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TWO PHASE FLOW ARISING IN HYDRAULICS

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Abstract. The aim of this paper is to proceed in the study of the system which will be specified below. The system concerns fluid flow in a simple hydraulic system consisting of a pipe with generator on one side and a valve or some more complicated hydraulic elements on the other end of the pipe. The purpose of the research is a rigorous mathematical analysis of the corresponding linearized system.

Here, we analyze the linearized problem near the fixed steady state which already have been explicitly described. The theory of mixed linear partial differential systems and other tools are applied to derive as explicit form of the solution as possible.

Keywords: compressible fluid; Navier-Stokes equations; hydraulic systems

MSC 2010: 76T10, 35L50, 35L45

1. Introduction

In this paper we analyze the mathematical model of a two phase flow of a real fluid in a pipe with in-flow and out-flow boundary conditions corresponding to hydraulic elements and given initial data. The theory and practice of two phase flow has a long history and there is a vast literature on this theme, from which we hopefully select a representative sample.

First of all we must mention the book [4], which is considered comprehensive in introducing general principles of physics of fluids. Second, the relatively recent excellent monograph [5] concentrates on mechanics of mixtures. For us, a very important source of information is its Chapter 7: "Mixture of two Newtonian fluids." For our concrete problem, the classic monograph [11] seems to be crucial for historical insight into this field of fluid mechanics. Finally, the paper [3] is relevant to our problem and the recent paper [6] concerns directly our problem.

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Note that [8] provides a good combination of the theoretical and practical approaches to mixtures suitable both for quick information in the theoretical approach and that aimed at engineering practice. As a general source of information about the connection between fluid mechanics and the theory of partial differential equations can serve for example the monograph [2].

In the next section we formulate the general problem which can be derived from physical balance laws applied in the system (see [7]).

Since the system is highly nonlinear and contains the features of general nonlinear hyperbolic systems which have not yet been managed even in much simpler cases, necessary reasonable physical simplification must be made to get at least a partial picture about what is going on in the real machinery.

One step has already been made in [10], where we found some solutions in a closed form: stationary solutions, purely time-dependent solutions and a mixture of both (called combined solutions).

Our ambition in the present paper is to analyze a *linearized system* around a steady state.

Let us note that our approach is slightly different from the more frequent one used e.g. in [5]. Instead of different components of two phases we consider a concept of concentration γ of bubbles in the fluid and assume that the fluid and bubbles have the same velocity w and the pressure p. Thus we have three unknowns w, p, γ which are functions of (x, t). Moreover, the flow is assumed isentropic.

We believe that there are further perspectives in the analysis of the problem in question. First of all, it is possible to look for so-called local solutions. This means that we assume either small T or 'small' initial conditions. Then we believe to find smooth solutions for smooth data in the time interval $(0, T)$ with T dependent on the data. Of course, it must be expected that larger T will require more restrictions on the data. Also, a problem is to find suitable function spaces to balance the technical obstructions. C-spaces would be probably more comfortable than Sobolev-like spaces, since the latter would probably bring the necessity of additional growth conditions for the data. On the other hand, we could get results with weaker data with, of course, weaker solutions.

An interesting approach to small solutions is the reduction to a single fixed point problem (to appear in [9]). This approach consists in successive expression of one variable in terms of others so that finally a suitable variable will satisfy one equation and at the same time the other excluded variables will be expressed in terms of this new variable. Then solving the equation for this (artificial) variable, we can hope for the solution of the complete problem. Again the question of suitable function spaces arises.

The paper is organized as follows. In Section 2 we formulate the general nonlinear problem and introduce the *stationary solution*, which has been derived explicitly in [10]. Then we linearize the general problem around this stationary solution. The product of this linearization is the main objective of our analysis here.

Section 3 is devoted to the analysis of the linearized system. In spite of restrictions we are forced to make, we believe that our approach is useful in the engineering practice, based on our experience in application of a similar mathematical approach to other problems.

