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DUNKL-GABOR TRANSFORM AND TIME-FREQUENCY CONCENTRATION

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Abstract. The aim of this paper is to prove two new uncertainty principles for the Dunkl-Gabor transform. The first of these results is a new version of Heisenberg's uncertainty inequality which states that the Dunkl-Gabor transform of a nonzero function with respect to a nonzero radial window function cannot be time and frequency concentrated around zero. The second result is an analogue of Benedicks' uncertainty principle which states that the Dunkl-Gabor transform of a nonzero function with respect to a particular window function cannot be time-frequency concentrated in a subset of the form $S \times \mathcal{B}(0,b)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. As a side result we generalize a related result of Donoho and Stark on stable recovery of a signal which has been truncated and corrupted by noise.

Keywords: time-frequency concentration; Dunkl-Gabor transform; uncertainty principles

MSC 2010: 42C20, 43A32, 46E22

1. Introduction

Heisenberg's uncertainty principle is usually understood as a relation between the simultaneous spreadings of a function and its Fourier transform. As well as its well-known original interpretation in quantum theory, it also has relevance to signal processing, as it gives a restriction on how well the instantaneous frequency of a signal can be measured. To be more precise, let $d \geqslant 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. Then Heisenberg's uncertainty inequality can be stated in the following version:

$$\||x|f\|_{L^2(\mathbb{R}^d)} \, \||\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)} \geqslant c(d)\|f\|^2_{L^2(\mathbb{R}^d)}, \tag{1.1}$$

where the Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) \mathcal{E}^{-\mathrm{i}\langle x, \xi \rangle} \,\mathrm{d}x,$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way.

In order to describe our results, we first need to introduce some notation (further details can be found in Section 2.1). In this paper we consider the Dunkl operators (see [6]) T_j; $j = 1, \ldots, d$ associated with an arbitrary finite reflection group G and a nonnegative multiplicity function k. These are differential-difference operators generalizing the usual partial derivatives and they play a useful role in the algebraic description of exactly solvable quantum many body systems of Calogero-Moser-Sutherland type; among the broad literature, we refer to [12] and [15].

The Dunkl kernel $\mathcal{K}$ on $\mathbb{R}^d \times \mathbb{R}^d$ associated with G and k has been introduced by C. F. Dunkl in [5], [6]. It generalizes the usual exponential function ($k = 0$) and can be characterized as the solution of a joint eigenvalue problem for the associated Dunkl operators. This kernel is especially of interest as it gives rise to a corresponding integral transform on $\mathbb{R}^d$. The Dunkl transform $\mathcal{F}_D$ associated with G and k involves a weight function w_k and is defined for an integrable function f on $\mathbb{R}^d$ with respect to the measure $\mathrm{d}\mu_k(x) = w_k(x)\,\mathrm{d}x$ by

$$\mathcal{F}_D(f)(\xi) := c_k \int_{\mathbb{R}^d} \mathcal{K}(-\mathrm{i}\xi, x) f(x) \,\mathrm{d}\mu_k(x), \quad \xi \in \mathbb{R}^d$$

and extended to $L_k^2(\mathbb{R}^d)$ by a Parseval-type relation with c_k being a suitable constant. Here, for $1 \leqslant p < \infty$, we denote by $L_k^p(\mathbb{R}^d)$ the Banach space consisting of measurable functions f on $\mathbb{R}^d$ equipped with the norms

$$\|f\|_{L_k^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}\mu_k(x) \right)^{1/p}.$$

This transformation generalizes the classical Fourier transform $\mathcal{F}$, to which it reduces in the case $k = 0$. Therefore Heisenberg's inequality (1.1) for the Dunkl transform leads to (see [16], [18])

$$\| |x| f \|_{L_k^2} \, \| |\xi| \, \mathcal{F}_D(f) \|_{L_k^2} \geqslant (\gamma(k) + d/2) \|f\|_{L_k^2}^2, \tag{1.2}$$

where $\gamma(k)$ is the index of k given by (2.1).

One way one may hope to overcome the lack of localization is to use the windowed Fourier transform, also known as the (continuous) Gabor transform, or the short-time Fourier transform. To be more precise, fix $g \in L^2(\mathbb{R}^d)$, a nonzero window function, and define for $f \in L^2(\mathbb{R}^d)$ its windowed Fourier transform with respect to the window g as

$$(1.3) \qquad \mathcal{F}_g(f)(x,\xi) = \mathcal{F}[f\,\overline{g(\cdot - x)}](\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(t)\overline{g(t-x)\mathcal{E}^{\mathrm{i}\langle t,\xi\rangle}}\,\mathrm{d}t.$$

In quantum mechanics and in signal analysis, uncertainty principles for the windowed Fourier transform are often discussed for simultaneous time-frequency representations on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (the so-called phase space or time-frequency plane), see for example [1], [3], [9], [19] and the references therein. The most famous of them is the following sharp Heisenberg type uncertainty inequality (see [1], Theorem 5.1):

$$(1.4) \qquad \||x|\mathcal{F}_g(f)\|_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\,\||\xi|\mathcal{F}_g(f)\|_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \geqslant C(d)\|g\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

