Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 3, 629–640

Persistent URL: http://dml.cz/dmlcz/144049

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Czechoslovak Mathematical Journal, 64 (139) (2014), 629–640

POINT-DISTINGUISHING CHROMATIC INDEX OF THE UNION OF PATHS

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(Received February 6, 2013)

Abstract. Let G be a simple graph. For a general edge coloring of a graph G (i.e., not necessarily a proper edge coloring) and a vertex v of G, denote by $S(v)$ the set (not a multiset) of colors used to color the edges incident to v. For a general edge coloring f of a graph G, if $S(u) \neq S(v)$ for any two different vertices u and v of G, then we say that f is a point-distinguishing general edge coloring of G. The minimum number of colors required for a point-distinguishing general edge coloring of G, denoted by $\chi_0(G)$, is called the point-distinguishing chromatic index of G. In this paper, we determine the point-distinguishing chromatic index of the union of paths and propose a conjecture.

Keywords: general edge coloring; point-distinguishing general edge coloring; point-distinguishing chromatic index

MSC 2010: 05C15

1. Introduction

We consider only finite, undirected graphs. We also consider only simple graphs except for the graph K_k^0 which is constructed from the complete graph K_k with $V(K_k) = \{1, 2, \ldots, k\}$ by adding exactly one loop at each vertex, G_2 in the definition of a packing of G_1 into G_2, and H, H^c described in the proof of Theorem 4.1. These four exceptional graphs may have loops (each vertex is incident to at most one loop) but no multiple edges. A non-loop edge of a graph G is called a link of G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For an edge coloring (proper or not necessarily proper) of a graph G and a vertex v of G, denote by $S(v)$ the set (not the multiset) of colors used to color the edges incident to v. The set $S(v)$ is called the color set of vertex v under the given edge coloring.

The research has been supported by the National Natural Science Foundation of China (Grant No. 61163037, 61163054)

A proper edge coloring of a graph G is said to be vertex-distinguishing if distinct vertices have distinct color sets. In other words, $S(u) \neq S(v)$ whenever $u \neq v$. A graph G has a vertex-distinguishing proper edge coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a vdec-graph. The minimum number of colors required for a vertex-distinguishing proper edge coloring of a vdec-graph G will be denoted $\chi'_s(G)$. The concept of vertex-distinguishing proper edge colorings has been considered in several papers [1]–[6], [9], [11]. For the vertex-distinguishing proper edge coloring, Burris and Schelp had proposed a conjecture (VDPEC Conjecture) in [5] as follows.

Conjecture 1.1 (VDPEC Conjecture). *If G is a* vdec-*graph and $\pi(G)$ is the minimum integer j such that $\binom{j}{i} \geqslant n_i$ with $\delta(G) \leqslant i \leqslant \Delta(G)$, then $\pi(G) \leqslant \chi'_s(G) \leqslant \pi(G) + 1$.*

A general edge coloring (not a necessarily a proper edge coloring) of a graph G is said to be point-distinguishing (or vertex-distinguishing) if $S(u) \neq S(v)$ for any two distinct vertices u, v. The point-distinguishing chromatic index of a vdec-graph G, denoted by $\chi_0(G)$, is the minimum number of colors required for a point-distinguishing general edge coloring of G. This parameter was introduced by Harary and Plantholt in [7]. Obviously we have $\chi_0(G) \leqslant \chi'_s(G)$ for any vdec-graph G. In spite of the fact that the structure of complete bipartite graphs is simple, it seems that the problem of determining $\chi_0(K_{m,n})$ is not easy, especially in the case $m = n$, as documented by Horňák and Soták [12], [10], Salvi [13], [14] and Horňák and Salvi [8].

As usual, we write K_n for a complete graph of order n. Write P_n for a path of length $n - 1$ (on n vertices) and $P(v_1, v_2, \ldots, v_r)$ for the trail with edges $v_i v_{i+1}$, $i = 1, 2, \ldots, r - 1$ (of length $r - 1$). We do not require v_i to be distinct. For any two graphs G_1 and G_2, write $G_1 \cup G_2$ for the vertex disjoint union of G_1 and G_2.

If G_1 and G_2 are graphs, a packing of G_1 into G_2 is a map $f\colon V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map $\operatorname{ind}(f)$ on edges defined by $xy \mapsto f(x)f(y)$ is an injection from $E(G_1)$ to $E(G_2)$. We do not require f to be injective on vertices, so if G_1 contains a cycle or path, its image in G_2 will be a circuit (closed trail) or a nonclosed trail.

In [7], Harary and Plantholt studied the point-distinguishing chromatic index for paths, cycles, complete graphs, cubes and complete bipartite graphs. In this paper we will propose a conjecture on the point-distinguishing chromatic index and determine the point-distinguishing chromatic index of a union of paths.

In Section 2 we will list several lemmas which are useful in the proofs of main results. In Section 3 we define a parameter and give some related results. In Section 4 we determine the point-distinguishing chromatic index of the union of some paths. In Section 5 we propose a conjecture. In Section 6 we give a remark.