2. Formulation of the problem and the main result

The equations for the two phase flow of real fluids in a pipe of the length l can be written in the form (see for instance [7]):

$$w_t + \varrho_0^{-1} p_x + f(w) = 0, \tag{2.1}$$

$$p_t + \varrho_0 c^2(p,\gamma) w_x = 0, \tag{2.2}$$

$$\gamma_t + w\gamma_x = g(\gamma, p), \quad x \in (0,l), \quad t \in (0,T),\ T > 0, \tag{2.3}$$

$$w(x,0) = w_0(x), \tag{2.4}$$

$$p(x,0) = p_0(x), \tag{2.5}$$

$$\gamma(x,0) = \gamma_0(x), \quad x \in [0,l], \tag{2.6}$$

$$C(p(0,t),\gamma(0,t)) + Q_V(p(0,t),H(t)) - S_0 w(0,t) + \varphi \dot{H}(t) = 0, \tag{2.7}$$

$$w(l,t) = h(t), \tag{2.8}$$

$$\ddot{H}(t) + \Phi(t, H(t), \dot{H}(t), p(0,t), p_t(0,t)) = 0, \quad t \in [0,T], \tag{2.9}$$

$$H(0) = H_0,\ \dot{H}(0) = H_1. \tag{2.10}$$

The quantities occurring in (2.1)–(2.10) have the following meaning:

$w = w(x,t)$	the velocity of the liquid at the point x and at the time t,
$p(x,t)$	the pressure,
$\gamma = \gamma(x,t)$	the mass of freed air per unit volume of the liquid,
ϱ_0	the density of the liquid,
$c = c(p,\gamma)$	the sound speed in the liquid and in the liquid containing the air (a given function of p and γ),
$f = f(w)$	the coefficient of resistance (the friction of the liquid on the wall of the duct),

$$g(\gamma,p) = \begin{cases} K_u((\overline{\gamma} - \gamma)/K_H - p), & \text{if } (\overline{\gamma} - \gamma)/K_H \geqslant p, \\ K_r((\overline{\gamma} - \gamma)/K_H - p), & \text{if } (\overline{\gamma} - \gamma)/K_H < p, \end{cases}$$

$K_u,\ K_r$	the constants characterizing the proportionality of the velocity of loosening and dissolution on the pressure gradient, respectively,
K_H	the coefficient of absorption,
$\overline{\gamma}$	the total mass of the air in the unit volume,
$w_0,\ p_0,\ \gamma_0$	the initial distribution of the velocity, the pressure of the loosened air in the unit volume and the concentration, respectively,
$C = C(p, \gamma)$	the hydraulic capacity (a given function of p and γ),
H	the throw of the valve,
$Q_V = Q_V(p, H)$	the flow through the valve (a given function of $p,\ H$),
S_0	the cross-section of the duct,
φ	the acting facing of the valve,
h	the flow rate caused by the hydrogenerator at the end of the duct,
$H_0,\ H_1$	the initial position and the velocity of the valve, respectively.

In what follows let us assume for simplicity that

$$H(t) \equiv H_0 = \text{const.} \tag{2.11}$$

If we wanted to employ equation (2.9), then we would have to assume something like

$$\Phi(t, H_0, 0, y, z) \equiv 0. \tag{2.12}$$

Nevertheless, we will assume (2.11) sharp, without any other discussion.

Now, the standard linearization procedure leads us to the following considerations. First, we consider the linearization around the stationary solution. Its existence and expression is analyzed in [10] and we remind the basics in the next section.

Denote by $(w_s, p_s, \gamma_s)^{\mathrm{T}}$ the *stationary solution,* and $(\overline{w}, \overline{p}, \overline{\gamma})^{\mathrm{T}} = (w - w_s, p - p_s, \gamma - \gamma_s)^{\mathrm{T}}$, where $(w, p, \gamma)^{\mathrm{T}}$ is the linearization variable dependent on x and t. Then, after some computations we obtain the following system.