In the present paper we are interested in proving an analogue of Heisenberg's inequality (1.4) for the Dunkl-Gabor transform introduced in [13], [14]. Precisely, we define the translation operator by

$$(1.5) \qquad \tau_x f = \mathcal{F}_D^{-1}[\mathcal{K}(\mathrm{i}x,\cdot)\mathcal{F}_D(f)]$$

and the modulation operator by

$$(1.6) \qquad \mathcal{M}_\xi g := \mathcal{F}_D\Big(\sqrt{\tau_\xi|g|^2}\Big).$$

Then for any nonzero radial window function $g \in L^2_k(\mathbb{R}^d)$, the Dunkl-Gabor transform of any signal $f \in L^2_k(\mathbb{R}^d)$ with respect to the window g is given by

$$(1.7) \qquad \mathcal{G}_g^D(f)(x,\xi) = \int_{\mathbb{R}^d} f(s)\overline{\tau_{-x}\mathcal{M}_\xi g(s)}\,\mathrm{d}\mu_k(s), \quad (x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Let us now be more precise and describe our results. To do so, we need to introduce some other notation. Throughout this paper, $L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$ will be the subspace of radial functions of $L^2_k(\mathbb{R}^d)$ and for $1 \leqslant p < \infty$, we denote by $L^p_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ the Banach space consisting of measurable functions F on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ equipped with the norms

$$\|F\|_{L^p_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \left(\iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |F(x,\xi)|^p\,\mathrm{d}\nu_k(x,\xi)\right)^{1/p},$$

where $\mathrm{d}\nu_k(x,\xi) = \mathrm{d}\mu_k(x)\,\mathrm{d}\mu_k(\xi)$.

Our main concern here is an uncertainty inequality like (1.4), which states in particular that if one concentrates $\mathcal{G}_g^D(f)$ in time (with respect to the x-variable), then one looses concentration in frequency (with respect to the ξ-variable). In other words, we are interested in the following adaptation of a well-known notion from Fourier analysis:

Definition 1.1. Let $0<\varepsilon<1$ and $s>0$. Let $f\in L^2_k(\mathbb{R}^d)$, $g\in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$ be two nonzero functions and Σ a measurable subset of $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$. Then:

(1) We say that $\mathcal{G}^D_g(f)$ is ε-time-concentrated of magnitude s around $x=0$, if

$$\||x|^s\,\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\leqslant\varepsilon\|f\|_{L^2_k}\|g\|_{L^2_k}. \tag{1.8}$$

(2) We say that $\mathcal{G}^D_g(f)$ is ε-frequency-concentrated of magnitude s around $\xi=0$, if

$$\||\xi|^s\,\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\leqslant\varepsilon\|f\|_{L^2_k}\|g\|_{L^2_k}. \tag{1.9}$$

(3) We say that $\mathcal{G}^D_g(f)$ is ε-time-frequency-concentrated of magnitude s around $(x,\xi)=(0,0)$, if

$$\||(x,\xi)|^s\,\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\leqslant\varepsilon\|f\|_{L^2_k}\|g\|_{L^2_k}. \tag{1.10}$$

(4) We say that $\mathcal{G}^D_g(f)$ is ε-time-frequency-concentrated on Σ, if

$$\|\mathcal{G}^D_g(f)\|_{L^2(\Sigma^c,\nu_k)}\leqslant\varepsilon\|f\|_{L^2_k}\|g\|_{L^2_k}, \tag{1.11}$$

where $\Sigma^c=(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)\setminus\Sigma$.

If we take $\varepsilon=0$ in inequality (1.11), then Σ will be the exact support of $\mathcal{G}^D_g(f)$, so that when $0<\varepsilon<1$, inequality (1.11) means that $\mathcal{G}^D_g(f)$ is "practically zero" outside Σ. Indeed Σ may be viewed as the "essential" support of $\mathcal{G}^D_g(f)$.

Our main result will be the following Heisenberg-type uncertainty inequality for the Dunkl-Gabor transform:

Theorem A. *Let $s>0$. Then there is a constant $c(k,s)$ such that, for every $f\in L^2_k(\mathbb{R}^d)$ and $g\in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$,*

$$\||x|^s\,\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\,\||\xi|^s\,\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\geqslant c(k,s)\|f\|^2_{L^2_k}\|g\|^2_{L^2_k}. \tag{1.12}$$

This theorem implies in particular that, if $\mathcal{G}^D_g(f)$ is ε-time-concentrated around $x=0$, then $\mathcal{G}^D_g(f)$ cannot be ε-frequency-concentrated around $\xi=0$.

As a side result we prove the following Benedicks-type uncertainty principle for the Dunkl-Gabor transform:

Theorem B. *Let $a, b > 0$. Let $g \in L^2_k(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a nonzero radial window function such that $\operatorname{supp} g \subset \mathcal{B}(0,a)$ and let $\Sigma = S \times \mathcal{B}(0,b) \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$. Then there exists a constant $C_k(\Sigma) > 0$ such that, for all functions $f \in L^2_k(\mathbb{R}^d)$*

$$\|f\|_{L^2_k} \|g\|_{L^2_k} \leqslant C_k(\Sigma) \|\mathcal{G}^D_g(f)\|_{L^2(\Sigma^c, \nu_k)}. \tag{1.13}$$

In particular, if $\operatorname{supp} \mathcal{G}^D_g(f)$ is supported in Σ, then f is necessarily the zero function.

