i \text{ and } j \text{ are odd.} \end{cases}$$

According to the partitions, the exact solution $u(x,y)$ is chosen as follows:

$$u(x,y) = \begin{cases} p_1(x,y)\sin(\pi x)\sin(\pi y)/\varrho(x,y) & \text{for } N = 2 \times 2, \\ p_2(x,y)\sin(2\pi x)\sin(2\pi y)/\varrho(x,y) & \text{for } N = 4 \times 4, \\ \sin(8\pi x)\sin(8\pi y)/\varrho(x,y) & \text{for } N = 8 \times 8, \end{cases}$$

where

$$p_1(x,y) = (x - \tfrac{1}{2})(y - \tfrac{1}{2}),$$
$$p_2(x,y) = (x - \tfrac{1}{4})(x - \tfrac{3}{4})(y - \tfrac{1}{4})(y - \tfrac{3}{4}).$$

From [22], a different mesh size in each subdomain is chosen according to the ratios of coefficients between neighboring subdomains, i.e.,

$$\frac{h_{ij}}{h_{kl}} \simeq \sqrt[4]{\frac{\varrho_{ij}}{\varrho_{kl}}},$$

where h_{ij} and H_{ij} are the mesh size and size of subdomain Ω_{ij}, respectively. We divide each subdomain into uniform meshes by the mesh sizes of these ratios. For the case of $N = 2 \times 2$ and $H/h = 16$, we obtain the noncomparable triangulations between neighboring subdomains, see Figure 3. In [22], a good approximation of the solution is obtained when the slave side is chosen to give a Lagrange multiplier space of a higher dimension. We can approximate the exact solution more accurately by choosing the subdomain with smaller ϱ_{ij} as the slave side.

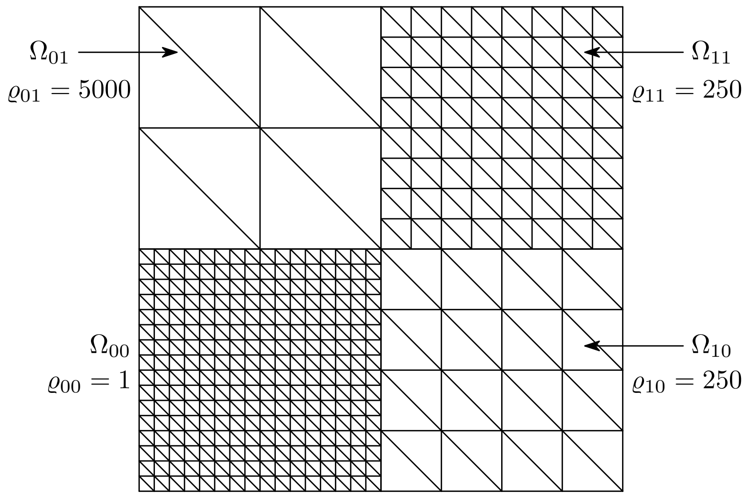

Figure 3. Triangulations for the case of $N = 2 \times 2$ and $H/h = 16$.

Table 3 gives the number of CG iterations for preconditioners $\widehat{F}_{DP}^{-1}$ and $\widehat{F}_{DW}^{-1}$ with the same stopping criterion 10^{-6} as before. By increasing H/h, we can see that the number of CG iterations of $\widehat{F}_{DP}^{-1}$ is much smaller than that of $\widehat{F}_{DW}^{-1}$. Thus, the condition number bound of $\widehat{F}_{DW}^{-1}$ depends on the ratio of meshes between neighboring subdomains and the preconditioner $\widehat{F}_{DW}^{-1}$ is inefficient for problems with noncomparable grids.

From the numerical tests, we conclude that our method gives a correct approximation of the model problem with nonmatching grids. The preconditioner $\widehat{F}_{DW}^{-1}$ gives a smaller number of iterations than our preconditioner $\widehat{F}_{DP}^{-1}$ for the case of continuous coefficients and comparable meshes across subdomain interfaces. However,

N	H/h	$\widehat{F}_{DP}^{-1}$ Iter	$\widehat{F}_{DW}^{-1}$ Iter
2×2	16	5	17
	32	6	26
	64	7	39
	128	8	50
	256	8	60
4×4	16	8	75
	32	7	81
	64	8	111
	128	8	130
8×8	16	8	113
	32	8	136
	64	8	168

Table 3. Comparison between $\widehat{F}_{DP}^{-1}$ and $\widehat{F}_{DW}^{-1}$ for the problem of highly discontinuous coefficients on noncomparable grids: Iter.

our preconditioner turns out to be much more efficient than $\widehat{F}_{DW}^{-1}$ for problems with highly discontinuous coefficients on noncomparable grids.

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