We write Q_n for an n-cube (with 2^n vertices) and C_n for a cycle with n vertices. Notation and terminology that are not defined here may be found in [2], [7].

2. Preliminaries

The following six results (Lemma 2.1 to 2.6) were proved in [7].

Lemma 2.1. $\chi_0(K_n) = \lceil \log_2 n \rceil + 1$.

Lemma 2.2. *If $n \geqslant 2$, then $\lceil \log_2(2n+1) \rceil \leqslant \chi_0(K_{n,n}) \leqslant \lceil \log_2 n \rceil + 2$.*

Lemma 2.3. *If $m < n$, $n \geqslant 4$, $m \geqslant \lceil \log_2 n \rceil + 1$, then $\lceil \log_2(m+n+1) \rceil \leqslant \chi_0(K_{m,n}) \leqslant \lceil \log_2 n \rceil + 2$.*

Lemma 2.4. *If $n \geqslant 2$, then $\chi_0(Q_n) = n + 1$.*

Lemma 2.5. *If $n \geqslant 3$, then*

$$\chi_0(P_n) = \min\Big\{2\big\lceil \tfrac{1}{4}(1+\sqrt{8n-9})\big\rceil - 1, 2\Big\lceil \sqrt{\tfrac{1}{2}(n-1)}\,\Big\rceil\Big\}.$$

Lemma 2.6. *If $n \geqslant 3$, then*

$$\chi_0(C_n) = \min\Big\{2\big\lceil \tfrac{1}{4}(1+\sqrt{8n+1})\big\rceil - 1, 2\Big\lceil \sqrt{\tfrac{1}{2}n}\,\Big\rceil\Big\}.$$

Lemma 2.7. (i) *If k is an odd number, $3 \leqslant l \leqslant \frac{1}{2}k(k+1) - 1$, then there exists a length l nonclosed trail in K_k^0 such that its initial and final edges are loops.*

(ii) *If k is an even number, $3 \leqslant l \leqslant \frac{1}{2}k(k+1) - \frac{1}{2}k + 1$, then there exists a length l nonclosed trail in K_k^0 such that its initial and final edges are loops.*

Proof. If $k = 2$, then $l = 3$; if $k = 3$, then $l = 3, 4, 5$; if $k = 4$, then $l = 3, 4, 5, 6, 7, 8, 9$. In these cases the results are trivial. Suppose $k \geqslant 5$.

(i) k is an odd number. K_k^0 has an Euler closed trail with length $\binom{k+1}{2}$. So there exists a nonclosed trail in K_k^0 such that its initial and final edges are loops and whose length is $\binom{k+1}{2} - 1$. (Such a trail can be easily constructed from an Eulerian trail in $K_k^0 - e$, where e is an edge of K_k^0.) By deleting some loops which are not initial and final loops from this nonclosed trail, we can obtain nonclosed trails in K_k^0 such that their initial and final edges are loops and their lengths are $\binom{k+1}{2}-2, \binom{k+1}{2}-3, \ldots, \binom{k+1}{2}-k+1$. Let T be a nonclosed trail obtained by deleting the initial and final edges (loops) from the length $\binom{k+1}{2}-k+1$ nonclosed trail we have obtained. Note that T has no loop and T has $\binom{k}{2}-1$ edges.

Suppose $3 \leqslant l \leqslant \frac{1}{2}k(k+1)-k$. Consider the subtrail T' formed from the first $l-2$ edges of T. If T' is not closed, then we can obtain a length l nonclosed trail in K_k^0 such that the initial and final edges are loops by adding two loops at the initial and final vertices from T'. If T' is closed, then we consider the subtrail T'' obtained from the second edge to the $(l-1)$-th edge. The subtrail T'' is not closed. We can obtain a length l nonclosed trail in K_k^0 such that its initial and final edges are loops by adding two loops at the initial and final vertices from T''.

(ii) k is an even number. K_k^0 has a nonclosed trail with length $\binom{k+1}{2}-\frac{1}{2}k+1$ and k loops. By deleting some loops which are not initial and final loops from this length $\binom{k+1}{2}-\frac{1}{2}k+1$ nonclosed trail, we can obtain nonclosed trails in K_k^0 such that their initial and final edges are loops and their lengths are $\binom{k+1}{2}-\frac{1}{2}k, \binom{k+1}{2}-\frac{1}{2}k-1, \ldots, \binom{k+1}{2}-\frac{1}{2}k-k+3$. Let T be a nonclosed trail obtained by deleting the initial and final edges (loops) from the length $\binom{k+1}{2}-\frac{1}{2}k-k+3$ nonclosed trail we have obtained. Note that T has no loop and T has $\binom{k}{2}-\frac{1}{2}k+1$ edges.

If $3 \leqslant l \leqslant \binom{k+1}{2}-\frac{1}{2}k-k+2$, we proceed similarly to the case (i) with $3 \leqslant l \leqslant \binom{k+1}{2}-k$.

