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ON SOME FREE SEMIGROUPS, GENERATED BY MATRICES

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Abstract. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}.$$

We call a complex number λ "semigroup free" if the semigroup generated by A and B_λ is free and "free" if the group generated by A and B_λ is free.

First families of semigroup free λ's were described by J. L. Brenner, A. Charnow (1978). In this paper we enlarge the set of known semigroup free λ's. To do it, we use a new version of "Ping-Pong Lemma" for semigroups embeddable in groups. At the end we present most of the known results related to semigroup free and free numbers in a common picture.

Keywords: free semigroup; semigroup of matrices

MSC 2010: 20M05, 20E05

1. Introduction

Let λ be any complex number and let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}, \quad C_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let X be a subset of any group. By $\mathrm{gp}(X)$ we mean a group generated by X and by $\mathrm{sgp}(X)$ a semigroup generated by X.

In many papers (for example, [2], [4], [7]–[9], [11], [14]) authors found the values of λ's for which the group $\mathrm{gp}(A, B_\lambda)$ is free. The problem "when the semigroup $\mathrm{sgp}(A, B_\lambda)$ is free" is similar to "when the group $\mathrm{gp}(A, B_\lambda)$ is free" but the knowledge in this field is relatively poor.

A *number* λ is called *free* if $\mathrm{gp}(A, B_\lambda)$ is free (otherwise it is called *nonfree*). If $\mathrm{sgp}(A, B_\lambda)$ is a free semigroup then λ is called *semigroup free* (otherwise it is called *semigroup nonfree*).

Any complex number λ can be viewed as a point in the plane. J. L. Brenner and A. Charnow in [3] found a set of semigroup free λ's (Figure 1).

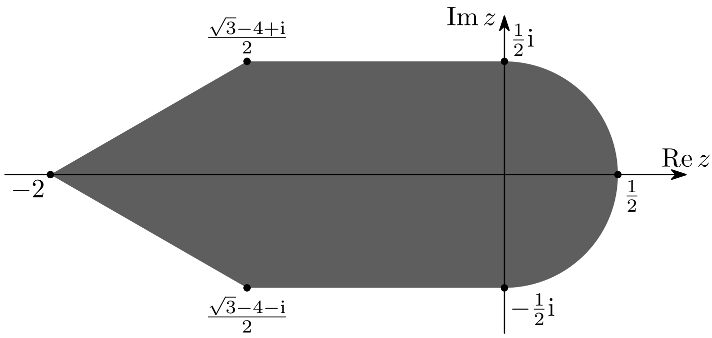

Figure 1. Points outside the grey area are free.

The authors also found some families of semigroup nonfree λ's; for example, they proved that the set of semigroup nonfree λ's is dense on the interval $(-2, 0)$ and that semigroup nonfree λ's are arbitrarily close to $1/2$.

2. Propositions

We start with some facts:

Proposition 2.1.

(i) *Let $A_1, A_2, \ldots$ be any square matrices of the same order over the same ring. If the group* $\mathrm{gp}(A_1, A_2, \ldots)$ *is free, then the semigroup* $\mathrm{sgp}(A_1, A_2, \ldots)$ *is free.*

(ii) *Let $2\lambda = \nu\mu$. Then the semigroup* $\mathrm{sgp}(A, B_\lambda)$ *is free if and only if* $\mathrm{sgp}(B_\mu, C_\nu)$ *is free.*

(iii) *The semigroup* $\mathrm{sgp}(B_\lambda, C_\lambda)$ *is free if and only if* $\mathrm{sgp}(B_\lambda, J)$ *is a free product of cyclic semigroups generated by B_λ and J.*

(iv) *Every transcendental λ is free.*

Note. (ii) and (iii) will be true if we write “group” instead of “semigroup”.

