Yan Lu; Yue Ping Zhu

Boundedness of some sublinear operators and commutators on Morrey-Herz spaces with variable exponents

Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 4, 969–987

Persistent URL: `http://dml.cz/dmlcz/144155`

Czechoslovak Mathematical Journal, 64 (139) (2014), 969–987

BOUNDEDNESS OF SOME SUBLINEAR OPERATORS AND COMMUTATORS ON MORREY-HERZ SPACES WITH VARIABLE EXPONENTS

Yan Lu, Yue Ping Zhu, Nantong

(Received September 3, 2013)

Abstract. We introduce a new type of variable exponent function spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ of Morrey-Herz type where the two main indices are variable exponents, and give some propositions of the introduced spaces. Under the assumption that the exponents α and p are subject to the log-decay continuity both at the origin and at infinity, we prove the boundedness of a wide class of sublinear operators satisfying a proper size condition which include maximal, potential and Calderón-Zygmund operators and their commutators of BMO function on these Morrey-Herz type spaces by applying the properties of variable exponent and BMO norms.

Keywords: Morrey-Herz space; variable exponent; sublinear operator; commutator

MSC 2010: 42B25, 42B35

1. Introduction

Function spaces with variable exponents showed up around 1990s (see [13]) and have been intensively studied in the recent years by a significant number of authors (i.e. [3], [7]). The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics, image restoration and PDE with non-standard growth conditions. A comprehensive review on this topic is given in the recent monograph [5].

It is well-known that Herz spaces play an important role in harmonic analysis. After they were introduced in [6], the theory of these spaces had a remarkable development in part due to its useful applications; we refer to [16] for more details. Herz

Supported by NNSF of China (Grant Nos. 11271209 and 11371370) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 14KJD11000).

spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot K^{\alpha(\cdot)}_{p(\cdot),q}$ with variable exponent p but fixed $\alpha \in \mathbb{R}$ and $q \in (0,\infty]$ were recently studied by Izuki [9] and these spaces with variable exponents α and p were studied by Almeida and Drihem [1], where they explored the boundedness of a class of classical operators on such spaces. The class of Morrey-Herz spaces $M\dot K^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponent was initially defined by Izuki in [10] and [11], and the boundedness of both the sublinear operators satisfying a proper size condition and the fractional integrals on $M\dot K^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ were proved. We also note that Morrey-Herz spaces with variable exponent are generalizations of Morrey-Herz spaces [15] and Herz spaces with variable exponent [9].

We consider sublinear operators satisfying the size condition

$$(1)\qquad |Tf(x)| \leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y,\quad x\notin \operatorname{supp} f,$$

for integrable and compactly supported functions f and commutators defined by

$$(2)\qquad [b,T]f(x) = T((b(x)-b)f)(x),\quad b\in \mathrm{BMO}(\mathbb{R}^n).$$

Condition (1) is satisfied by several classical operators in harmonic analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator [1]. The main purpose of this paper is to discuss the boundedness of sublinear operators and their commutators satisfying a proper size condition on Morrey-Herz spaces $M\dot K^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents α and p.

The layout of the paper is as follows. In Section 2 we recall the definitions of some function spaces with variable exponents. In Section 3 we give some key lemmas needed in the proofs of the main statements. The main results are formulated in Section 4, where we establish the boundedness of sublinear operators and their commutators on Morrey-Herz spaces $M\dot K^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents α and p.

2. Definitions of function spaces with variable exponents

Definition 1. Let $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \Big\{f \text{ is measurable: } \varrho_p\Big(\frac{f}{\lambda}\Big) < \infty \text{ for some constant } \lambda > 0\Big\},$$

where $\varrho_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x$.

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\Big\{\lambda > 0\colon\ \varrho_p\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\}.$$

We denote

$$p_- := \operatorname{ess\,inf}\{p(x)\colon\ x \in \mathbb{R}^n\}, \quad p_+ := \operatorname{ess\,sup}\{p(x)\colon\ x \in \mathbb{R}^n\}.$$

The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. Let $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, namely $1/p(x) + 1/p'(x) = 1$ holds. We also note that the generalized Hölder's inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

is true for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, where $r_p := 1 + 1/p_- - 1/p_+$ ([13] and [14]).

We say that a function $g := \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, if there exists a constant $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(\mathrm{e} + 1/|x-y|)}$$

for all $x, y \in \mathbb{R}^n$. If

$$|g(x) - g(0)| \leqslant \frac{c_{\log}}{\log(\mathrm{e} + 1/|x|)} \tag{3}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at the origin (or has a log decay at the origin). If for some $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$ we have

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\log(\mathrm{e} + |x|)} \tag{4}$$

for all $x \in \mathbb{R}^n$, then we say that g is log-Hölder continuous at infinity (or has a log decay at infinity). By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which have a log decay at the origin and at infinity, respectively.