The proof of Lemma 2.7 is completed. $\square$

Lemma 2.8. *Suppose $k, p, s_1, s_2, \ldots, s_p$ are integers with $k \geqslant 2$, $p \geqslant 1$ and $s_i \geqslant 3$, $i = 1, 2, \ldots, p$. The following two statements are equivalent:*

(i) *The graph $P_{s_1} \cup P_{s_2} \cup \ldots \cup P_{s_p}$ has a point-distinguishing general edge coloring that uses k colors.*

(ii) *There exists a packing g of the graph $P = P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into K_k^0 such that $\operatorname{ind}(g)$ maps all pendant edges of P to loops of K_k^0.*

Proof. Let $G = P_{s_1} \cup P_{s_2} \cup \ldots \cup P_{s_p}$ and suppose that the consecutive vertices of P_{s_i} are $u_1^{(i)}, u_2^{(i)}, \ldots, u_{s_i}^{(i)}$, while those of P_{s_i+1} are $v_1^{(i)}, v_2^{(i)}, \ldots, v_{s_i+1}^{(i)}$, $i = 1, 2, \ldots, p$.

(i) $\Rightarrow$ (ii): Consider a point-distinguishing general edge coloring $f\colon E(G) \to \{1, 2, \ldots, k\}$. The map $g\colon V(P) \to \{1, 2, \ldots, k\} = V(K_k^0)$ defined, for $i = 1, 2, \ldots, p$, by

$$g(v_j^{(i)}) = f(u_{j-1}^{(i)} u_j^{(i)}), \quad j = 2, 3, \ldots, s_i, \quad g(v_1^{(i)}) = g(v_2^{(i)}), g(v_{s_i+1}^{(i)}) = g(v_{s_i}^{(i)}),$$

is a required packing of P into K_k^0. Indeed, the map $\operatorname{ind}(g)$ assigns to the edge $v_j^{(i)} v_{j+1}^{(i)}$ the color set of the vertex $u_j^{(i)}$ under f, it is therefore injective and maps any pendant edge of P to a loop of K_k^0.

(ii) $\Rightarrow$ (i): Suppose $g\colon V(P) \to V(K_k^0)$ is such a packing of P into K_k^0 that $\mathrm{ind}(g)$ maps all pendant edges of P to loops of K_k^0. Note that $g(v_1^{(i)}) = g(v_2^{(i)})$, $g(v_{s_i}^{(i)}) = g(v_{s_i+1}^{(i)})$, $i = 1, 2, \ldots, p$. The general edge coloring $f\colon E(G) \to \{1, 2, \ldots, k\}$ that is defined, for $i = 1, 2, \ldots, p$, by

$$f(u_{j-1}^{(i)} u_j^{(i)}) = g(v_j^{(i)}), \quad j = 2, 3, \ldots, s_i,$$

is point distinguishing, since the color set of the vertex $u_j^{(i)}$ under f is equal to the image of the edge $v_j^{(i)} v_{j+1}^{(i)}$ under the map $\mathrm{ind}(g)$. $\square$

The following lemma is obvious.

Lemma 2.9. *For any real number a, we have $\lceil a \rceil \leqslant 2\lceil \frac{1}{2}(a+1) \rceil - 1 \leqslant \lceil a \rceil + 1$.*

3. A parameter and some related results

Suppose G is a vdec-graph. Let $n_d = n_d(G)$ denote the number of vertices of degree d in G, $\delta \leqslant d \leqslant \Delta$, where δ, Δ are the minimum and maximum degree of G, respectively. When G has one isolated vertex, let

$$\varrho(G) = \min \left\{ \theta;\ \delta \leqslant s \leqslant \Delta \Rightarrow \sum_{i=0}^{s} \binom{\theta}{i} \geqslant \sum_{i=\delta}^{s} n_i(G) \right\}.$$

Otherwise, when G has no isolated vertex we let

$$\varrho(G) = \min \left\{ \theta;\ \delta \leqslant s \leqslant \Delta \Rightarrow \sum_{i=1}^{s} \binom{\theta}{i} \geqslant \sum_{i=\delta}^{s} n_i(G) \right\}.$$

Proposition 3.1. *For a* vdec-*graph G, we have $\chi_0(G) \geqslant \varrho(G)$.*

P r o o f. Suppose f is a point-distinguishing general edge coloring, of G, using l colors $1, 2, \ldots, l$.

(i) If G has one isolated vertex, then for any s with $\delta \leqslant s \leqslant \Delta$, the color set of each one of the vertices of degree $\delta(=0), \delta+1, \ldots$, or s is an 0-set, 1-set,…, or an s-set of $\{1, 2, \ldots, l\}$. The number of 0-sets, 1-sets,…, and s-sets of $\{1, 2, \ldots, l\}$ is $\binom{l}{0} + \binom{l}{1} + \ldots + \binom{l}{s}$. Since different vertices have different color sets, we have $\binom{l}{0} + \binom{l}{1} + \ldots + \binom{l}{s} \geqslant n_\delta(G) + n_{\delta+1}(G) + \ldots + n_s(G)$. Note that $\varrho(G)$ is the minimum integer θ such that $\binom{\theta}{0} + \binom{\theta}{1} + \ldots + \binom{\theta}{s} \geqslant n_\delta(G) + n_{\delta+1}(G) + \ldots + n_s(G)$ is valid for any s with $\delta \leqslant s \leqslant \Delta$. So $l \geqslant \varrho(G)$ and $\chi_0(G) \geqslant \varrho(G)$.