The rest of the paper is organized as follows. The next section is devoted to some preliminaries on the Dunkl-Gabor transform. In Section 3, we prove the Heisenberg uncertainty inequality for the Dunkl-Gabor transform and in Section 4 we prove our Benedicks-type uncertainty principle.

2. Preliminaries

2.1. The Dunkl transform and Dunkl translation. Let us fix some notation and present some necessary material on the Dunkl transform. Let G be a finite reflection group on $\mathbb{R}^d$, associated with a root system R and the positive subsystem R_+ of R (see [2], [5], [17]). We denote by k a nonnegative multiplicity function defined on R with the property that k is G-invariant. We associate with k the index

$$\gamma := \gamma(k) = \sum_{\xi \in R_+} k(\xi) \geqslant 0 \tag{2.1}$$

and the weight function w_k defined by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x \rangle|^{2k(\xi)}.$$

Further we introduce the Mehta-type constant c_k by

$$c_k = \left(\int_{\mathbb{R}^d} \mathcal{E}^{-|x|^2/2} \, \mathrm{d}\mu_k(x) \right)^{-1}.$$

Moreover,

$$\int_{\mathbb{S}^{d-1}} w_k(x) \, \mathrm{d}\sigma(x) = \frac{c_k^{-1}}{2^{\gamma + d/2 - 1} \Gamma(\gamma + d/2)} = d_k.$$

By using the homogeneity of w_k it is shown in [17] that for a radial function $f \in L^1(\mathbb{R}^d, \mu_k)$ the function $\tilde{f}$ defined on $\mathbb{R}^+$ by $f(x) = \tilde{f}(|x|)$ for all $x \in \mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}\,\mathrm{d}r$. More precisely,

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) w_k(x)\,\mathrm{d}x &= \int_{\mathbb{R}^+} \Big(\int_{\mathbb{S}^{d-1}} w_k(ry)\,\mathrm{d}\sigma(y) \Big) \tilde{f}(r) r^{d-1}\,\mathrm{d}r \\ &= d_k \int_{\mathbb{R}^+} \tilde{f}(r) r^{2\gamma+d-1}\,\mathrm{d}r. \end{aligned} \tag{2.2}$$

Introduced by C. F. Dunkl in [6], the Dunkl operators T_j, $1 \leqslant j \leqslant d$ on $\mathbb{R}^d$ associated with the reflection group G and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j} + \sum_{\xi \in R_+} k(\xi)\xi_j \frac{f(x) - f(\sigma_\xi(x))}{\langle \xi, x \rangle}, \quad x \in \mathbb{R}^d,$$

where f is an infinitely differentiable function on $\mathbb{R}^d$, $\xi_j = \langle \xi, e_j \rangle$, $(e_1, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$, and σ_ξ denotes the reflection with respect to the hyperplane orthogonal to ξ.

The Dunkl kernel $\mathcal{K}$ on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C. F. Dunkl in [5]. For $\xi \in \mathbb{R}^d$ the function $x \mapsto \mathcal{K}(x, \xi)$ can be viewed as the solution on $\mathbb{R}^d$ of the initial problem

$$T_j u(x, \xi) = \xi_j u(x, \xi), \quad 1 \leqslant j \leqslant d; \quad u(0, \xi) = 1.$$

Therefore, for all $\lambda \in \mathbb{C}$, z, $z' \in \mathbb{C}^d$ and x, $\xi \in \mathbb{R}^d$

$$\mathcal{K}(z, z') = \mathcal{K}(z', z), \quad \mathcal{K}(\lambda z, z') = \mathcal{K}(z, \lambda z'), \quad \overline{\mathcal{K}(-\mathrm{i}\xi, x)} = \mathcal{K}(\mathrm{i}\xi, x), \quad |\mathcal{K}(-\mathrm{i}\xi, x)| \leqslant 1.$$

According to [2], [17] we have for all $f \in L_k^1(\mathbb{R}^d)$,

$$\|\mathcal{F}_D(f)\|_\infty \leqslant c_k \|f\|_{L_k^1}, \tag{2.3}$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm and $L^\infty(\mathbb{R}^d)$ denotes the usual space of essentially bounded functions. Moreover, the Dunkl transform $\mathcal{F}_D$ extends uniquely to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$:

$$\|\mathcal{F}_D(f)\|_{L_k^2} = \|f\|_{L_k^2} \quad \text{and} \quad \mathcal{F}_D^{-1}(f)(\xi) = \mathcal{F}_D(f)(-\xi).$$

The Dunkl translation operator $f \to \tau_x f$ is defined on $L_k^2(\mathbb{R}^d)$ by

$$\tau_x f = \mathcal{F}_D^{-1}[\mathcal{K}(\mathrm{i}x, \cdot)\mathcal{F}_D(f)]. \tag{2.4}$$

The function $\tau_x f$ belongs to $L^2_k(\mathbb{R}^d)$ and we have

$$\|\tau_x f\|_{L^2_k} \leqslant \|f\|_{L^2_k}.$$

Note also that if f is supported in $\mathcal{B}_r \subset \mathbb{R}^d$, the ball of center 0 and radius r, then $\tau_x f$ is supported in $\mathcal{B}_{r+x}$.