Next we define the Morrey-Herz space with variable exponents motivated by [1] and [10]. We use the following notation. For each $k \in \mathbb{Z}$, we denote

$$B_k := \{x \in \mathbb{R}^n\colon\ |x| \leqslant 2^k\}, \quad R_k := B_k \setminus B_{k-1}, \quad \text{and} \quad \chi_k := \chi_{R_k}.$$

Definition 2. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leqslant \lambda < \infty$ and $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The Morrey-Herz space with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \big\{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})\colon\ \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/q}.$$

If α is a constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ defined in [10]. If both α and p are constants, and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ are classical Herz spaces ([16]).

Given a function $f \in L_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|\,{\rm d}y,$$

where $B(x,r) = \{y \in \mathbb{R}^n\colon |x-y| < r\}$. Let $\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

If $p(\cdot)$ has a log-decay at the origin and at infinity, so does $1/p(\cdot)$. The following proposition was initially proved by Cruz-Uribe et al. [4], when $p_+ < \infty$. Later Cruz-Uribe et al. [2] and Diening et al. [5] have independently extended the result even to the case of $p_+ = \infty$.

Proposition 1. *If $p(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then we have $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*

3. Properties of function spaces with variable exponents

Lemma 1 ([1]). *Let $p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and $R = B(0,r) \setminus B(0,r/2)$. If $|R| \geqslant 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{1/p(x)} \approx |R|^{1/p_\infty}$$

with the implicit constants independent of r and $x \in R$. The left hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, $p \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n) \cap \mathcal{P}^{\log}_0(\mathbb{R}^n)$.

Lemma 2 ([1]). *Let $\alpha \in L^\infty(\mathbb{R}^n)$ and $r_1 > 0$. If α is log-Hölder continuous, both at the origin and at infinity, then*

$$r_1^{\alpha(x)} \leqslant C r_2^{\alpha(y)} \times \begin{cases} \Big(\dfrac{r_1}{r_2}\Big)^{\alpha_+} & \text{if } 0 < r_2 \leqslant r_1/2;\\ 1 & \text{if } r_1/2 < r_2 \leqslant 2r_1;\\ \Big(\dfrac{r_1}{r_2}\Big)^{\alpha_-} & \text{if } r_2 \geqslant 2r_1; \end{cases}$$

for any $x \in B(0,r_1) \setminus B(0,r_1/2)$ *and* $y \in B(0,r_2) \setminus B(0,r_2/2)$*, with the implicit constant not depending on* x, y, r_1 *and* r_2*.*

The following are the key lemmas due to the authors of [8] and [12].

Lemma 3. *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$*, then there exist constants* $0 < \varepsilon < 1$ *and* $C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$*,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\frac{|B|}{|S|},$$
$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C\Big(\frac{|S|}{|B|}\Big)^{\varepsilon}.$$

Lemma 4. *If* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$*, then for all balls* B *in* $\mathbb{R}^n$ *we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C.$$

Lemma 5. *For all* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and all* $j, i \in \mathbb{Z}$ *with* $j > i$ *we have,*

$$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant \sup_{B:\mathrm{ball}} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$
$$\|(b-b_{B_i})\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C(j-i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following statement gives some basic embeddings between Morrey-Herz spaces.

Proposition 2. *Let* $\alpha \in L^{\infty}(\mathbb{R}^n)$*,* $0 \leqslant \lambda < \infty$*,* $p \in \mathcal{P}(\mathbb{R}^n)$ *and* q_0*,* $q_1 \in (0,\infty]$*. If* $q_0 \leqslant q_1$*, then*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q_0,p(\cdot)}(\mathbb{R}^n) \hookrightarrow M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p(\cdot)}(\mathbb{R}^n).$$

P r o o f. The embeddings are immediate consequences of the embedding $l^{q_0} \hookrightarrow l^{q_1}$ for $0 < q_0 \leqslant q_1 \leqslant \infty$. □

Proposition 3. *Let* $\alpha \in L^{\infty}(\mathbb{R}^n)$*,* $0 \leqslant \lambda < \infty$*,* $p_0, p_1 \in \mathcal{P}(\mathbb{R}^n)$ *and* $q_0, q_1 \in (0,\infty]$*. If* $p_0 \leqslant p_1$ *and* $1/p_0 - 1/p_1$ *is log-Hölder continuous, both at the origin and at infinity, then*

$$M\dot{K}^{\alpha(\cdot)+n/p_0(\cdot)-n/p_1(\cdot),\lambda}_{q,p_1(\cdot)}(\mathbb{R}^n) \hookrightarrow M\dot{K}^{\alpha(\cdot),\lambda}_{q,p_0(\cdot)}(\mathbb{R}^n).$$