(ii) When G has no isolated vertex, we can prove that $\chi_0(G) \geqslant \varrho(G)$ similarly. $\square$

From Lemmas 2.1, 2.2, 2.3, 2.4 we may obtain easily the following Propositions 3.2, 3.3, 3.4, 3.5.

Proposition 3.2. *Suppose k, n are integers, $n \geqslant 3$. If $2^{k-1} < n < 2^k$, then $\chi_0(K_n) = k+1 = \varrho(G)+1$; if $n = 2^k$, then $\chi_0(K_n) = k+1 = \varrho(G)$.*

Proposition 3.3. *Suppose k, n are integers, $n \geqslant 2$. If $2^{k-1} < n < 2^k$, then $k+1 \leqslant \chi_0(K_{n,n}) \leqslant k+2$, $\varrho(K_{n,n}) = k+1$, and $\varrho(K_{n,n}) \leqslant \chi_0(K_{n,n}) \leqslant \varrho(K_{n,n})+1$; if $n = 2^k$, then $\chi_0(K_{n,n}) = k+2 = \varrho(K_{n,n})$.*

Proposition 3.4. *Suppose $m < n$, $n \geqslant 4$, $m \geqslant \lceil \log_2 n \rceil + 1$. If $2^{k-1} < n < 2^k$ and $m+n \geqslant 2^k$ or $n = 2^k$, then $\varrho(K_{m,n}) = k+1$, and $\varrho(K_{m,n}) \leqslant \chi_0(K_{m,n}) \leqslant \varrho(K_{m,n})+1$.*

Proposition 3.5. *If $n \geqslant 2$, then $\chi_0(Q_n) = n+1 = \varrho(Q_n)$.*

Proposition 3.6. *If $n \geqslant 3$, then $\varrho(P_n) \leqslant \chi_0(P_n) \leqslant \varrho(P_n)+1$.*

Proof. Let $f(n) = \frac{1}{2}(-1+\sqrt{8n-9})$, $g(n) = \frac{1}{2}(-1+\sqrt{8n+1})$; clearly, $f(n) < g(n)$, and so $\lceil f(n) \rceil \leqslant \lceil g(n) \rceil$. Note that $\varrho(P_n)$ is the minimum integer r such that $\binom{r}{1} + \binom{r}{2} \geqslant n$, hence $\varrho(P_n) = \lceil g(n) \rceil$. By Lemma 2.5 we have $\chi_0(P_n) \leqslant f_1(n) = 2\lceil \frac{1}{4}(1+\sqrt{8n-9}) \rceil - 1 = 2\lceil \frac{1}{2}(f(n)+1) \rceil - 1$, where $f_1(n)$ is the smallest odd integer bounded from below by $\lceil f(n) \rceil$, which necessarily satisfies $f_1(n) \leqslant \lceil f(n) \rceil + 1$ (see Lemma 2.9). Then from $\lceil f(n) \rceil \leqslant \lceil g(n) \rceil = \varrho(P_n)$ it follows that $f_1(n) \leqslant \lceil f(n) \rceil + 1 \leqslant \varrho(P_n)+1$ and $\varrho(P_n) \leqslant \chi_0(P_n) \leqslant \varrho(P_n)+1$. □

Note that Proposition 3.6 will also be obtained from Theorem 4.1 (see Corollary 4.5).

Proposition 3.7. *If $n \geqslant 3$, then $\varrho(C_n) \leqslant \chi_0(C_n) \leqslant \varrho(C_n)+1$.*

Proof. We adopt the notation used in the proof of Proposition 3.6. First note that $\varrho(C_n) = \varrho(P_n) = \lceil g(n) \rceil$. Lemma 2.6 yields $\chi_0(C_n) \leqslant g_1(n) = 2\lceil \frac{1}{4}(1+\sqrt{8n+1}) \rceil - 1 = 2\lceil \frac{1}{2}(g(n)+1) \rceil - 1$, where $g_1(n)$ is the smallest odd integer bounded from below by $\lceil g(n) \rceil$, which evidently satisfies $g_1(n) \leqslant \lceil g(n) \rceil + 1 = \varrho(C_n)+1$ (see Lemma 2.9). Thus we obtain $\varrho(C_n) \leqslant \chi_0(C_n) \leqslant \varrho(C_n)+1$. □

4. Point-distinguishing chromatic index of the union of paths

In this section we will prove that $\chi_0(G) = \varrho(G)$ or $\varrho(G)+1$ in the case when G is a vertex-disjoint union of p paths $P_{s_1} \cup P_{s_2} \cup \ldots \cup P_{s_p}$, where $s_i \geqslant 3$.

We solve this problem by considering first the packing from the graph $P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into K_k^0 with $2p$ pendant edges of the paths mapped to loops in K_k^0. Note in particular that we must have $k \geqslant 2p$. Write $L = \sum\limits_{i=1}^{p} s_i$ and note that $n_1(G) = 2p$ and $n_1(G) + n_2(G) = L$.