The Dunkl convolution $f *_D g$ of two functions f and g is defined by

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-t) g(t) \, \mathrm{d}\mu_k(t) = g *_D f(x), \quad x \in \mathbb{R}^d.$$

Let $1 \leqslant p, q, r \leqslant \infty$ be such that $1/p + 1/q - 1 = 1/r$. If $f \in L^p_k(\mathbb{R}^d)$ and $g \in L^q_{k,\mathrm{rad}}(\mathbb{R}^d)$, then $f *_D g \in L^r_k(\mathbb{R}^d)$ and

$$\|f *_D g\|_{L^r_k} \leqslant \|f\|_{L^p_k} \|g\|_{L^q_k}.$$

In particular, if $f \in L^1_k(\mathbb{R}^d)$ and $g \in L^2_k(\mathbb{R}^d)$, then $f *_D g \in L^2_k(\mathbb{R}^d)$ and

$$\mathcal{F}_D(f *_D g) = \mathcal{F}_D(f)\mathcal{F}_D(g). \tag{2.5}$$

Moreover, for $f, g \in L^2_k(\mathbb{R}^d)$ the function $f *_D g$ belongs to $L^2_k(\mathbb{R}^d)$ if and only if the function $\mathcal{F}_D(f)\mathcal{F}_D(g)$ belongs to $L^2_k(\mathbb{R}^d)$ and then (2.5) holds.

2.2. The Dunkl-Gabor transform. Following [13], [14] for every radial function $g \in L^2_k(\mathbb{R}^d)$ the modulation of g by $\xi \in \widehat{\mathbb{R}}^d$ is defined by

$$\mathcal{M}_\xi g := g_\xi := \mathcal{F}_D\Big(\sqrt{\tau_\xi |g|^2}\Big). \tag{2.6}$$

Then

$$\|g_\xi\|_{L^2_k} = \|g\|_{L^2_k}. \tag{2.7}$$

For $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\xi \in \widehat{\mathbb{R}}^d$, we consider the family $g_{x,\xi}$ defined by:

$$g_{x,\xi} = \tau_{-x} g_\xi.$$

Then, for any function $f \in L^2_k(\mathbb{R}^d)$, we define its Dunkl-Gabor transform with respect to the "window" g by

$$\mathcal{G}^D_g(f)(x,\xi) = \int_{\mathbb{R}^d} f(s) \overline{g_{x,\xi}(s)} \, \mathrm{d}\mu_k(s), \quad (x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,$$

which can also be written in the form

$$\mathcal{G}^D_g(f)(x,\xi) = f *_D \overline{\mathcal{F}_D^{-1}\Big(\sqrt{\tau_\xi |g|^2}\Big)}(x). \tag{2.8}$$

The Dunkl-Gabor transform possesses the following properties (see [14]).

Proposition 2.1. *Let $g \in L^2_k(\mathbb{R}^d)$ be a nonzero radial function. Then we have:*

(1) A Plancherel's formula: *For every $f \in L^2_k(\mathbb{R}^d)$,*

$$\|\mathcal{G}_g^D(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|f\|_{L^2_k}\|g\|_{L^2_k}. \tag{2.9}$$

(2) *For every $f \in L^2_k(\mathbb{R}^d)$,*

$$\|\mathcal{G}_g^D(f)\|_{\infty} \leqslant \|f\|_{L^2_k}\|g\|_{L^2_k}. \tag{2.10}$$

2.3. The dilation operator. For $\lambda > 0$, we define the dilation operator δ_λ on $L^2_k(\mathbb{R}^d)$ by

$$\delta_\lambda f(x) = \frac{1}{\lambda^{\gamma+d/2}} f\Big(\frac{x}{\lambda}\Big).$$

Then we have immediately the following properties:

(1) $\delta_{1/\lambda}\delta_\lambda f = \delta_\lambda\delta_{1/\lambda} f$;
(2) $\|\delta_\lambda f\|_{L^2_k} = \|f\|_{L^2_k}$;
(3) $\mathcal{F}_D\delta_\lambda = \delta_{1/\lambda}\mathcal{F}_D$;
(4) $\delta_\lambda|f|^2 = \lambda^{\gamma+d/2}|\delta_\lambda f|^2$;
(5) $\sqrt{\delta_\lambda|f|} = \lambda^{2\gamma+d/4}\delta_\lambda\sqrt{|f|}$;
(6) $\tau_x\delta_\lambda = \delta_\lambda\tau_{x/\lambda}$.