Definition 2.1. By a *linearized problem* of the problem (2.1)–(2.3) we mean the system

$$\begin{gathered}\overline{w}_t + \varrho_0^{-1}\overline{p}_x + f'(w_s)\overline{w} = 0,\\ \overline{p}_t + \varrho_0 w_{sx}\frac{\partial c^2}{\partial p}(p_s, \gamma_s)\overline{p} + \varrho_0 w_{sx}\frac{\partial c^2}{\partial \gamma}(p_s, \gamma_s)\overline{\gamma} + \varrho_0 c^2(p_s, \gamma_s)\overline{w}_x = 0,\\ \overline{\gamma}_t + \gamma_{sx}\overline{w} + w_s\overline{\gamma}_x = \frac{\partial g}{\partial \gamma}(\gamma_s, p_s)\overline{\gamma} + \frac{\partial g}{\partial p}(\gamma_s, p_s)\overline{p}.\end{gathered} \tag{2.13}$$

Now we are going to define rigorously what we mean by a solution of (2.1)–(2.8) with $H(t) \equiv H_0$. To this end we need some standard notation which we now shortly describe.

If $\Omega \subset \mathbb{R}^n$, then $C^k(\Omega)$ denotes the space of all functions continuous on Ω together with all derivatives up to the order k.

We make the following assumptions:

A1) $w_0, p_0, \gamma_0 \in C([0, l])$,

A2) $h \in C([0, T])$,

A3) the compatibility condition $w_0(l) = h(0)$ is satisfied,

A4) H_0 is a given nonnegative constant.

Definition 2.2. By a solution to the problem (2.1)–(2.8) (with $H \equiv H_0$) we mean a triplet (w, p, γ) with $w, p, \gamma \in C^1((0, l) \times (0, T))$ satisfying (2.1)–(2.8) pointwise.

Definition 2.3. By a *stationary solution* (with $H \equiv H_0$) we mean the functions w, p, γ independent of t and satisfying equations (2.1), (2.2), (2.3) and (2.7) with $H \equiv H_0 = \text{const}$.

Definition 2.4. By solution of the *linearized problem* we mean a triplet $(\overline{w}, \overline{p}, \overline{\gamma})$ satisfying equations (2.13).

Now let us formulate the main result.

Theorem 2.1 (Main result)**.** *Let assumptions* A1)–A4) *be satisfied, let* f, f' *be continuous in* $(-\infty, \infty)$ *and let* (4.1) *below hold true. Then there exists a unique solution* $(\overline{w}, \overline{p}, \overline{\gamma})$ *of the system* (2.1)–(2.8) *with* $H \equiv H_0$.

Moreover, having solved the system (6.27), (6.28) *below, find* $\ell_i(x)$ *according to* (6.12)*. Then define* L *by* (6.16) *and* Λ *by* (6.19)*. After that solve* (6.25) *with respect to* z*, put* $v(x, t) = z(x, t; t)$*, and* $u = L^{-1}v$ *in accordance with* (6.15)*. Then the sought solution is given by* $(\overline{w}(x, t), \overline{p}(x, t), \overline{\gamma}(x, t)) = u(x, t)$.

To be selfcontained, in the next section we shortly present results of the analysis of the stationary solution derived in [10].

3. Stationary solution

For this case, the equations reduce to a simple system of three ordinary differential equations and a scalar equation [10]

$$\varrho_0^{-1}p' + f(w) = 0, \tag{3.1}$$

$$\varrho_0 c^2(p,\gamma)w' = 0, \tag{3.2}$$

$$w\gamma' = g(\gamma, p), \quad x \in (0, l), \tag{3.3}$$

$$C(p(0),\gamma(0)) + Q_V(p(0),\gamma(0)) - S_0 W(0) = 0, \tag{3.4}$$

where $p' = \mathrm{d}p/\mathrm{d}x$, etc. Analogously as in [7], the function $c(p,\gamma)$ is assumed in the form

$$c(p,\gamma) = \frac{c_1 p^2}{c_2 p^2 + \gamma + c_3}, \tag{3.5}$$

where $c_i > 0$, $i = 1, 2, 3$ are constants. Then a trivial physically reasonable argument shows us that

$$w = w_0 = \text{constant}, \tag{3.6}$$

$$p(x) = p_0 - \varrho_0 f(w_0)x, \tag{3.7}$$

$$\gamma' = \frac{1}{w_0} g(\gamma, p_0 - \varrho_0 f(w_0)x), \tag{3.8}$$

$$C(p(0),\gamma(0)) + Q_V(p(0), H_0) - S_0 w(0) = 0. \tag{3.9}$$

The constants w_0, p_0 may be chosen arbitrarily. Also the integration of (3.8) gives us an additional free integration constant γ_0. So, given constants w_0, γ_0, the number p_0 in (3.9) is then determined from