Theorem 4.1. *Suppose $k = 2p + r$, and $L = \sum_{i=1}^{p} s_i$. The following conditions are both necessary and sufficient for the packing $P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into K_k^0 with the $2p$ pendant edges of the paths mapped to loops in K_k^0.*

$L = \binom{k+1}{2}$ or $L \leqslant \binom{k+1}{2} - 3$ if $r = 0$;
$L \leqslant \binom{k+1}{2} - \frac{1}{2}r$ if $r > 0$ and r is even;
$L \leqslant \binom{k+1}{2} - p$ if r is odd.

Proof. Let $P = P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$.

1. Necessity. Consider the image H of P under a given packing of P into K_k^0. The degrees of $2p$ vertices in H must be odd and the remaining r vertices must have an even degree. Now consider the edge complement H^c of H in K_k^0, i.e., $H^c = K_k^0 - E(H)$. If r (and hence k) is odd, H^c will have $2p$ odd degree vertices which are not incident to loops in H^c. Hence H^c will have at least p edges (links). If r is a positive even integer then H^c will have r odd degree vertices and at least $\frac{1}{2}r$ edges (links). If $r = 0$ then H^c has no loops and every vertex of H^c has even degree. So H^c has either no edges or at least three edges.

2. Sufficiency. Order the paths P_{s_i+1} so that $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_p \geqslant 3$. Since K_k^0 contains a set of $\lfloor \frac{1}{2}k \rfloor \geqslant p$ independent edges (not loops), we are done in the case when all s_i are equal to 3, so we may assume $s_1 \geqslant 4$. We proceed by induction on p.

(1) Consider the case $p = 1$. From $\binom{k+1}{2} \geqslant s_1 \geqslant 4$ it follows that $2 + r = k \geqslant 3$, hence $r > 0$ and a required packing of P_{s_1+1} into K_k^0 exists by Lemma 2.7.

(2) Now assume that $p \geqslant 2$, $s_1 \geqslant 4$ and the conditions are sufficient for any union of $p - 1$ paths. From $\binom{k+1}{2} \geqslant s_1 + s_2 \geqslant 7$ it follows that $k \geqslant 4$. Let $\lambda = s_1 + s_2 - 4$, so that $\lambda \geqslant s_2$. We shall try to pack paths of lengths $\lambda, s_3, \ldots, s_p$ into K_{k-2}^0 by induction. This may fail due to the fact that the total length is too large, so we will reduce the lengths. If $\lambda \geqslant 6$ and k is even, first reduce λ by three. Now reduce each λ or s_i, $i \geqslant 3$, by multiples of four until we have removed the total length of $2k - 5$ (k even), or $2k - 6$ (k odd) or until we have reduced all the lengths to at most 6 and at least 3 (if $\lambda < 6$, then $s_i \leqslant 4$ for all $i \geqslant 3$). Call these reduced lengths $\lambda', s_3', \ldots, s_p'$ and pack trails of these lengths into K_{k-2}^0. We will show that this will always succeed.

Let $k' = k - 2$, $p' = p - 1$, $r' = r$, $L' = L - (2k - 1)$ when k is even and $L' = L - (2k - 2)$ when k is odd.

▷ We have removed the total length of $2k - 5$ (k even), or $2k - 6$ (k odd).

If k is even and $r' = 0$, then $L' = L - (2k - 1) = \binom{k+1}{2} - (2k - 1) = \binom{k-1}{2}$ or $L' = L - (2k - 1) \leqslant \binom{k+1}{2} - 3 - (2k - 1) = \binom{k-1}{2} - 3$; if k is even and $r' > 0$, then $L' = L - (2k - 1) \leqslant \binom{k+1}{2} - \frac{1}{2}r - (2k - 1) = \binom{k-1}{2} - \frac{1}{2}r'$; if k is odd and $r' > 0$, then $L' = L - (2k-2) \leqslant \binom{k+1}{2} - p - (2k-2) = \binom{k-1}{2} - p'$. Thus we may apply the induction hypothesis.

$\triangleright$ We have reduced all the lengths to at most 6 and at least 3.

If k' is even and $r' = 0$, then $L' \leqslant 6(p-1) = 3(k-2) \leqslant \binom{k-1}{2} - 3$ when $k \geqslant 8$; if k' is even and $r' > 0$, then $L' \leqslant 6(p-1) = 3(k-r-2) \leqslant \binom{k-1}{2} - \frac{1}{2}r$ when $k \geqslant 6$; if k' is odd and $r' > 0$, then $L' \leqslant 6(p-1) = 3(k-r-2) \leqslant \binom{k-1}{2} - (p-1)$ when $k \geqslant 7$ or $r \geqslant 3$. We may apply the induction hypothesis in these cases.

What is left to be analyzed are three cases. In each of them we present directly packings of those unions of paths for which we are unable to use the induction hypothesis. All unions not mentioned are "covered" by the induction.

Case 1. $k = 6$, $p = 3$, $r = 0$, $k' = 4$, $p' = 2$, $r' = 0$.