From this we deduce the following lemma:

Lemma 2.2. *Let $\lambda > 0$ and let $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$ be a nonzero window function. Then for every $f \in L^2_k(\mathbb{R}^d)$ and $(x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$,*

$$\mathcal{G}^D_{\delta_{\lambda^{-1}}g}(\delta_\lambda f)(x,\xi) = \mathcal{G}_g^D(f)\Big(\frac{x}{\lambda}, \lambda\xi\Big). \tag{2.11}$$

P r o o f. First, we have for every $\xi \in \widehat{\mathbb{R}}^d$ we have

$$\begin{aligned}\mathcal{F}_D^{-1}\Big(\sqrt{\tau_\xi|\delta_{\lambda^{-1}}g|^2}\Big) &= \mathcal{F}_D^{-1}\Big(\sqrt{\lambda^{(\gamma+d/2)}\tau_\xi\delta_{\lambda^{-1}}|g|^2}\Big) = \mathcal{F}_D^{-1}\Big(\lambda^{(2\gamma+d)/4}\sqrt{\delta_{\lambda^{-1}}\tau_{\lambda\xi}|g|^2}\Big)\\ &= \mathcal{F}_D^{-1}\Big(\delta_{\lambda^{-1}}\sqrt{\tau_{\lambda\xi}|g|^2}\Big) = \delta_\lambda\mathcal{F}_D^{-1}\Big(\sqrt{\tau_{\lambda\xi}|g|^2}\Big).\end{aligned}$$

Thus for every $(x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we have

$$\begin{aligned}
\mathcal{G}^D_{\delta_{\lambda^{-1}}g}(\delta_\lambda f)(x,\xi) &= \delta_\lambda f *_D \overline{\mathcal{F}_D^{-1}\Big(\sqrt{\tau_\xi|\delta_{\lambda^{-1}}g|^2}\Big)}(x) \\
&= \int_{\mathbb{R}^d} \tau_x \delta_\lambda f(-t)\overline{\delta_\lambda \mathcal{F}_D^{-1}\Big(\sqrt{\tau_{\lambda\xi}|g|^2}\Big)(t)}\,\mathrm{d}\mu_k(t) \\
&= \int_{\mathbb{R}^d} \delta_\lambda \tau_{x/\lambda} f(-t)\overline{\delta_\lambda \mathcal{F}_D^{-1}\Big(\sqrt{\tau_{\lambda\xi}|g|^2}\Big)(t)}\,\mathrm{d}\mu_k(t) \\
&= \frac{1}{\lambda^{2\gamma+d}} \int_{\mathbb{R}^d} \tau_{x/\lambda} f\Big(-\frac{t}{\lambda}\Big)\overline{\mathcal{F}_D^{-1}\Big(\sqrt{\tau_{\lambda\xi}|g|^2}\Big)\Big(\frac{t}{\lambda}\Big)}\,\mathrm{d}\mu_k(t).
\end{aligned}$$

Now by a change of variable $t = \lambda s$, we get the desired result. $\square$

3. A Heisenberg-type uncertainty inequality for the Dunkl-Gabor transform

First we will recall the following theorem which limits the concentration of the Dunkl-Gabor transform in any small set. This result can be found in [14], Theorem 5.1, or [13], Theorem 4.4. Nonetheless, we can deduce this result easily from (2.9) and (2.10).

Theorem 3.1. *Let $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be such that $0 < \nu_k(\Sigma) < 1$. Then for all $f \in L^2_k(\mathbb{R}^d)$ and $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$,*

$$\|f\|_{L^2_k}\|g\|_{L^2_k} \leqslant \frac{1}{\sqrt{1-\nu_k(\Sigma)}}\|\mathcal{G}^D_g(f)\|_{L^2(\Sigma^c,\nu_k)}. \tag{3.1}$$

P r o o f. From Plancherel's theorem (2.9) we have

$$\|f\|^2_{L^2_k}\|g\|^2_{L^2_k} = \|\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|\mathcal{G}^D_g(f)\|^2_{L^2(\Sigma,\nu_k)} + \|\mathcal{G}^D_g(f)\|^2_{L^2(\Sigma^c,\nu_k)}. \tag{3.2}$$

Now by (2.10),

$$\|\mathcal{G}^D_g(f)\|^2_{L^2(\Sigma,\nu_k)} \leqslant \nu_k(\Sigma)\|\mathcal{G}^D_g(f)\|^2_{\infty} \leqslant \nu_k(\Sigma)\|f\|^2_{L^2_k}\|g\|^2_{L^2_k}. \tag{3.3}$$

Thus the result follows immediately by integrating (3.3) in (3.2). $\square$

In particular, if $\mathcal{G}^D_g(f)$ is supported in Σ, then $f \equiv 0$ or $g \equiv 0$. On the other hand, Theorem 3.1 implies the following version of Heisenberg uncertainty inequality for the Dunkl-Gabor transform.