P r o o f. The proof is similar to Proposition 3.5 in [1]. □

Proposition 4. *Let $\alpha \in L^{\infty}(\mathbb{R}^n)$, $0 \leqslant \lambda < \infty$, $p \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (0,\infty]$. If α is log-Hölder continuous, both at the origin and at infinity, then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\Bigg\{\sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\bigg(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^{1/q},$$

$$\sup_{L\geqslant 0,L\in\mathbb{Z}}\bigg[2^{-L\lambda}\bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^{1/q} + 2^{-L\lambda}\bigg(\sum_{k=0}^{L} 2^{k\alpha_\infty q}\|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^{1/q}\bigg]\Bigg\}.$$

P r o o f. If α is log-Hölder continuous at infinity, then for $k \geqslant 0$ and $x \in C_k$ we have

$$k|\alpha(x) - \alpha_\infty| \geqslant \frac{k}{\log(\mathrm{e} + |x|)} \geqslant 1.$$

Therefore $2^{k\alpha(x)} \approx 2^{k\alpha_\infty}$ with constants independent of k and x, and hence

$$\|2^{k\alpha(\cdot)} f\chi_k\|_{L^{p(\cdot)}} \approx 2^{k\alpha_\infty}\|f\chi_k\|_{L^{p(\cdot)}}.$$

If, in addition, α is log-Hölder continuous at the origin, then for $k < 0$ and $x \in C_k$ we have $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$. Thus

$$\|2^{k\alpha(\cdot)} f\chi_k\|_{L^{p(\cdot)}} \approx 2^{k\alpha(0)}\|f\chi_k\|_{L^{p(\cdot)}}$$

and hence the result follows. □

4. Main results

Theorem 1. *Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < n/\lambda$, $0 \leqslant \lambda < n$, and let $\alpha \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous, both at the origin and at infinity, with*

$$\lambda - n/p_+ < \alpha_- \leqslant \alpha_+ < n(1 - 1/p_-). \tag{5}$$

Suppose that T is a sublinear operator satisfying (1). If T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then T is bounded on $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$.

P r o o f. Denoting $f_j := f\chi_j$ for each $j \in \mathbb{Z}$, we split f into $f = \sum\limits_{j=-\infty}^{\infty} f_j$. Thus we have

$$\|Tf(x)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\bigg(\sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} Tf\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)$$

$$
\begin{aligned}
&\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\Big(\sum_{k=-\infty}^{L}\Big\|2^{k\alpha(\cdot)}\Big(\sum_{j=-\infty}^{k-2}|Tf_j(x)|\Big)\chi_k\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\Big)\\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\Big(\sum_{k=-\infty}^{L}\Big\|2^{k\alpha(\cdot)}\Big(\sum_{j=k-1}^{k+1}|Tf_j(x)|\Big)\chi_k\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\Big)\\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\Big(\sum_{k=-\infty}^{L}\Big\|2^{k\alpha(\cdot)}\Big(\sum_{j=k+2}^{\infty}|Tf_j(x)|\Big)\chi_k\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\Big)\\
&=: U_1+U_2+U_3.
\end{aligned}
$$

First we estimate U_1. We note that $|x-y|>|x|-|y|>2^k/4$, if $x\in R_k$, $y\in R_j$ for every $j,k\in\mathbb{Z}$ with $k\leqslant L$ and $j\leqslant k-2$. Using Lemma 2, we have

$$
\begin{aligned}
2^{k\alpha(x)}\sum_{j=-\infty}^{k-2}|Tf_j(x)|\chi_k(x) &\leqslant C\sum_{j=-\infty}^{k-2}2^{k\alpha(x)}\int_{R_j}\frac{|f_j(y)|}{|x-y|^n}\,\mathrm{d}y\chi_k(x)\\
&\leqslant C\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha_+}2^{-kn}\int_{R_j}2^{j\alpha(y)}|f_j(y)|\,\mathrm{d}y\chi_k(x).
\end{aligned}
$$

Applying Hölder's inequality and Lemma 1, we obtain

$$
\begin{aligned}
&\|2^{k\alpha(x)}\sum_{j=-\infty}^{k-2}|Tf_j(x)|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha_+}2^{-kn}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\approx C\sum_{j=-\infty}^{k-2}2^{(k-j)\alpha_+}2^{-kn}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{jn}|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}\\
&\quad\leqslant C\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha_++n/p_--n)}\|2^{j\alpha(\cdot)}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where we use the fact that $|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}\geqslant 2^{(k-j)n/p_-}$ for $j\leqslant k-2$ which appears in the proof of Theorem 4.12 in [1].