$$C(p_0,\gamma_0) + Q_V(p_0, H_0) - S_0 w_0 = 0, \tag{3.10}$$

supposing that equation (3.10) is solvable with respect to p_0. It remains to determine the function γ from (3.8). This is substantially not difficult but a little bit lengthy computation. To this end we refer to our paper [10]. After doing that, the steady state problem is completely solved. Let us denote the *stationary solution* by $w = w_s$, $p = p_s$, and $\gamma = \gamma_s$, where according to the previous analysis

$$w_s(x) \equiv w_0 = \text{const.}, \quad p_s(x) = p_0 - \varrho_0 f(w_0)x, \quad \gamma_s \text{ is given by (3.8).} \tag{3.11}$$

4. Oscillatory solutions

Definition 4.1. By an oscillatory solution we mean a solution of (2.1)–(2.3) which is independent of x, i.e., $w = w(t)$, $p = p(t)$, $\gamma = \gamma(t)$.

Proposition 4.1. *Let f and f' be continuous in $\mathbb{R}$ and*

$$m = \inf_{z \in \mathbb{R}} f'(z) > -\infty. \tag{4.1}$$

Then for any $w_0, p_0, \gamma_0 \in \mathbb{R}$ there exists a unique oscillatory solution (w, p, γ) of the problem (2.1)–(2.6).

P r o o f. Indeed, in this case, the system (2.1)–(2.3) takes the form

$$\dot{w} + f(w) = 0, \tag{4.2}$$

$$\dot{p} = 0, \tag{4.3}$$

$$\dot{\gamma} = g(\gamma, p), \quad t > 0. \tag{4.4}$$

Equation (4.2) has a unique solution w due to elementary existence and uniqueness results for ordinary differential equations. Equation (4.3) implies $p \equiv p_0$, and for equation (4.4) again the theory of ordinary differential equations can be applied. □

Let us note that the condition (2.7) in this case reads

$$C(p_0, \gamma(t)) + Q_V(p_0, H_0) - S_0 w(t) = 0$$

and would make the problem overdetermined.

For example, for the frequent case (see [1])

$$f(w) = k|w|w$$

we obtain

$$w(t) = \frac{w_0}{1 + k|w_0|t}.$$

5. Combined solutions

Definition 5.1. By a *combined solution* we mean a solution of equations (2.1)–(2.3) which is neither stationary nor oscillatory, and for which at least one of the functions w, p, γ depends only on x or on t.

In [10] we considered the "Anzatz" $w = w(t)$, $p = p(x)$. In this case the equations (2.1), (2.2), (2.3) have the form

$$\dot{w} + \frac{1}{\varrho_0} p' + f(w) = 0, \tag{5.1}$$

$$\gamma_t + w\gamma_x = g(\gamma, p). \tag{5.2}$$

Proposition 5.1. *Let f and f' be continuous in $\mathbb{R}$ and the condition*

$$\inf_{z\in\mathbb{R}} f'(z) > -\infty$$

be satisfied. Then for any $w_0, p_0, \gamma_0 \in \mathbb{R}$ there exists a unique combined solution (w, p, γ) of the problem (2.1)–(2.6) *with $H \equiv H_0$.*

Proof. We easily obtain that p is a linear function of x independent of t, the function w satisfies the equation

$$\dot{w} + f(w) = \frac{p(l) - p_0}{\varrho_0 l},$$

which is solvable by the same argument as above, and the equation (2.3) is solvable by the method of characteristics. For details see [10]. Proof is complete. □

On the other hand, for the quite frequent friction near the wall modelled by $f(w) = k|w|w$ (see e.g. [1]) we are able to solve the problem explicitly. The result is

$$w(t) = -\frac{\alpha_0}{k}\Big(1 - \Big(1 - \frac{k^2}{\alpha_0^2} w_0^2\Big) \exp\Big(-\frac{\alpha_0 t}{2}\Big)\Big)^{1/2},$$

where

$$\alpha_0 = \Big(\frac{k(p_1 - p_0)}{\varrho_0 l}\Big)^{1/2}.$$

Now it is possible to express the whole combined solution via quadratures (see [10]).