Corollary 3.2. *Let $s > 0$. Then there exists a constant $c_{k,s} > 0$ such that, for all $f \in L^2_k(\mathbb{R}^d)$ and $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$,*

$$\||(x,\xi)|^s \mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \geqslant c_{k,s}\|g\|_{L^2_k}\|f\|_{L^2_k}. \tag{3.4}$$

Proof. Let $0 < r \leqslant 1$ be a real number and $\mathcal{B}_r = \{(x,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\colon\ |(x,\xi)| < r\}$ the ball of center 0 and radius r in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Fix $0 < r_0 \leqslant 1$ small enough such that $\nu_k(\mathcal{B}_{r_0}) < 1$. Therefore by inequality (3.1) we obtain

$$\begin{aligned}\|f\|^2_{L^2_k}\|g\|^2_{L^2_k} &\leqslant \frac{1}{r_0^{2s}(1-\nu_k(\mathcal{B}_{r_0}))}\int_{|(x,\xi)|\geqslant r} r_0^{2s}\,|\mathcal{G}^D_g(f)(x,\xi)|^2\,\mathrm{d}\nu_k(x,\xi)\\ &\leqslant \frac{1}{r_0^{2s}(1-\nu_k(\mathcal{B}_{r_0}))}\int_{|(x,\xi)|\geqslant r_0} |(x,\xi)|^{2s}\,|\mathcal{G}^D_g(f)(x,\xi)|^2\,\mathrm{d}\nu_k(x,\xi)\\ &\leqslant \frac{1}{r_0^{2s}(1-\nu_k(\mathcal{B}_{r_0}))}\||(x,\xi)|^s\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}.\end{aligned}$$

This allows to conclude with $c_{k,s} = r_0^s(1-\nu_k(\mathcal{B}_r))^{1/2}$. $\square$

Corollary 3.3. *Let $s > 0$. Then the following uncertainty inequalities hold.*

(1) A Heisenberg-type uncertainty inequality for the Dunkl-Gabor transform: *There exists a constant $c(k,s) > 0$ such that, for all $f \in L^2_k(\mathbb{R}^d)$ and $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$*

$$\||x|^s\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\,\||\xi|^s\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \geqslant c(k,s)\|g\|^2_{L^2_k}\|f\|^2_{L^2_k}. \tag{3.5}$$

(2) A local uncertainty inequality for the Dunkl-Gabor transform: *There exists a constant $c(s,k) > 0$ such that for every $f \in L^2_k(\mathbb{R}^d)$, $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d)$ and every measurable subset Σ of finite measure, $0 < \nu_k(\Sigma) < \infty$*

$$\|\mathcal{G}^D_g(f)\|_{L^2(\Sigma,\nu_k)} \leqslant c(s,k)[\nu_k(\Sigma)]^{1/2}\||(x,\xi)|^s\mathcal{G}^D_g(f)\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}. \tag{3.6}$$

Proof. From the fact that $|a+b|^s \leqslant 2^s(|a|^s+|b|^s)$, we deduce by inequality (3.4) that

$$\||x|^s\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} + \||\xi|^s\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \geqslant \frac{c^2_{k,s}}{2^{2s}}\|g\|^2_{L^2_k}\|f\|^2_{L^2_k}.$$

Replacing f and g by $\delta_\lambda f$ and $\delta_{\lambda^{-1}}g$, respectively, in the previous inequality, we obtain by (2.11) and by a suitable change of variables:

$$\lambda^{2s}\||x|^s\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} + \lambda^{-2s}\||\xi|^s\mathcal{G}^D_g(f)\|^2_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \geqslant \frac{c^2_{k,s}}{2^{2s}}\|g\|^2_{L^2_k}\|f\|^2_{L^2_k}.$$

Then (3.5) follows by minimizing the left hand side of that inequality over $\lambda > 0$.

On the other hand, as

$$\|\mathcal{G}_g^D(f)\|_{L^2(\Sigma,\nu_k)} \leqslant [\nu_k(\Sigma)]^{1/2}\|\mathcal{G}_g^D(f)\|_\infty,$$

then from inequality (2.10), we obtain

$$\|\mathcal{G}_g^D(f)\|_{L^2(\Sigma,\nu_k)} \leqslant [\nu_k(\Sigma)]^{1/2}\|g\|_{L_k^2}\|f\|_{L_k^2}.$$

Thus from inequality (3.4), we deduce the desired result. □

Inequality (3.6) is known as the local uncertainty inequality which extends a result of Faris [7]. It implies, in particular, that if the Dunkl-Gabor transform is ε-time-frequency-concentrated of magnitude s around zero, then it cannot be ε-time-frequency-concentrated in the subset Σ of finite measure but it disperses in Σ^c.

4. Concentration in sets of finite measures

We introduce a pair of orthogonal projections on $L_k^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. The first, denoted P_g, is the orthogonal projection from $L_k^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ onto $\mathcal{G}_g^D[L_k^2(\mathbb{R}^d)]$ and the other is the time-frequency limiting operator defined by

$$P_\Sigma F = F\chi_\Sigma; \quad F \in L_k^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d),$$

where $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a subset of finite measure $0 < \nu_k(\Sigma) < \infty$ and χ_Σ denotes the characteristic function of Σ.

4.1. Benedicks-type uncertainty principle

Definition 4.1. Let Σ a measurable subset of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and $g \in L_k^2(\mathbb{R}^d)$ a nonzero radial window function. Then:

(1) We say that Σ is weakly annihilating, if any function $f \in L_k^2(\mathbb{R}^d)$ vanishes when its Dunkl-Gabor transform $\mathcal{G}_g^D(f)$ with respect to the window g is supported in Σ.