Proof. (i) is trivial. For the proof of (ii) we note that if $S := \begin{bmatrix} \nu/2 & 0 \\ 0 & 1 \end{bmatrix}$, then $S^{-1} C_\nu S = A$ and $S^{-1} B_\mu S = B_\lambda$. (iii) easily follows from the equality $B_\lambda = J C_\lambda J$. (iv) was proved in [6]. $\square$

It is known that homographic functions $f(z) = (az + b)/(cz + d)$, $a, b, c, d \in \mathbb{C}$; $ad - cb \neq 0$; $z \in \mathbb{C} \cup \{\infty\}$, which map $\mathbb{C} \cup \{\infty\}$ into $\mathbb{C} \cup \{\infty\}$ form a group under superposition $(f_1 \circ f_2)(z) = f_1(f_2(z))$ [12], which is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$; it means that there exists an epimorphism φ of $\mathrm{GL}(2,\mathbb{C})$ onto the group of homographic functions such that

$$\varphi\Bigl(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Bigr) = \Bigl(z \mapsto \frac{az+b}{cz+d}\Bigr).$$

Let $\alpha = (x \mapsto x+2)$, $\beta_\lambda = (x \mapsto x/(\lambda x+1))$, $\gamma_\lambda = (x \mapsto x+\lambda)$, $\iota = (x \mapsto 1/x)$.

Then $\varphi(A) = \alpha$, $\varphi(B_\lambda) = \beta_\lambda$, $\varphi(C_\lambda) = \gamma_\lambda$, $\varphi(J) = \iota$.

As was mentioned in [15], $\mathrm{gp}(A, B_\lambda)$ is free if and only if $\mathrm{gp}(\alpha, \beta_\lambda)$ is free. It remains true if we write "sgp" instead of "gp".

In the following text we will consider homographic functions instead of matrices.

Now let us write down some properties of homographic functions.

Proposition 2.2. *Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$, $b_1 \neq 0$ and $b_2 \neq 0$. If*

$$z = a_1 + a_2\mathrm{i} + (b_1 + b_2\mathrm{i})t, \quad t \in \mathbb{R}$$

is the parametric equation of a line in the complex plane $\mathbb{C}$, not including the origin, then the transformation $\iota\colon z \mapsto 1/z$ maps this line into a circumference (which includes origin) defined as

$$\Bigl|z + \frac{b_2 + b_1\mathrm{i}}{2(a_2b_1 - a_1b_2)}\Bigr| = \frac{|b_1 + b_2\mathrm{i}|}{2|a_2b_1 - a_1b_2|}.$$

P r o o f. The transformation $\iota\colon z \mapsto 1/z$ maps any line not including origin into a circle [13]. We will find its centre and radius.

Three points $\mathrm{i}(a_2b_1 - a_1b_2)/b_1$, $-(a_2b_1 - a_1b_2)/b_2$ and ∞ belong to the line $z = a_1 + a_2\mathrm{i} + (b_1 + b_2\mathrm{i})t$. Observe that $\iota(\mathrm{i}(a_2b_1 - a_1b_2)/b_1) = -b_1\mathrm{i}/(a_2b_1 - a_1b_2)$, $\iota(-(a_2b_1 - a_1b_2)/b_2) = -b_2/(a_2b_1 - a_1b_2)$ and $\iota(\infty) = 0$. Thus the centre of the circle is $-\frac{1}{2}(b_2 + b_1\mathrm{i})/(a_2b_1 - a_1b_2)$ and its radius is $\frac{1}{2}|b_1 + b_2\mathrm{i}|/|a_2b_1 - a_1b_2|$. $\square$

Proposition 2.3. *Let*

$$|z - a| = r, \quad |a| > r, \quad a \in \mathbb{C},\ r \in \mathbb{R}_+$$

be the equation of a circle in the complex plane $\mathbb{C}$. Then the transformation $\iota\colon z \mapsto 1/z$ maps this circle into the circle

$$\Bigl|z - \frac{\overline{a}}{|a|^2 - r^2}\Bigr| = \frac{r}{|a|^2 - r^2}.$$

P r o o f. The transformation $\iota\colon z \mapsto 1/z$ maps the circle $|z - a| = r$, $|a| > r$, into a circle [13]. We denote this circle by C.

