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ON GEOMETRIC CONVERGENCE OF DISCRETE GROUPS

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Abstract. One of the basic questions in the Kleinian group theory is to understand both algebraic and geometric limiting behavior of sequences of discrete subgroups. In this paper we consider the geometric convergence in the setting of the isometric group of the real or complex hyperbolic space. It is known that if Γ is a non-elementary finitely generated group and $\varrho_i\colon \Gamma \to \mathrm{SO}(n,1)$ a sequence of discrete and faithful representations, then the geometric limit of $\varrho_i(\Gamma)$ is a discrete subgroup of $\mathrm{SO}(n,1)$. We generalize this result by showing that for a sequence of discrete and non-elementary subgroups $\{G_j\}$ of $\mathrm{SO}(n,1)$ or $\mathrm{PU}(n,1)$, if $\{G_j\}$ has uniformly bounded torsion, then its geometric limit is either elementary, or discrete and non-elementary.

Keywords: discrete group; geometric convergence; uniformly bounded torsion

MSC 2010: 30C62, 30F40, 20H10

1. Introduction

In this paper, we consider the group G of isometries of the real or complex hyperbolic space H^n. One of the basic questions is to understand limiting behavior of sequences of discrete subgroups of G. Given such a sequence of discrete groups Γ_i, it is known that the analysis of this sequence has both algebraic and geometric aspects. More precisely, one says $\Gamma_i = \langle g_{1_i}, g_{2_i}, \ldots\rangle$ converges algebraically to Γ if $\lim g_{k_i} = g_k$ exists for all k and $\Gamma = \langle g_1, g_2, \ldots\rangle$. On the other hand, Γ_i converges geometrically to H if the following two conditions hold:

(a) for every $h \in H$, $h = \lim h_i$, $h_i \in \Gamma_i$;

(b) if $h_{i_j} \in \Gamma_{i_j}$ are such that $\lim h_{i_j} = h$ exists, then $h \in H$.

For either algebraically or geometrically convergent sequence of discrete groups, the best one can hope is to show that its limit is also discrete. There are many

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discussions in this direction (cf. [1], [4], [6], [7]). For example, if a sequence of finitely generated, discrete and non-elementary subgroups Γ_i has uniformly bounded torsion (i.e., there is a universal upper bound on the orders of all elements in $\{\Gamma_i\}$ of finite order), then the algebraic limit Γ of Γ_i is also discrete and non-elementary. This was proved by Martin for the $\mathrm{SO}(n,1)$ case in [7], and for the isometric group of negatively pinched Hadamard manifolds in [6]. In addition, Martin [7] constructed a counter-example to show the limit group Γ may not be discrete if $\{\Gamma_i\}$ does not have uniformly bounded torsion for $G = \mathrm{SO}(n,1)$ when $n \geqslant 4$.

The geometric convergence for $G = \mathrm{PSL}(2,C)$ was first studied by Jørgensen and Marden in [3], where they proved that if discrete groups Γ_i converge geometrically to H, then H is either a Kleinian group, or is elementary. For the higher dimensional case, the following is known (cf. [5], Proposition 8.9):

Theorem 1.1. *Let Γ be a non-elementary finitely generated group and $\varrho_i\colon \Gamma \to \mathrm{SO}(n,1)$ a sequence of discrete and faithful representations that converges to $\varrho_\infty\colon \Gamma \to \mathrm{SO}(n,1)$. Assume that $\lim^{\mathrm{geo}}_{j\to\infty} \varrho_i(\Gamma) = H$. Then H is a discrete subgroup of $\mathrm{SO}(n,1)$.*

Here a discrete and faithful representation $\varrho\colon \Gamma \to \mathrm{SO}(n,1)$ means ϱ is a monomorphism and the image $\varrho(\Gamma)$ is a discrete group. Hence the sequence $\varrho_i(\Gamma)$ has uniformly bounded torsion since Γ is finitely generated.

The purpose of this paper is to generalize Theorem 1.1 to the case when Γ is not finitely generated. More precisely, we will prove

Theorem 1.2. *Let Γ_j be a sequence of discrete non-elementary subgroups in $G = \mathrm{Isom}(H^n)$ which has uniformly bounded torsion. Assume that $\lim^{\mathrm{geo}}_{j\to\infty}(\Gamma_j) = H$, then H is either discrete and non-elementary, or it is elementary.*

Corollary 1.3. *Let Γ be a non-elementary group and $\varrho_i\colon \Gamma \to G$ a sequence of discrete and faithful representations with uniformly bounded torsion that converges to $\varrho_\infty\colon \Gamma \to G$. Assume that $\lim^{\mathrm{geo}}_{j\to\infty} \varrho_i(\Gamma) = H$, then H is a discrete non-elementary subgroup of G.*

Remark 1.4. In the proof of Theorem 1.2, we essentially use another equivalent characterization of geometric convergence of closed groups in G, that is, Γ_j converges geometrically to H if and only if for every compact subgroup $K \subset G$, the sequence $\Gamma_j \cap K$ converges to $H \cap K$ in the Hausdorff topology (cf. [2], Fact 8.3).