(2) We say that Σ is strongly annihilating, if there exists a constant $C_k(\Sigma) > 0$ such that for every function $f \in L_k^2(\mathbb{R}^d)$,

$$\|f\|_{L_k^2}\|g\|_{L_k^2} \leqslant C_k(\Sigma)\|\mathcal{G}_g^D(f)\|_{L^2(\Sigma^c,\nu_k)}. \tag{4.1}$$

The constant $C_k(\Sigma)$ will be called the annihilation constant of Σ.

Of course, every strongly annihilating set is also a weakly one and from Theorem 3.1, we see that any set $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with $0 < \nu_k(\Sigma) < 1$ is strongly annihilating.

Now let $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$, then from [13], (33), $P_\Sigma P_g$ is Hilbert-Schmidt, since

$$\|P_\Sigma P_g\|_{HS} \leqslant \sqrt{\nu_k(\Sigma)}. \tag{4.2}$$

According to [10], I.1.3.2.A, page 90, if $P_\Sigma P_g$ is compact (in particular if $P_\Sigma P_g$ is Hilbert-Schmidt), then if Σ is weakly annihilating, it is also strongly annihilating (see also [13], Theorem 4.5). Moreover, we will recall the following well-known lemma (see e.g. [10], page 90, and [11], Proposition 5.1.2).

Lemma 4.2. *Let g be a nonzero radial window function. Then:*

(1) *If $\|P_\Sigma P_g\| := \|P_\Sigma P_g\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)\to L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} < 1$, then for all $f \in L^2_k(\mathbb{R}^d)$,*

$$\|f\|_{L^2_k}\|g\|_{L^2_k} \leqslant \frac{1}{\sqrt{1-\|P_\Sigma P_g\|^2}}\|\mathcal{G}^D_g(f)\|_{L^2(\Sigma^c,\nu_k)}. \tag{4.3}$$

(2) *If Σ is strongly annihilating, then $\|P_\Sigma P_g\| < 1$.*

In this section we will prove that any subset Σ of the form $\Sigma = S \times \mathcal{B}_R \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with $0 < \mu_k(S) < \infty$ is weakly annihilating (and then strongly annihilating).

We denote by $\operatorname{Im}\mathcal{P}$ the range of a linear operator $\mathcal{P}$. Then we have the following lemma.

Lemma 4.3. *Let $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$ and let $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a nonzero window function. Then*

$$\dim(\operatorname{Im} P_g \cap \operatorname{Im} P_\Sigma) \leqslant [\nu_k(\Sigma)]^2 < \infty. \tag{4.4}$$

Proof. This follows from [13], Proposition 4.3, and [19], Lemma 3.1. □

Theorem 4.4 (Benedicks-type uncertainty principle for $\mathcal{G}^D_g$). *Let $r, R > 0$. Let $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a nonzero window function such that $\operatorname{supp} g \subset \mathcal{B}_r$ and let $\Sigma = S \times \mathcal{B}_R \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$. Then*

$$\operatorname{Im} P_g \cap \operatorname{Im} P_\Sigma = \{0\}, \tag{4.5}$$

i.e., Σ is weakly annihilating.

P r o o f. Let $F \in \operatorname{Im} P_g \cap \operatorname{Im} P_\Sigma$, then there exists a function $f \in L^2_k(\mathbb{R}^d)$ such that $F = \mathcal{G}_g^D(f)$ and $\operatorname{supp} F \subset \Sigma$. Let $\xi \in \mathcal{B}(0,R)$ and let $\Phi_{\xi,\, g}$ be the function defined on $\mathbb{R}^d$ by

$$\Phi_{\xi,\, g}(t) = \mathcal{F}_D(f)(-t)\sqrt{\tau_\xi |g|^2(t)}.$$

Then for all $(x,\xi) \in \Sigma$,

$$F(x,\xi) = \mathcal{F}_D(\Phi_{\xi,\, g})(x). \tag{4.6}$$

Thus $\operatorname{supp} \mathcal{F}_D(\Phi_{\xi,\, g}) \subset S$, with $\mu_k(S) < \infty$.

On the other hand, as $\operatorname{supp} g \subset \mathcal{B}(0,r)$, we have

$$\operatorname{supp} \Phi_{\xi,\, g} \subset \operatorname{supp} \tau_\xi |g|^2 \subset \mathcal{B}_{R+r}.$$

Hence by the Benedicks theorem for the Dunkl transform [8], Theorem 4.4 (2), we deduce that $\Phi_{\xi,\, g} \equiv 0$, and then $F \equiv 0$. $\square$

Consequently, we obtain the following improvement.

Corollary 4.5. *Let $r, R > 0$. Let $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a nonzero window function such that $\operatorname{supp} g \subset \mathcal{B}_r$ and let $\Sigma = S \times \mathcal{B}_R \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$. Then there exists a constant $C_k(\Sigma) > 0$ such that, for all functions $f \in L^2_k(\mathbb{R}^d)$,*

$$\|f\|_{L^2_k} \|g\|_{L^2_k} \leqslant C_k(\Sigma) \|\mathcal{G}_g^D(f)\|_{L^2(\Sigma^c, \nu_k)}. \tag{4.7}$$

4.2. Application: Stable reconstruction from incomplete noisy data

Now we will derive a sufficient condition by means of which one can recover a signal $F \in L^2_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ from the knowledge of a truncated version of it, following the Donoho-Stark criterion [4].