There is a required packing of the graph $P_7 \cup P_7 \cup P_7$ into K_6^0, under which images of consecutive vertices of the three components P_7 form three sequences $(1,1,6,4,3,2,2)$, $(3,3,5,6,2,4,4)$ and $(5,5,2,1,3,6,6)$. This fact will be for simplicity coded by

$$P_7 \cup P_7 \cup P_7 \to (1,1,6,4,3,2,2), (3,3,5,6,2,4,4), (5,5,2,1,3,6,6).$$

The remaining unions of three paths can be packed into K_6^0 as follows:

$P_8 \cup P_7 \cup P_6 \to (1,1,6,3,5,6,2,2), (3,3,1,2,5,4,4), (5,5,1,4,6,6)$,
$P_9 \cup P_6 \cup P_6 \to (1,1,2,6,5,3,4,2,2), (3,3,2,5,4,4), (5,5,1,4,6,6)$,
$P_7 \cup P_6 \cup P_6 \to (1,1,6,3,4,2,2), (3,3,2,5,4,4), (5,5,1,4,6,6)$,
$P_9 \cup P_5 \cup P_5 \to (1,1,5,4,6,5,3,2,2), (3,3,1,4,4), (5,5,2,6,6)$,
$P_8 \cup P_6 \cup P_5 \to (1,1,6,5,3,4,2,2), (3,3,1,5,4,4), (5,5,2,6,6)$,
$P_7 \cup P_7 \cup P_5 \to (1,1,6,3,4,2,2), (3,3,1,5,6,4,4), (5,5,2,6,6)$,
$P_8 \cup P_7 \cup P_4 \to (1,1,5,2,4,6,2,2), (3,3,1,6,3,4,4), (5,5,6,6)$,
$P_9 \cup P_6 \cup P_4 \to (1,1,6,2,5,4,3,2,2), (3,3,1,2,4,4), (5,5,6,6)$,
$P_{10} \cup P_5 \cup P_4 \to (1,1,6,2,5,1,4,3,2,2), (3,3,5,4,4), (5,5,6,6)$,
$P_{11} \cup P_4 \cup P_4 \to (1,1,6,2,3,1,2,5,4,2,2), (3,3,4,4), (5,5,6,6)$,
$P_6 \cup P_6 \cup P_6 \to (1,1,3,6,2,2), (3,3,2,5,4,4), (5,5,1,4,6,6)$,
$P_8 \cup P_5 \cup P_5 \to (1,1,5,6,4,3,2,2), (3,3,1,4,4), (5,5,2,6,6)$,
$P_7 \cup P_6 \cup P_5 \to (1,1,6,5,3,2,2), (3,3,1,5,4,4), (5,5,2,6,6)$,
$P_7 \cup P_7 \cup P_4 \to (1,1,6,4,5,2,2), (3,3,4,1,2,4,4), (5,5,6,6)$,
$P_8 \cup P_6 \cup P_4 \to (1,1,6,3,4,5,2,2), (3,3,2,1,4,4), (5,5,6,6)$,
$P_9 \cup P_5 \cup P_4 \to (1,1,6,2,5,1,4,2,2), (3,3,5,2,4,4), (5,5,6,6)$,
$P_{10} \cup P_4 \cup P_4 \to (1,1,5,2,4,5,3,1,2,2), (3,3,4,4), (5,5,6,6)$,
$P_6 \cup P_5 \cup P_5 \to (1,1,6,4,2,2), (3,3,1,4,4), (5,5,2,6,6)$,
$P_5 \cup P_5 \cup P_5 \to (1,1,3,2,2), (3,3,6,4,4), (5,5,2,6,6)$,
$P_7 \cup P_4 \cup P_4 \to (1,1,6,4,5,2,2), (3,3,4,4), (5,5,6,6)$,
$P_6 \cup P_5 \cup P_4 \to (1,1,6,4,2,2), (3,3,1,4,4), (5,5,6,6)$,

$P_6 \cup P_4 \cup P_4 \to (1,1,6,4,2,2), (3,3,4,4), (5,5,6,6),$

$P_5 \cup P_5 \cup P_4 \to (1,1,6,2,2), (3,3,1,4,4), (5,5,6,6).$

Case 2. $k = 5$, $p = 2$, $r = 1$, $k' = 3$, $p' = 1$, $r' = 1$.

$P_6 \cup P_6 \to (1,1,3,5,2,2), (3,3,2,1,4,4),$

$P_7 \cup P_5 \to (1,1,5,4,3,2,2), P(3,3,1,4,4),$

$P_7 \cup P_5 \to (1,1,2,4,5,3,2,2), P(3,3,4,4).$

Case 3. $k = 4$, $p = 2$, $r = 0$, $k' = 2$, $p' = 1$, $r' = 0$.

Now $s_1 + s_2 \in \{7, 10\}$ and $\lambda' = 3$, hence we can apply the induction hypothesis.