Hence we have

$$
U_1\leqslant C\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=-\infty}^{k-2}2^{(k-j)\delta}\|2^{j\alpha_{(\cdot)}}f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q
$$

where we put $\delta=\alpha_+-n+n/p_-$ for short. It follows from the condition (5) that $\delta<0$. We consider two cases $0<q\leqslant 1$ and $1<q<\infty$. When $0<q\leqslant 1$, we apply

the inequality

$$\left(\sum_{h=1}^{\infty} a_h\right)^q \leqslant \sum_{h=1}^{\infty} a_h^q (a_1, a_2, \ldots \geqslant 0) \tag{6}$$

and obtain

$$\begin{aligned}
U_1 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)\delta q} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \\
&= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=-\infty}^{L-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \sum_{k=j+2}^{L} 2^{(k-j)\delta q} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=-\infty}^{L-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $1 < q < \infty$, we use Hölder's inequality and obtain

$$\begin{aligned}
U_1 &\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)\delta q/2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)\delta q'/2}\right)^{q/q'} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=-\infty}^{k-2} 2^{(k-j)\delta q/2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \\
&= C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=-\infty}^{L-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \sum_{k=j+2}^{L} 2^{(k-j)\delta q/2} \\
&\leqslant C \sup_{L\in\mathbb{Z}} 2^{-L\lambda q} \left(\sum_{j=-\infty}^{L-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right) \\
&\leqslant C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next we estimate U_2. By Proposition 4, we get

$$\begin{aligned}
U_2 \approx \max\Bigg\{ &\sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left\| 2^{k\alpha(0)} \left(\sum_{j=k-1}^{k+1} |Tf_j(x)|\right)\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
&\sup_{L\geqslant 0, L\in\mathbb{Z}} \Bigg[2^{-L\lambda q} \sum_{k=-\infty}^{-1} \left\| 2^{k\alpha(0)} \left(\sum_{j=k-1}^{k+1} |Tf_j(x)|\right)\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + 2^{-L\lambda q} \sum_{k=0}^{L} \left\| 2^{k\alpha_\infty} \left(\sum_{j=k-1}^{k+1} |Tf_j(x)|\right)\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg] \Bigg\}.
\end{aligned}$$

Using the sublinearity and the boundedness of T on $L^{p(\cdot)}$, we have

$$\begin{aligned}\Big\|2^{k\alpha(0)}\Big(\sum_{j=k-1}^{k+1}|Tf_j(x)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant \Big\|2^{k\alpha(0)}\Big(\sum_{j=k-1}^{k+1}|Tf_j(x)|\Big)\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leqslant \sum_{j=k-1}^{k+1}\|2^{k\alpha(0)}|Tf_j(x)|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\leqslant \sum_{j=k-1}^{k+1}\|2^{k\alpha(0)}|f_j|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\begin{aligned}U_2 \approx \max\Big\{&C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\Big(\sum_{j=k-1}^{k+1}\|2^{k\alpha(0)}|f_j|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big),\\ &C\sup_{L\geqslant 0,L\in\mathbb{Z}}\Big[2^{-L\lambda q}\Big(\sum_{k=-\infty}^{-1}\|2^{k\alpha(0)}|f\chi_k|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\ &\qquad+2^{-L\lambda q}\Big(\sum_{k=0}^{L}\|2^{k\alpha_\infty}|f\chi_k|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\Big]\Big\}\\ \leqslant \max\Big\{&C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\|2^{k\alpha(0)}|f\chi_k|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},\\ &C\sup_{L\geqslant 0,L\in\mathbb{Z}}\Big[2^{-L\lambda q}\Big(\sum_{k=-\infty}^{-1}\|2^{k\alpha(0)}|f\chi_k|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\ &\qquad+2^{-L\lambda q}\Big(\sum_{k=0}^{L}\|2^{k\alpha_\infty}|f\chi_k|\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\Big]\Big\}\\ \leqslant C&\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