Finally, if we assume $f \equiv 0$, $w = w_1 x + w_0$, $p = p(t)$, and $\gamma = \gamma(t)$, where w_0, w_1 are constants, then the system reduces to two ordinary differential equations, namely,

$$\dot{p}(t) = \varrho_0 c^2(p(t), \gamma(t)),$$
$$\dot{\gamma}(t) = g(p(t), \gamma(t)), \quad t > 0,$$

with the initial conditions

$$p(0) = p_0 = \text{const.},$$
$$\gamma(0) = \gamma_0 = \text{const.}$$

6. Proof of Theorem 2.1

In this section we will prove the existence and uniqueness for the linearized system (2.13). The system (2.13) is a first order linear system of the type

$$u_t + A(x)u_x + B(x)u = G, \tag{6.1}$$

where $u = u(x,t) = (\overline{w}(x,t), \overline{p}(x,t), \overline{\gamma}(x,t))^{\mathrm{T}}$ is an unknown three-dimensional vector, and $A(x)$ and $B(x)$ are given 3×3 matrices and G a given three-dimensional vector, namely,

$$A(x) = \begin{pmatrix} 0 & \varrho_0^{-1} & 0 \\ \varrho_0 c^2(p_s, \gamma_s) & 0 & 0 \\ 0 & 0 & w_s \end{pmatrix}, \tag{6.2}$$

$$B(x) = \begin{pmatrix} f'(w_s) & 0 & 0 \\ 0 & \dfrac{\partial c^2}{\partial p}(p_s, \gamma_s) & \varrho_0 w_{sx} \dfrac{\partial c^2}{\partial \gamma}(p_s, \gamma_s) \\ \gamma_{sx} & -\dfrac{\partial g}{\partial p}(\gamma_s, p_s) & -\dfrac{\partial g}{\partial \gamma}(\gamma_s, p_s) \end{pmatrix}, \tag{6.3}$$

$$G(x) = (0, 0, g(p_s(x), \gamma_s(x)))^{\mathrm{T}}. \tag{6.4}$$

Coming out of the theory of first order partial differential equations, we are interested in the properties of the matrix $A(x)$. Solving the equation

$$\det(\lambda I - A(x)) = 0,$$

where I is the identity matrix, we get eigenvalues of the matrix $A(x)$:

$$\lambda_1(x) = c(p_s(x), \gamma_s(x)), \quad \lambda_2(x) = -c(p_s(x), \gamma_s(x)), \quad \lambda_3(x) = w_s(x). \tag{6.5}$$

Let us impose initial conditions on the system (2.13):

$$\overline{w}(x,0) = \overline{w}_0(x), \quad \overline{p}(x,0) = \overline{p}_0(x), \quad \overline{\gamma}(x,0) = \overline{\gamma}_0(x), \quad x \in [0,l]. \tag{6.6}$$

It remains to input a linearized boundary condition corresponding to the condition (2.7), which in this particular case is of the form

$$C(p(0,t), \gamma(0,t)) + Q_V(p(0,t), H_0) - S_0 w(0,t) = 0. \tag{6.7}$$

By an easy manipulation in the spirit of preceeding linearization procedures we arrive at the following *linearized boundary condition* derived from (6.7):

$$
\begin{aligned}
(6.8)\quad & \Big(\frac{\partial C}{\partial p}(p_s(0),\gamma_s(0)) + \frac{\partial Q_v(p_s(0),H_0)}{\partial p}\Big)(p(0,t)-p_s(0)) \\
& + \Big(\frac{\partial C}{\partial \gamma}(p_s(0),\gamma_0(0)) - \frac{\partial C}{\partial \gamma}(p_s(0),\gamma_0(0))\Big)(\gamma(0,t)-\gamma_0(0)) \\
& - S_0(w(0,t)-w_s(0)) = S_0 w_s(0) - C(p_s(0),\gamma_s(0)) - Q_V(p_s(0),\gamma_s(0)).
\end{aligned}
$$