Let g be a nonzero radial window function. A signal $F \in L^2_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is transmitted to a receiver who knows that $F \in \mathcal{G}_g^D[L^2_k(\mathbb{R}^d)]$. Suppose that the observation of F is corrupted by a noise $n \in L^2_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ (which is nonetheless assumed to be small) and unregistered values on $\Sigma \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Thus, the observable function r satisfies

$$r(x) = \begin{cases} F(x) + n(x), & x \in \Sigma^c; \\ 0, & x \in \Sigma. \end{cases} \tag{4.8}$$

Here we have assumed without loss of generality that $n = 0$ on Σ. Equivalently,

$$r = (I - P_\Sigma)F + n. \tag{4.9}$$

We say that F can be stably reconstructed from r, if there exists a linear operator

$$K_{\Sigma,g} : L^2_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \to L^2_k(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$$

and a constant $C_{g,\Sigma}$ such that

$$\|F - K_{\Sigma,g}\, r\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \leqslant C_{\Sigma,g}\|n\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}. \tag{4.10}$$

The estimate (4.10) shows that the noise n is at most amplified by a factor $C_{\Sigma,g}$.

Theorem 4.6. *Let $g \in L^2_{k,\mathrm{rad}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a nonzero window function such that $\operatorname{supp} g \subset \mathcal{B}_r$ and let $\Sigma = S \times \mathcal{B}_R \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a subset of finite measure $0 < \nu_k(\Sigma) < \infty$. Then F can be stably reconstructed from r. The constant $C_{\Sigma,g}$ in (4.10) is not larger than $(1 - \|P_\Sigma P_g\|)^{-1}$.*

Proof. From Corollary 4.5, Σ is strongly annihilating, hence from Lemma 4.2 we have $\|P_\Sigma P_g\| < 1$. Therefore $I - P_\Sigma P_g$ is invertible. Let

$$K_{g,\Sigma} = (I - P_\Sigma P_g)^{-1}.$$

Since $F \in \mathcal{G}^D_g[L^2_k(\mathbb{R}^d)]$, we have $(I - P_\Sigma)F = (I - P_\Sigma P_g)F$. Hence

$$\begin{aligned} F - K_{\Sigma,g}\, r &= F - K_{\Sigma,g}((I - P_\Sigma)F + n) = F - K_{\Sigma,g}(I - P_\Sigma P_g)F - K_{\Sigma,g}n \\ &= F - (I - P_\Sigma P_g)^{-1}(I - P_\Sigma P_g)F - K_{\Sigma,g}n = -K_{\Sigma,g}n. \end{aligned}$$

So

$$\begin{aligned} \|F - K_{\Sigma,g}r\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} &= \|K_{\Sigma,g}n\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \leqslant \|(I - P_\Sigma P_g)^{-1}\|\|n\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \\ &\leqslant \sum_{k=0}^{\infty} \|P_\Sigma P_g\|^k \|n\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = (1 - \|P_\Sigma P_g\|)^{-1}\|n\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}, \end{aligned}$$

which allows to conclude the proof. □

Remark 4.7. Since $\|P_\Sigma P_g\| \leqslant \|P_\Sigma P_g\|_{HS}$, one can deduce from inequality (4.2) that for any subset Σ with $0 < \nu_k(\Sigma) < 1$, any signal F can be also stably reconstructed from r, and $C_{\Sigma,g}$ is not larger than $\big(1 - \sqrt{\nu_k(\Sigma)}\big)^{-1}$.

Remark 4.8 (An algorithm for computing $K_{\Sigma,g}\, r$)**.** The so-called Neumann series $K = \sum\limits_{k=0}^{\infty}(P_\Sigma P_g)^k$ suggests the following algorithm for computing Kr. Put

$$F^{(n)} = \sum_{k=0}^{n}(P_\Sigma P_g)^k r,$$

then

$$F^{(0)} = r, \quad F^{(n+1)} = r + P_\Sigma P_g F^{(n)} \quad \text{and} \quad F^{(n)} \longrightarrow K_{\Sigma,g}\, r \quad \text{as } n \to \infty.$$

As $F = P_g F$ we deduce that

$$F^{(n+1)} - F = P_\Sigma P_g (F^{(n)} - F). \tag{4.11}$$

So that, if Σ is strongly annihilating, then by virtue of (4.11), the following error estimate holds:

$$\|F - F^{(n)}\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \leqslant \|P_\Sigma P_g\|^n \|F - r\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}, \tag{4.12}$$

and particularly, if $\nu_k(\Sigma) < 1$, then (4.2) yields

$$\|F - F^{(n)}\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \leqslant [\nu_k(\Sigma)]^{n/2} \|F - r\|_{L^2_k(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}. \tag{4.13}$$

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