We now add back the two remaining vertices a and b of K_k^0 and construct trails of the required original lengths. Let the trail T of length λ' go from the vertex u to the vertex v in K_{k-2}^0. Let u' be any vertex on this trail which is a distance $t_1 \leqslant s_1 - 2$ (along T) from u, a distance $t_2 \leqslant s_2 - 2$ (along T) from v and such that $t_2 \equiv s_2 - 2 \pmod 2$. Such a vertex exists since $\lambda' \leqslant (s_1 - 2) + (s_2 - 2)$ and $\lambda' \geqslant s_2 \geqslant 3$. The trail of length λ' is composed of two edge-disjoint subtrails: one subtrail (from u to u'), denoted by T_1, is of length t_1 and its initial edge is a loop at u, the other one (from u' to v), denoted by T_2, is of length t_2 and its final edge is a loop at v.

Let $I = \{i \in \{3, 4, \dots, p\};\ s_i' < s_i\}$. For each $i \in I$ pick an endvertex $v_i \neq u'$ of the trail T_i of length s_i' in K_{k-2}^0 and let $V_I = \{v_i;\ i \in I\}$. Further, for each $i \in I$ consider a set $\mathcal{T}_i$ of $(s_i - s_i' - 2)/2$ paths of the form $P(a, x, b)$, where $x \notin V_I \cup \{u'\}$. Since $(s_i - s_i' - 2)/2$ is odd, the subgraph G_i of K_k^0 induced by the set of edges $\{v_i a, v_i b\} \cup \bigcup_{W \in \mathcal{T}_i} E(W)$ is Eulerian, and so (v_i is a common vertex of T_i and G_i) there is a trail of length s_i in K_k^0 having the same initial and final edges as T_i. Of course, we have (and are able) to suppose that paths in $\mathcal{T}_i$ and $\mathcal{T}_j$ are edge-disjoint whenever $i, j \in I$, $i \neq j$.

We now construct trails corresponding to P_{s_1+1} and P_{s_2+1}.

$\triangleright$ k is odd, $\lambda \geqslant 7$ and $s_1 - (t_1 + 2) \equiv s_2 - (t_2 + 2) \equiv 0 \pmod 4$.

Linking up T_1 with the edge $u'a$ and the unused $\frac{1}{2}(s_1 - (t_1 + 2))$ paths of length two between a and b and the edge aa (the loop incident to a) gives a trail of length s_1 from u to a with initial edge uu and final edge aa. Similarly, linking up bb and the unused $\frac{1}{2}(s_2 - (t_2 + 2))$ paths of length two between a and b with the edge $u'b$ and T_2 gives a trail of length s_2 from b to v with initial edge bb and final edge vv.

$\triangleright$ k is odd, $\lambda \geqslant 7$ and $s_1 - (t_1 + 2) \equiv s_2 - (t_2 + 2) \equiv 2 \pmod 4$.

Linking up T_1 with the edge $u'a$ and the unused $\frac{1}{2}(s_1 - (t_1 + 2))$ paths of length two between a and b and the edge bb gives a trail of length s_1 from u to b with initial edge uu and final edge bb. Similarly, linking up aa and the unused $\frac{1}{2}(s_2 - (t_2 + 2))$ paths of length two between a and b with the edge bu' and T_2 gives a trail of length s_2 from a to v with initial edge aa and final edge vv.

$\triangleright$ k is even, $\lambda \geqslant 6$ and $s_1 - (t_1 + 2) \equiv 3 \pmod 4$, $s_2 - (t_2 + 2) \equiv 0 \pmod 4$.

Linking up T_1 with the edge $u'b$ and the unused $\frac{1}{2}(s_1 - (t_1 + 3))$ (odd number of) paths of length two between a and b and two edges ab and bb gives a trail of length s_1 from u to b with initial edge uu and final edge bb. Linking up aa and the unused $\frac{1}{2}(s_2 - (t_2 + 2))$ paths of length two between a and b with the edge au' and T_2 gives a trail of length s_2 from a to v with initial edge aa and final edge vv.

▷ k is even, $\lambda \geqslant 6$ and $s_1 - (t_1 + 2) \equiv 1 \pmod 4$, $s_2 - (t_2 + 2) \equiv 2 \pmod 4$.

Linking up T_1 with the edge $u'b$ and the unused $\frac{1}{2}(s_1 - (t_1 + 3))$ (even number of) paths of length two between a and b and two edges ba and aa gives a trail of length s_1 from u to a with initial edge uu and final edge aa. Linking up bb and the unused $\frac{1}{2}(s_2 - (t_2 + 2))$ paths of length two between a and b with the edge au' and T_2 gives a trail of length s_2 from b to v with initial edge bb and final edge vv.

▷ k is even, $\lambda \leqslant 5$ or k is odd, $\lambda \leqslant 6$.

This time $\lambda' = \lambda$, $s_i = s'_i$, $i = 3, 4, \ldots, p$. By the induction hypothesis in K^0_{k-2} there are edge-disjoint trails of lengths λ', $s'_3, s'_4, \ldots, s'_p$, in which initial and final edges are loops. The trail of length s'_i is the required trail of length $s_i, i = 3, 4, \ldots, p$. The lengths of T_1 and T_2 are exactly $s_1 - 2$ and $s_2 - 2$. We can obtain two trails of lengths s_1 and s_2 by linking T_1 with $u'a$ and aa, and by linking T_2 with $u'b$ and bb, respectively.