Remark 1.5. Note that Martin's example in [7] shows that the "uniformly bounded torsion" condition in our Theorem 1.2 is necessary, too.

2. Main results

The following is the classification of nontrivial elements in G:

(i) f is elliptic if it has a fixed point in H^n;
(ii) f is parabolic if it has exactly one fixed point in ∂H^n;
(iii) f is loxodromic if it has exactly two fixed points in ∂H^n.

Among the three types only loxodromic elements have the property that if f_i is a sequence converging to a loxodromic element, then f_i is loxodromic for sufficiently large i (cf. [8], Lemma 2C). Thus we have

Lemma 2.1. *The set of all loxodromic elements is open in G.*

We also need the following lemma, which can be found as Corollary 4.5.1 in [3].

Lemma 2.2. *Let Γ be a closed subgroup of G, then one of the following is true:*

(i) *Γ is discrete;*
(ii) *the elements of Γ have a common fixed point, or Γ leaves invariant a proper hyperbolic space;*
(iii) *$\Gamma = G$.*

Denote by $L(\Gamma)$ the limit set of Γ and $L_\Gamma(I)$ the set $\{f \in \Gamma;\ f(x) = x,\ \forall\, x \in L(\Gamma)\}$. Note that each nontrivial element in $L_\Gamma(I)$ is elliptic if Γ is non-elementary. Now we can prove

Lemma 2.3. *A non-elementary subgroup Γ of G is discrete if and only if the following two conditions are satisfied:*

(i) *$L_\Gamma(I)$ is finite;*
(ii) *each non-elementary subgroup of Γ generated by two loxodromic elements is discrete.*

P r o o f. The necessary part is obvious since $L_\Gamma(I)$ is a compact group.

Conversely, suppose that $L_\Gamma(I)$ is finite and each non-elementary subgroup of Γ generated by two loxodromic elements is discrete.

Since Γ is non-elementary, there exists a minimal totally geodesic submanifold M containing $L(\Gamma)$ whose dimension $\dim(M) = k$ is at least two. Obviously, M is Γ-invariant. Denote by $\Gamma|_M$ and $f|_M$ the restriction to M of Γ and $f \in \Gamma$, respectively. It is easy to see $f|_M$ is loxodromic if and only if f is loxodromic.

We claim that $\Gamma|_M$ is discrete; otherwise, there exists a sequence $\{f_i|_M\}$ of distinct loxodromic elements converging to the identity by Lemmas 2.1 and 2.2. Since the fixed point set $\{a_i, b_i\}$ of the loxodromic element $\{f_i|_M\}$ lies in a compact set, we may assume that $a_i \to a$ and $b_i \to b$. Notice that since Γ is non-elementary, one can select a loxodromic element $f \in \Gamma$ whose fixed point set is disjoint from $\{a, b\}$ and thus disjoint from $\{a_i, b_i\}$ for large i. Therefore, $\langle f, f_i \rangle$ is non-elementary and thus discrete by the assumption. Now we want to show that $\langle f|_M, f_i|_M \rangle$ is also discrete. Suppose not, then there is a sequence $\{g_j|_M\}$ in $\langle f|_M, f_i|_M \rangle$ converging to the identity. Note that g_j is in a discrete convergence group $\langle f, f_i \rangle$; see [8] for the definition of convergence groups. Thus there exist two points $a, b \in \partial H^n$ and a subsequence g_{j_k} such that

$$g_{j_k}|_{\bar{H}^n \setminus \{b\}} \to a$$

uniformly on compact subsets of $\bar{H}^n \setminus \{b\}$. This contradicts that $g_j|_M \to \mathrm{id}$.

On the other hand, $\langle f|_M, f_i|_M \rangle$ is also non-elementary because $\langle f, f_i \rangle$ is non-elementary. However, by Jørgensen's inequality $\langle f|_M, f_i|_M \rangle$ cannot be both discrete and non-elementary since $f_i|_M$ converges to the identity. This completes the proof that $\Gamma|_M$ is discrete.

Finally, suppose that Γ is not discrete, that is, there are distinct $g_i \in \Gamma$ such that $g_i \to \mathrm{id}$ as $i \to \infty$. Obviously, $g_i|_M$ also converges to the identity map. Since $\Gamma|_M$ is discrete, $g_i|_M = \mathrm{id}$ for any large i. This contradicts the assumption that $L_\Gamma(I)$ is finite. □

Theorem 2.4. *Let Γ_j be a sequence of discrete non-elementary subgroups in G which has uniformly bounded torsion. Assume that $\lim^{\mathrm{geo}}_{j\to\infty}(\Gamma_j) = H$, then H is either discrete and non-elementary, or it is elementary.*

Proof. Assume that H is non-elementary.