In terms of $\overline{w}$, $\overline{p}$, $\overline{\gamma}$ condition (6.8) reads

$$
\begin{aligned}
(6.9)\qquad & \Big(\frac{\partial C}{\partial p}(p_s(0),\gamma_s(0)) + \frac{\partial Q_v(p_s(0),H_0)}{\partial p}\Big)\overline{p}(0,t) \\
& + \Big(\frac{\partial C}{\partial \gamma}(p_s(0),\gamma_0(0)) - \frac{\partial C}{\partial \gamma}(p_s(0),\gamma_0(0)\Big)\overline{\gamma}(0,t) - S_0\overline{w}(0,t) \\
& = S_0 w_s(0) - C(p_s(0),\gamma_s(0)) - Q_V(p_s(0),\gamma_s(0)).
\end{aligned}
$$

We intend to adapt some procedures known from the theory of linear hyperbolic systems to the problem defined by (2.13), or in more concise equivalent form (6.1), with initial conditions (6.6) and boundary condition (6.9). To this end we employ the left eigenvectors

$$
(6.10)\qquad \ell^i(x) = (\ell_1^i(x), \ell_2^i(x), \ell_3^i(x)), \quad i=1,2,3,
$$

corresponding to the eigenvalues λ_i, $i=1,2,3$, of the matrix $A(x)$. These eigenvectors are computed from the equations

$$
(6.11)\qquad (\ell_1^i(x), \ell_2^i(x), \ell_3^i(x))(\lambda_i(x)I - A(x)) = 0.
$$

Elementary algebra leads us to the result

$$
(6.12)\qquad
\begin{aligned}
\ell_1(x) &= (c(p_s(x),\gamma_s(x)), \varrho_0^{-1}, 0), \quad \text{if } w_s(x) \neq c(p_s(x),\gamma_s(x)), \\
\ell_2(x) &= (-c(p_s(x),\gamma_s(x)), \varrho_0^{-1}, 0), \quad \text{if } w_s(x) \neq -c(p_s(x),\gamma_s(x)), \\
\ell_3(x) &= (0,0,1), \quad \text{if } w_s(x) \neq \pm c(p_s(x),\gamma_s(x)).
\end{aligned}
$$

Denote for short

$$
(6.13)\qquad c_s = c_s(x) = c_s(p_s(x),\gamma_s(x)).
$$

Multiply system (6.1) by ℓ_i, $i=1,2,3$. Then we get

$$
(6.14)\qquad \ell_i u_t + \lambda_i \ell_i \cdot u_x + \ell_i \cdot (B \cdot u) = \ell_i \cdot G, \quad i=1,2,3.
$$

Substituting

$$v = L \cdot u, \tag{6.15}$$

where the matrix L is given by

$$L = (\ell_{ij})_{i,j=1}^{3}, \quad \text{with } \ell_{ij} = \ell_i^j, \tag{6.16}$$

we obtain system (6.14) in the form

$$(v_i)_t + \lambda_i (v_i)_x + \ell_i \cdot ((I + B) \cdot (L^{-1} v)) = \ell_i \cdot G, \quad i = 1, 2, 3, \tag{6.17}$$

which in vector notation reads

$$v_t + \Lambda v_x + L \cdot (I + B) \cdot (L^{-1} v)) = L \cdot G, \tag{6.18}$$

denoting by Λ the diagonal matrix given by

$$\Lambda = (\lambda_{ij})_{i,j=1}^{3}, \quad \text{where } \lambda_{ij} = \lambda_i \delta_{ij},\ i, j = 1, 2, 3. \tag{6.19}$$

Denote

$$C = L \cdot (I + B) \cdot L^{-1} \quad \text{and} \quad F = L \cdot G. \tag{6.20}$$

Then system (6.18) reads

$$v_t + \Lambda v_x + C v = F. \tag{6.21}$$

Now we are in the position to use comfortably the method of characteristics. Define functions $\xi_i = \xi_i(x, t; \tau)$ as the solutions of the problems

$$\begin{aligned} \frac{\mathrm{d}\xi_i(x, t; \tau)}{\mathrm{d}\tau} &= \lambda_i(\xi_i(x, t; \tau)), \quad x \in [0, l),\ t \geqslant 0,\ \tau \geqslant 0, \\ \xi_i(x, t; t) &= x, \quad i = 1, 2, 3. \end{aligned} \tag{6.22}$$

Clearly, by our assumptions and the theory of ODE's, the problem (6.22) has a unique solution for each (x, t).