Obviously we have enough paths $P(a, x, b)$ of length two from a to b for constructing the required trails of lengths from s_1 to s_p. □

Corollary 4.2. *If $L \leqslant \binom{k}{2}$ then there exists a packing from $P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into K^0_k with the $2p$ pendant edges of the paths mapped to loops in K^0_k.*

Proof. Let $r = k - 2p$. From $L \leqslant \binom{k}{2} = \binom{k+1}{2} - k$ we know that $L \leqslant \binom{k+1}{2} - 3$ when $r = 0$ ($k \geqslant 4$), $L \leqslant \binom{k+1}{2} - \frac{1}{2}r$ when $r > 0$ and r is an even number, $L \leqslant \binom{k+1}{2} - p$ when $r > 0$ and r is an odd number. Thus this corollary follows by Theorem 4.1. □

Corollary 4.3. *Let G be the vertex-disjoint union of p paths $P_{s_1}, P_{s_2}, \ldots, P_{s_p}$, where $s_i \geqslant 3$, $i = 1, 2, \ldots, p$. Then $\chi_0(G) = \varrho(G)$ or $\varrho(G) + 1$.*

Proof. Let $L = \sum\limits_{i=1}^{p} s_i$. From the definition of $\varrho(G)$ we know that $\binom{\varrho(G)}{1} \geqslant 2p$, $\binom{\varrho(G)+1}{2} = \binom{\varrho(G)}{1} + \binom{\varrho(G)}{2} \geqslant L$. By Corollary 4.2 there exists a packing from $P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into $K^0_{\varrho(G)+1}$ with the $2p$ pendant edges of the paths mapped to loops in $K^0_{\varrho(G)+1}$. Thus $\chi_0(G) \leqslant \varrho(G) + 1$ by Lemma 2.8. Of course we have $\chi_0(G) \geqslant \varrho(G)$. The proof is completed. □

The following two lemmas are obvious.

Corollary 4.4. *Let G be the vertex-disjoint union of p paths $P_{s_1}, P_{s_2}, \ldots, P_{s_p}$, where $s_i \geqslant 3$, $i = 1, 2, \ldots, p$, $L = \sum\limits_{i=1}^{p} s_i$. Then $\chi_0(G) = \varrho(G)$ when $\varrho(G) = 2p, L = \binom{\varrho(G)+1}{2}$ or $\varrho(G) = 2p, L \leqslant \binom{\varrho(G)+1}{2} - 3$ or $\varrho(G) - 2p$ is a positive even number, $L \leqslant \binom{\varrho(G)+1}{2} - \frac{1}{2}\varrho(G) + p$ or $\varrho(G) - 2p$ is an odd number, $L \leqslant \binom{\varrho(G)+1}{2} - p$; $\chi_0(G) = \varrho(G) + 1$ otherwise.*

Corollary 4.5. *Let P_s be a path with $s (\geqslant 3)$ vertices. Then $\chi_0(P_s) = \varrho(P_s)$ when $\varrho(P_s) = 2$, $s = 3$ or $\varrho(P_s) > 2$ is an even number, $s \leqslant \binom{\varrho(P_s)+1}{2} - \frac{1}{2}\varrho(P_s) + 1$ or $\varrho(P_s) > 2$ is an odd number, $s \leqslant \binom{\varrho(P_s)+1}{2} - 1$; $\chi_0(P_s) = \varrho(P_s) + 1$ otherwise.*

5. A conjecture

First we give an example.

Example 5.1. Suppose mC_3 is the disjoint union of m cycles of order 3. Then $\varrho(3C_3) = \min\left\{\theta;\ \binom{\theta}{1} + \binom{\theta}{2} \geqslant 9\right\} = 4$. However, $\chi_0(G) = 6 = \varrho(G) + 2$.

Based on Propositions 3.2 to 3.7, the results in Section 4 and Example 5.1, we propose the following conjecture.

Conjecture 5.2 (VDGEC Conjecture)**.** *If G is a* vdec-*graph, then $\varrho(G) \leqslant \chi_0(G) \leqslant \varrho(G) + 2$ and*

$$\lim_{n\to\infty} \frac{|\mathcal{G}^{(3)}(n)|}{|\mathcal{G}^{(1)}(n)| + |\mathcal{G}^{(2)}(n)| + |\mathcal{G}^{(3)}(n)|} = 0$$

where $\mathcal{G}^{(i)}(n)$ denotes the set of all order n vdec graphs with point-distinguishing general chromatic indices $\varrho(G) + i - 1$, $i = 1, 2, 3$.

6. A remark

Remark 6.1. Balister et al. in [2] obtained a result about a packing of $P_{s_1+1} \cup P_{s_2+1} \cup \ldots \cup P_{s_p+1}$ into K_k with the $2p$ endpoints mapped to distinct vertices (and then verified VDPEC Conjecture for the union of paths).

Acknowledgement. The author wishes to thank the anonymous referee whose suggestions helped to improve the presentation of the results obtained.

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