We now estimate U_3. By Proposition 4, we get

$$\begin{aligned}U_3 \approx \max\Big\{&\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\Big(\sum_{k=-\infty}^{L}\|2^{k\alpha(0)}\Big(\sum_{j=k+2}^{\infty}|Tf_j(x)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big),\\ &\sup_{L\geqslant 0,L\in\mathbb{Z}}\Big[2^{-L\lambda q}\Big(\sum_{k=-\infty}^{-1}\Big\|2^{k\alpha(0)}\Big(\sum_{j=k+2}^{\infty}|Tf_j(x)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\ &\qquad+2^{-L\lambda q}\Big(\sum_{k=0}^{L}\Big\|2^{k\alpha_\infty}\Big(\sum_{j=k+2}^{\infty}|Tf_j(x)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\Big]\Big\}\\ =:\ &\max\{E,F\}.\end{aligned}$$

For U_3, we note that $|x-y|>2^j/4$ for $x\in R_k, y\in R_j$ and $j\geqslant k+2$, hence we have

$$|Tf_j(x)|\leqslant C\int_{R_j}\frac{|f_j(y)|}{|x-y|^n}\,\mathrm{d}y\leqslant C2^{-jn}\int_{R_j}|f_j(y)|\,\mathrm{d}y.$$

Applying Hölder's inequality to the last integral, we get

$$\Big\|\sum_{j=k+2}^{\infty}|Tf_j(x)|\chi_k\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\sum_{j=k+2}^{\infty}2^{-jn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

It follows from $p\in\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ that $p'\in\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$. Lemma 1 tells us

$$\|\chi_j\|_{L^{p'(\cdot)}}\approx|R_j|^{1/p'(x_j)},\quad x_j\in R_j,\quad\text{and}\quad\|\chi_k\|_{L^{p(\cdot)}}\approx|R_k|^{1/p(x_k)},\quad x_k\in R_k.$$

Hence

$$\|\chi_j\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\approx|R_j|^{1/p'(x_j)}|R_k|^{1/p(x_k)}\leqslant 2^{jn}|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}.$$

Now we can distinguish three cases as follows:

Case I: $j\geqslant k+2\geqslant 0$. By Lemma 1 we have

$$|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}\approx|R_j|^{-1/p_\infty}|R_k|^{1/p_\infty}\approx 2^{(k-j)n/p_\infty}\geqslant 2^{(k-j)n/p_+}.$$

Case II: $j>0\geqslant k+2$. In this case we obtain

$$|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}\geqslant|R_j|^{-1/p_+}|R_k|^{1/p_+}\geqslant 2^{(k-j)n/p_+}.$$

Case III: $0\geqslant j\geqslant k+2$. Here we have

$$|R_j|^{-1/p(x_j)}|R_k|^{1/p(x_k)}\approx\Big(\frac{|R_k|}{|R_j|}\Big)^{1/p(x_k)}|R_j|^{1/p(x_k)-1/p(x_j)}\geqslant 2^{(k-j)n/p_+}.$$

Indeed, since $|x_j|<2^j$, $|x_k|<2^k<2^j$, we make use of the local log-Hölder continuity of p at the origin and get, for $j<0$,

$$\Big|\frac{1}{p(x_k)}-\frac{1}{p(x_j)}\Big|\log\frac{1}{|R_j|}\geqslant\frac{\log(1/2^j)}{\log(\mathrm{e}+1/2^j)}\leqslant C$$

with $C>0$ independent of k,j,χ_k,χ_j.

Therefore, in all cases we have essentially the same bound and hence, combining the estimates above, we arrive at the inequality

$$\Big\|\sum_{j=k+2}^{\infty}|Tf_j(x)|\chi_k\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\sum_{j=k+2}^{\infty}2^{(k-j)n/p_+}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

When $0<q\leqslant 1$, we have

$$\begin{aligned}E&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n/p_+}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q\\&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n/p_+q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\Big(\sum_{j=k+2}^{L-1}2^{(k-j)n/p_+q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\&\quad+C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\Big(\sum_{j=L}^{\infty}2^{(k-j)n/p_+q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)\\&=:E_1+E_2.\end{aligned}$$

Observing that $n/p_+ + \alpha(0) > n/p_+ + \alpha_- > 0$ by (5), we have an estimate of E_1:

$$\begin{aligned}E_1&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L-1}2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{j-2}2^{(k-j)(n/p_++\alpha(0))q}\\&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L-1}2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Since $n/p_+ + \alpha(0) > 0$ and $\lambda-\alpha(0)-n/p_+<0$, the estimate of E_2 is obtained:

$$\begin{aligned}E_2&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{k\alpha(0)q}\sum_{j=L}^{\infty}2^{(k-j)n/p_+q}2^{-j\alpha(0)q}2^{j\lambda q}\\&\quad\times 2^{-j\lambda q}\sum_{l=-\infty}^{j}2^{l\alpha(0)q}\|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}\Big(\sum_{k=-\infty}^{L}2^{k(\alpha(0)+n/p_+)q}\Big)\Big(\sum_{j=L}^{\infty}2^{j(\lambda-\alpha(0)-n/p_+)q}\Big)\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\\&=C\sup_{L<0,L\in\mathbb{Z}}2^{-L\lambda q}2^{L(\alpha(0)+n/p_+)q}2^{L(\lambda-\alpha(0)-n/p_+)q}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\\&=C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