First we will show that $L_H(I)$ is finite. Otherwise, $L_H(I)$ is non-discrete since it is compact.

Let $\varepsilon > 0$ be arbitrary and U the closed neighborhood of $L_H(I)$ with radius 3ε. It is obvious that U is also compact.

Since $L_H(I)$ is not discrete, there is a nontrivial $g \in L_H(I)$ with $\|g - \mathrm{id}\| < \varepsilon$. We want to show there is a sequence of distinct elements $g_j \in \Gamma_j$ converging to g which eventually have finite order. Suppose not for the contradiction. We may assume that $g_j \to g$ such that each g_j is loxodromic or parabolic (because Γ_j is discrete), and $\|g_j - g\| < \varepsilon$ for each g.

Fix such a g_j. Obviously it is in the compact set U. Then there exists a minimal positive integer k such that

$$g_j^{k-1} \in U \quad \text{while } g_j^k \notin U.$$

This easily implies that

$$g_j^{k-1} \in \Gamma_j \cap U \quad \text{and} \quad g^{k-1} \in H \cap U.$$

Note that Γ_j converges geometrically to H. Therefore, for any compact subgroup $K \subset G$ and for any $\varepsilon > 0$, there exists an integer N, such that $\Gamma_j \cap K$ lies in the ε-neighborhood of $H \cap K$, and $H \cap K$ lies in the ε-neighborhood of $\Gamma_j \cap K$ for $j > N$. Hence for any large j, the Hausdorff distance between Γ_j and H on the above U is less than ε. This implies that $\|g_j^{k-1} - g^{k-1}\| < \varepsilon$. Then we have

$$\begin{aligned}\|g_j^k - g^{k-1}\| &\leqslant \|g_j^k - g_j^{k-1}\| + \|g_j^{k-1} - g^{k-1}\| \\ &= \|g_j - \mathrm{id}\| + \|g_j^{k-1} - g^{k-1}\| \\ &\leqslant \|g_j - g\| + \|g - \mathrm{id}\| + \|g_j^{k-1} - g^{k-1}\| \\ &< 3\varepsilon.\end{aligned}$$

This together with $g^{k-1} \in L_H(I)$ implies that $g_j^k \in U$, which is a contradiction.

As Γ_j has uniformly bounded torsion, say N, we see that $g_j^{N!} = \mathrm{id}$ and then $g^{N!} = \mathrm{id}$. By letting $\varepsilon \to 0$, we get a sequence of nontrivial elements in $L_H(I)$ converging to the identity whose orders are less than $N!$. This is impossible because if $\{h_i\}$ is a sequence of nontrivial elements converging to the identity, then the orders of $\{h_i\}$ converge to infinity.

Finally, let $\langle f, g\rangle$ be a non-elementary subgroup of H where both f and g are loxodromic, then f and g have disjoint fixed point sets. Since H is the geometric limit of Γ_j, there are f_j and g_j in Γ_j which converge to f and g, respectively. By Lemma 2.1 we see that both f_j and g_j are loxodromic for large j, which means that the fixed points of f_j and g_j converge to those of f and g, respectively. Hence f_j and g_j also have disjoint fixed point sets, namely, $\langle f_j, g_j\rangle$ is non-elementary. Notice that the discrete groups $\langle f_j, g_j\rangle$ converge algebraically to $\langle f, g\rangle$. Then $\langle f, g\rangle$ is discrete by [6], Theorem 4.2. Hence H is discrete by Lemma 2.3. □

Corollary 2.5. *Let Γ be a non-elementary group and $\varrho_i\colon \Gamma \to G$ a sequence of discrete and faithful representations with uniformly bounded torsion that converges to $\varrho_\infty\colon \Gamma \to G$. Assume that $\lim^{\mathrm{geo}}_{j\to\infty} \varrho_i(\Gamma) = H$, then H is a discrete non-elementary subgroup of G.*

P r o o f. Since Γ is non-elementary, there is a Schottky subgroup $\langle f, g\rangle$ in Γ. Note that a discrete subgroup of G is elementary if and only if it contains no free non-abelian subgroups. Then $\langle \varrho_i(f), \varrho_i(g)\rangle$ is discrete and non-elementary for each i because ϱ_i is a discrete and faithful representation. From [6], Theorem 4.7, it follows that $\langle \varrho_\infty(f), \varrho_\infty(g)\rangle$, the algebraic limit of $\langle \varrho_i(f), \varrho_i(g)\rangle$, is non-elementary. Recall that the algebraic limit $\varrho_\infty(\Gamma)$ is a subgroup of the corresponding geometric limit H. Hence H is also non-elementary and then discrete by Theorem 2.4. □

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