Then the functions

$$z_i(x, t; \tau) = v_i(x + \lambda_i(\xi_i(x, t; \tau))(\tau - t), \tau), \quad i = 1, 2, 3, \tag{6.23}$$

($z = (z_1, z_2, z_3)$) satisfy the relations

$$\begin{aligned}(6.24)\qquad \frac{\mathrm{d}z_i(x,t;\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau} v_i(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau) \\ &= \frac{\partial v_i}{\partial \tau}(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau) \\ &\quad + \frac{\partial v_i}{\partial y}(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau))(\lambda_i(\xi_i(x,t;\tau)) \\ &\quad + (\tau - t)\frac{\mathrm{d}\lambda_i}{\mathrm{d}y}(\xi_i(x,t;\tau)\lambda_i(\xi_i(x,t;\tau))) \\ &= F_i(\xi_i(x,t;\tau)) - \sum_{j=1}^{3} c_{ij}(\xi_i(x,t;\tau)) \cdot z_j(\xi_i(x,t;\tau), \tau)).\end{aligned}$$

Finally, we arrive at the system

$$\begin{aligned}(6.25)\qquad \frac{\mathrm{d}z_i(x,t;\tau)}{\mathrm{d}\tau} &= F_i(\xi_i(x,t;\tau)) - \sum_{j=1}^{3} c_{ij}(\xi_i(x,t;\tau)) \cdot z_j(\xi_i(x,t;\tau)), \\ z_i(x,t;0) &= (L^{-1}u_0)(\xi_i(x,t;0), 0) \\ \frac{\mathrm{d}\xi_i(x,t;\tau)}{\mathrm{d}\tau} &= \lambda_i(\xi_i(x,t;\tau)), \\ \xi_i(x,t;t) &= x,\end{aligned}$$

with

$$\begin{aligned}(6.26)\qquad \lambda_{1,2}(x) &= \pm c(p_s(x), \gamma_s(x)) = \pm \frac{c_1 p_s(x)^2}{c_2 p_s(x)^2 + \gamma_s(x) + c_3}, \\ \lambda_3(x) &= w_s(x).\end{aligned}$$

This implies that the explicit form of the system (6.22) is

$$\begin{aligned}(6.27)\qquad \frac{\mathrm{d}\xi_i^{1,2}}{\mathrm{d}\tau}(x,t;\tau) &= \pm \frac{c_1 p_s(\xi_i^{1,2}(x,t;\tau))^2}{c_2 p_s(\xi_i^{1,2}(x,t;\tau))^2 + \gamma_s(\xi_i^{1,2}(x,t;\tau)) + c_3}, \\ \frac{\mathrm{d}\xi_i^3}{\mathrm{d}\tau}(x,t;\tau) &= w_s(\xi_i(x,t;\tau)), \\ \xi_i(x,t;t) &= x,\ \ i = 1,2,3.\end{aligned}$$

Since $w_s \equiv w_0 = \text{const}$, we find

$$(6.28)\qquad \xi_i^3(x,t;\tau) = w_0(\tau - t) + x.$$

As far as the solutions of equations (6.27) are concerned, we know that the solutions $\xi_i^{1,2}$ exist. We can decompose the open set $\{\tau \in (0,\infty)\}$ into components where g is given by one prescription and by the other on the complement. On these intervals the solution is given by one formula which inserted into (6.27) yields the equation given by one analytical expression.

Let us express the equations (6.27) in a more explicit form. To simplify notation, write ξ instead of $\xi^{1,2}$, and K instead of K_u or K_H. Then, in the respective interval, γ satisfies the equation

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\xi} = g(\gamma, p) = g(\gamma, p_0\xi + p_1) = K\Big(\frac{\overline{\gamma} - \gamma}{K} - p_0\xi - p_1\Big), \tag{6.29}$$

and (6.29) can be solved independently of other equations of the system, thus making it possible to resolve the complete system.

The proof of Theorem 2.1 is finished.

A c k n o w l e d g e m e n t. I would like to say in this place that these remarks and other changes in the first version of the paper were inspired by deep insight of a referee who extended my rather narrow view of the problem.

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