When $1 < q < \infty$, we have

$$
\begin{aligned}
E &\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=k+2}^{L} 2^{(k-j)n/p_+} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\quad + C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \left(\sum_{j=L+1}^{\infty} 2^{(k-j)n/p_+} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&=: E_3 + E_4.
\end{aligned}
$$

We use Hölder's inequality and obtain

$$
\begin{aligned}
E_3 &\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k+2}^{L} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(n/p_+ + \alpha(0))q/2} \\
&\quad \times \left(\sum_{j=k+2}^{L} 2^{(k-j)(n/p_+ + \alpha(0))q'/2} \right)^{q/q'} \\
&\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-2} 2^{(k-j)(n/p_+ + \alpha(0))q/2} \\
&\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Since $\alpha(0) + n/p_+ - \lambda > 0$, we have

$$
\begin{aligned}
E_4 &= C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=L+1}^{\infty} 2^{j\alpha(0)} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right. \\
&\quad \left. \times 2^{(k-j)(n/p_+ + \alpha(0) + \lambda)/2} 2^{(k-j)(n/p_+ + \alpha(0) - \lambda)/2} \right)^q \\
&\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=L+1}^{\infty} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{(k-j)(n/p_+ + \alpha(0) + \lambda)q/2} \\
&\quad \times \left(\sum_{j=L+1}^{\infty} 2^{(k-j)(n/p_+ + \alpha(0) - \lambda)q'/2} \right)^{q/q'} \\
&\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=L+1}^{\infty} 2^{(k-j)(n/p_+ + \alpha(0) + \lambda)q/2} 2^{j\lambda q} 2^{-j\lambda q} \\
&\quad \times \sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}
$$

$$\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{k\lambda q} \sum_{j=L+1}^{\infty} 2^{(k-j)(n/p_+ + \alpha(0)-\lambda)q/2} \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}$$
$$\leqslant C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Hence we arrive at the inequality

$$E \leqslant C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimate of F since it is essentially similar to that of E. Consequently, we have proved Theorem 1. □

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (3) and (4). Then so does $p'(\cdot)$. In particular, we see that $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Therefore applying Lemma 3 we can take a constant $0 < r < 1/(p')_+$ so that

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant C\Big(\frac{|S|}{|B|}\Big)^r \tag{7}$$

for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$.

Theorem 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions* (3) *and* (4)*, and take a constant $0 < r < 1/(p')_+$ such that* (7) *holds. Let $\alpha \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous, both at the origin and at infinity, with*

$$\lambda < \alpha_- \leqslant \alpha_+ < nr. \tag{8}$$

Suppose that T is a sublinear operator satisfying (1)*. If commutators $[b, T]$ are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, we have for all $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$*

$$\|[b,T]f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Proof. By Proposition 4 we have

$$\|[b,T]f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\Big\{ \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Big(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\|[b,T]f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/q},$$
$$\sup_{L\geqslant 0, L\in\mathbb{Z}} \Big[2^{-L\lambda}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|[b,T]f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/q}$$
$$+ 2^{-L\lambda}\Big(\sum_{k=0}^{L} 2^{k\alpha_\infty q}\|[b,T]f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/q}\Big]\Big\}$$
$$=: \max\{I, J\}.$$

First we estimate I:

$$\begin{aligned} I &\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\biggl\{\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=-\infty}^{k-2} \|[b,T]f_j\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\biggr\}^{1/q} \\ &\quad + C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\biggl\{\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \sum_{j=k-1}^{\infty} \|[b,T]f_j\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\biggr\}^{1/q} \\ &=: I_1 + I_2. \end{aligned}$$

We note that $|x-y| > 2^k/4$ for $x \in R_k$, $y \in R_j$ and $j \leqslant k-2$. Using the generalized Hölder's inequality, we get that

$$\begin{aligned} |[b,T]f(x)\chi_k(x)| &\leqslant C \int_{R_j} |b(x)-b(y)|\frac{|f_j(y)|}{|x-y|^n}\,\mathrm{d}y\,\chi_k(x) \\ &\leqslant C2^{-kn}\int_{R_j} |b(x)-b(y)||f_j(y)|\,\mathrm{d}y\,\chi_k(x) \\ &\leqslant C2^{-kn}\biggl\{|b(x)-b_{B_j}|\int_{R_j}|f_j(y)|\,\mathrm{d}y + \int_{R_j}|b_{B_j}-b(y)||f_j(y)|\,\mathrm{d}y\biggr\} \\ &\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\{|b(x)-b_{B_j}|\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + \|(b_{B_j}-b)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\}. \end{aligned}$$