A line passing through the origin and a includes two points of the circle $|z-a|=r$, namely $a+ra/|a|$ and $a-ra/|a|$. They are the ends of the diameter. Their images by ι are $\overline{a}/(|a|(|a|+r))$ and $\overline{a}/(|a|(|a|-r))$, respectively, and they are the ends of the diameter of C. It gives us the radius of C:

$$\frac{1}{2}\Big|\frac{\overline{a}}{|a|(|a|+r)}-\frac{\overline{a}}{|a|(|a|-r)}\Big|=\frac{r}{|a|^2-r^2}$$

and its center

$$\frac{1}{2}\Big(\frac{\overline{a}}{|a|(|a|+r)}+\frac{\overline{a}}{|a|(|a|-r)}\Big)=\frac{\overline{a}}{|a|^2-r^2}.$$

□

In [5] a version of "Ping-Pong Lemma" for semigroups was presented. If we assume that semigroups occurring in this lemma are cyclic or torsion, then this lemma remains valid even if one of the assumptions is omitted.

Lemma 2.4. *Let H_1, H_2 be both cyclic or torsion subsemigroups of the group G. Then every nontrivial relation satisfied in $\operatorname{sgp}(H_1, H_2)$ implies a relation $w_1 = w_2$ with the property: let $i \in \{1, 2\}$. If any of the words w_1 or w_2 starts (ends) with the element from the semigroup H_i then the second word starts (ends, respectively) with the element from H_{3-i}.*

Proof. Let $h_1, h_1' \in H_1$, $h, h' \in \operatorname{sgp}(H_1, H_2)$ be any nontrivial elements.

If H_1 is cyclic and $a \geqslant b$ then any relation of the form $h_1^a h = h_1^b h'$ implies $h_1^{a-b} h = h'$ and if H_1 is torsion then any relation of the form $h_1 h = h_1' h$ implies $(h_1')^{n-1} h_1 h = h'$ for some $n \in \mathbb{N}$.

Therefore $h_1 h = h_1' h'$ implies one of the relations in $\operatorname{sgp}(H_1, H_2)$:

$$h_1 h'' = h_2, \quad h_1 h'' = h_2 h''', \quad h_1 = h_2 h'''$$

for some $h'', h''' \in \operatorname{sgp}(H_1, H_2)$, $h_2, h_2' \in H_2$.

Similarly, if $h_2, h_2' \in H_2$, $h, h' \in \operatorname{sgp}(H_1, H_2)$ then any relation $hh_2 = h'h_2'$ implies one of the $h''h_2 = h_1$, $h''h_2 = h'''h_1$, $h_2 = h'''h_1$ for some $h'', h''' \in \operatorname{sgp}(H_1, H_2)$, $h_1, h_1' \in H_1$. □

Lemma 2.5 (A version of "Ping-Pong Lemma" for semigroups embeddable in groups.)**.** *Let G be a group which acts on the set X and let H_1, H_2 be both cyclic or both torsion subsemigroups of the group G. Let X_1, X_2 be two nonempty disjoint subsets of the set X such that*

(i) *for every $h_1 \in H_1$, $h_1(X_1 \cup X_2) \subset X_2$,*
(ii) *for every $h_2 \in H_2$, $h_2(X_2) \subset X_1$.*

Then the semigroup generated by H_1 and H_2 is a free product of H_1 and H_2.

P r o o f. Let $h_1, h_1' \in H_1$, $h_2, h_2' \in H_2$, $h, h' \in \operatorname{sgp}(H_1, H_2)$ be nontrivial elements. We consider seven cancelled relations:

(i) $h_1 = 1$,
(ii) $h_2 = 1$,
(iii) $h_1 h h_2 = 1$,
(iv) $h_1 h h_2 = h_2' h' h_1'$,
(v) $h_1 h h_1' = h_2$,
(vi) $h_2 h h_2' = h_1$,
(vii) $h_2 h h_2' = h_1 h' h_1'$.

Thanks to Lemma 2.4, it suffices to show that none of these relations can be satisfied.

We consider a relation in each form and show that it leads to a contradiction.