By virtue of Lemma 5, we have

$$\begin{aligned} \|[b,T]f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\{\|(b-b_{B_j})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|(b_{B_j}-b)\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\} \\ &\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\{(k-j)\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\} \\ &\leqslant C2^{-kn}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(k-j)\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Using Lemma 3 and Lemma 4, we obtain

$$\begin{aligned} 2^{-kn}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} &\leqslant C2^{-kn}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C2^{nr(j-k)}. \end{aligned}$$

Thus, we have the estimate

$$\begin{aligned} I_1 &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\ &\quad \times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\biggl\{\sum_{k=-\infty}^{L}\biggl(\sum_{j=-\infty}^{k-2} 2^{\alpha(0)j}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}(k-j)2^{(k-j)(\alpha(0)-nr)}\biggr)^q\biggr\}^{1/q}. \end{aligned}$$

We consider two cases $0 < q \leqslant 1$ and $1 < q < \infty$. In the case of $0 < q \leqslant 1$, noting that $\alpha(0) - nr < 0$, we apply inequality (6) and obtain

$$\begin{aligned}
I_1 &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad\times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}(k-j)^q 2^{(k-j)(\alpha(0)-nr)q}\right)\right\}^{1/q} \\
&= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad\times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{L}(k-j)^q 2^{(k-j)(\alpha(0)-nr)q}\right\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

In the case of $1 < q < \infty$, we use Hölder's inequality and obtain

$$\begin{aligned}
I_1 &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad\times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha(0)-nr)q/2}\right)\right. \\
&\quad\times \left.\left(\sum_{j=-\infty}^{k-2}(k-j)^{q'}2^{(k-j)(\alpha(0)-nr)q'/2}\right)^{q/q'}\right\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad\times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(k-j)(\alpha(0)-nr)q/2}\right)\right\}^{1/q} \\
&= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad\times \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{L}2^{(k-j)(\alpha(0)-nr)q/2}\right\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-2} 2^{\alpha(0)jq}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next we estimate I_2. By the $L^{p(\cdot)}$-boundedness of $[b,T]$, we get that

$$\begin{aligned}
I_2 &\leqslant C \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left(\sum_{j=k-1}^{\infty}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{1/q} \\
&= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{j=k-1}^{\infty} 2^{j\alpha(0)}2^{\alpha(0)(k-j)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{1/q}.
\end{aligned}$$

Now we consider two cases $0 < q \leqslant 1$ and $1 < q < \infty$. When $0 < q \leqslant 1$, we use inequality (6) again and get

$$\begin{aligned} I_2 &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{j=k-1}^{\infty} 2^{j\alpha(0)q}2^{\alpha(0)(k-j)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{j=k-1}^{L-1} 2^{j\alpha(0)q}2^{\alpha(0)(k-j)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &\quad + C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{j\alpha(0)q}2^{\alpha(0)(k-j)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &=: I_{21} + I_{22}. \end{aligned}$$

The estimate of I_{21} is obtained as follows:

$$\begin{aligned} I_{21} &= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+1} 2^{\alpha(0)(k-j)q}\right\}^{1/q} \\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Because $\alpha(0) > 0$ and $\lambda - \alpha(0) < 0$, we have estimate of I_{22}:

$$\begin{aligned} I_{22} &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\ &\quad \times \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{\alpha(0)(k-j)q}2^{j\lambda q}2^{-j\lambda q} \sum_{l=-\infty}^{j} 2^{l\alpha(0)q}\|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{j=L}^{\infty} 2^{\alpha(0)(k-j)q}2^{j\lambda q}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\sum_{j=L}^{\infty} 2^{(\lambda-\alpha(0))jq}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0,L\in\mathbb{Z}} 2^{-L\lambda}2^{L\alpha(0)}2^{(\lambda-\alpha(0))L}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Because $\lambda < \alpha(0)$, we can take a constant $\theta > 1$ such that $\lambda - \alpha(0)/\theta < 0$. When $1 < q < \infty$, we use Hölder's inequality and obtain

$$\begin{aligned}
I_2 &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \bigg(\sum_{j=k-1}^{\infty} 2^{j\alpha(0)q} 2^{\alpha(0)(k-j)q/\theta} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg) \\
&\qquad \times \bigg(\sum_{j=k-1}^{\infty} 2^{\alpha(0)(k-j)q\theta-1/\theta} \bigg)^{\theta/\theta'} \Bigg\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \sum_{j=k-1}^{\infty} 2^{j\alpha(0)q} 2^{\alpha(0)(k-j)q/\theta} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \sum_{j=k-1}^{L-1} 2^{j\alpha(0)q} 2^{\alpha(0)(k-j)q/\theta} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q} \\
&\quad + C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{j\alpha(0)q} 2^{\alpha(0)(k-j)q/\theta} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q} \\
&=: I_{23} + I_{24}.
\end{aligned}$$

Because $\alpha(0) > 0$, we get

$$\begin{aligned}
I_{23} &= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j+1} 2^{\alpha(0)(k-j)q/\theta} \Bigg\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{j=-\infty}^{L-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, it follows from $\lambda - \alpha(0)/\theta < 0$ that

$$\begin{aligned}
I_{24} &\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha(0)(k-j)q/\theta} 2^{j\lambda q} 2^{-j\lambda q} \\
&\qquad \times \bigg(\sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg) \Bigg\}^{1/q} \\
&\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} \sum_{j=L}^{\infty} 2^{\alpha(0)(k-j)q/\theta} 2^{j\lambda q} \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q} \\
&= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg\{ \sum_{k=-\infty}^{L} 2^{k\alpha(0)q/\theta} \sum_{j=L}^{\infty} 2^{j(\lambda-\alpha(0)/\theta)q} \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \Bigg\}^{1/q}
\end{aligned}$$

$$= C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda} 2^{L\alpha(0)/\theta} 2^{L(\lambda-\alpha(0)/\theta)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}$$
$$\leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

Hence we arrive at the inequality

$$I \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimate of J since it is essentially similar to that of I. The proof of Theorem 2 is complete. □

Acknowledgement. The authors would like to thank the referee for carefully reading the manuscript and making several valuable suggestions.

References

[1] *A. Almeida, D. Drihem*: Maximal, potential and singular type operators on Herz spaces with variable exponents. J. Math. Anal. Appl. *394* (2012), 781–795.
[2] *D. Cruz-Uribe, L. Diening, A. Fiorenza*: A new proof of the boundedness of maximal operators on variable Lebesgue spaces. Boll. Unione Mat. Ital. (9) *2* (2009), 151–173.
[3] *D. Cruz-Uribe, A. Fiorenza, J. M. Martell, C. Pérez*: The boundedness of classical operators on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. *31* (2006), 239–264.
[4] *D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer*: The maximal function on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. *28* (2003), 223–238.
[5] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011.
[6] *C. Herz*: Lipschitz spaces and Berstein's theorem on absolutely convergent Fourier transforms. J. Math. Mech. *18* (1968), 283–323.
[7] *K.-P. Ho*: The fractional integral operators on Morrey spaces with variable exponent on unbounded domains. Math. Inequal. Appl. *16* (2013), 363–373.
[8] *M. Izuki*: Boundedness of commutators on Herz spaces with variable exponent. Rend. Ciec. Mat. Palermo (2) *59* (2010), 199–213.
[9] *M. Izuki*: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. *36* (2010), 33–50.
[10] *M. Izuki*: Fractional integrals on Herz-Morrey spaces with variable exponent. Hiroshima Math. J. *40* (2010), 343–355.
[11] *M. Izuki*: Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent. Math. Sci. Res. J. *13* (2009), 243–253.
[12] *M. Izuki, Y. Sawano*: Variable Lebesgue norm estimates for BMO functions. Czech. Math. J. *62* (2012), 717–727.
[13] *O. Kováčik, J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czech. Math. J. *41* (1991), 592–618.
[14] *A. K. Lerner*: On some questions related to the maximal operator on variable L^p spaces. Trans. Am. Math. Soc. *362* (2010), 4229–4242.
[15] *S. Lu, L. Xu*: Boundedness of rough singular integral operators on the homogeneous Morrey-Herz spaces. Hokkaido Math. J. *34* (2005), 299–314.

[16] *S. Z. Lu, D. C. Yang, G. Hu*: Herz Type Spaces and Their Applications. Science Press, Beijing, 2008.

Authors' addresses: Yan Lu, School of Xinglin, Nantong University, No. 9 Seyuan Rouad, Nantong, Jiangsu Province, 226007, P. R. China, e-mail: luyan851212@ntu.edu.cn; Yue Ping Zhu (corresponding author), School of Science, Nantong University, No. 9 Seyuan Rouad, Nantong, Jiangsu Province, 226007, P. R. China, e-mail: zhuyueping@ntu.edu.cn.