(i) $h_1(X_1) \subset X_2$ and $1(X_1) = X_1 \neq h_1(X_1)$.
(ii) $h_2(X_2) \subset X_1$ and $1(X_2) = X_2 \neq h_2(X_2)$.

Now let $h_1'' \in H_1$, $h_2'' \in H_2$ be nontrivial elements. If $h = h_2'' h_1'' h''$ for some $h'' \in \operatorname{sgp}(H_1, H_2)$ then $h_1 h h_2(X_2) = h_1 h_2'' h_1'' h'' h_2(X_2) \subset h_2'' h_1'' h'' h_2(X_2) \subset X_1$. We see now that

(iii) $1(X_2) = X_2 \neq h_1 h h_2(X_2)$;
(iv) $h_2' h' h_1'(X_2) \subset X_2$ is not equal to $h_1 h h_2(X_2)$;
(v) $h_1 h h_1'(X_2) \subset X_2$ is not equal to $h_2(X_2)$;
(vi) $h_2 h h_2'(X_2) \subset X_1$ is not equal to $h_1(X_2)$;
(vii) $h_2 h h_2'(X_2) \subset X_1$ while $h_1 h h_1'(X_2) \subset X_2$. □

Based on the proof we have an immediate corollary:

Corollary 2.6. *If H_1 and H_2 are infinite cyclic semigroups satisfying assumptions of Lemma* 2.5, *then the semigroup generated by H_1 and H_2 is free.*

Now we recall Theorem 2.4 from [3] and prove it using Lemma 2.5:

Proposition 2.7. *If* $\operatorname{Re}(\lambda) > 0$ *and* $|\lambda| > \frac{1}{2}$, *then the semigroup generated by A, B_λ is free (that is λ is semigroup free).*

P r o o f. Let

$$X_1 = \{z\colon\ 0 < \operatorname{Re}(z) \leqslant 2\}, \quad X_2 = \{z\colon\ 2 < \operatorname{Re}(z)\}, \quad H_1 = \operatorname{gp}\langle\alpha\rangle, \quad H_2 = \operatorname{gp}\langle\beta_\lambda\rangle.$$

Then for every natural m, n:

$$\alpha^m X_1 = \{z\colon\ 2m < \operatorname{Re}(z) \leqslant 2m+2\} \subset X_2,$$

so it is a vertical strip of width 2; $\alpha^m X_2 \subset X_2$ and $\beta_\lambda^n X_2$ is a circle (included in X_1), so by Corollary 2.6 the subsemigroup generated by H_1 and H_2 is free. □

3. Main theorem

We will show now that the set of semigroup free λ's, described by inequalities $\mathrm{Re}(\lambda) > 0$ and $|\lambda| > \frac{1}{2}$, can be enlarged.

We consider $\lambda' = \varepsilon\lambda$, where $0 < \varepsilon < 1$, $\mathrm{Re}(\lambda) > 0$ and $|\lambda| = \frac{1}{2}$. Without loss of generality, we assume that $\mathrm{Im}(\lambda) \geqslant 0$.

We use the construction similar to that used in [8].

We will find $\beta^n_{\lambda'}(X_2)$ (note that $\beta^n_{\lambda'} = \iota\gamma^n_{\lambda'}\iota$). By Proposition 2.2, $\iota(X_2)$ is the circle defined by $|z - \frac{1}{4}| < \frac{1}{4}$. For each natural n, $\iota\gamma^n_{\lambda'}(X_2)$ is the circle of radius $\frac{1}{4}$ and with the centre lying on the line $z = \frac{1}{4} + \lambda t$, $t \in \mathbb{R}$ (which is parallel to the line including the origin and λ). Two lines tangent to all these circles are $k\colon\ z = \frac{1}{4} + \frac{1}{2}\mathrm{i}\lambda + \lambda t$, $t \in \mathbb{R}$ and $j\colon\ z = \frac{1}{4} - \frac{1}{2}\mathrm{i}\lambda + \lambda t$, $t \in \mathbb{R}$ (see Figure 2).

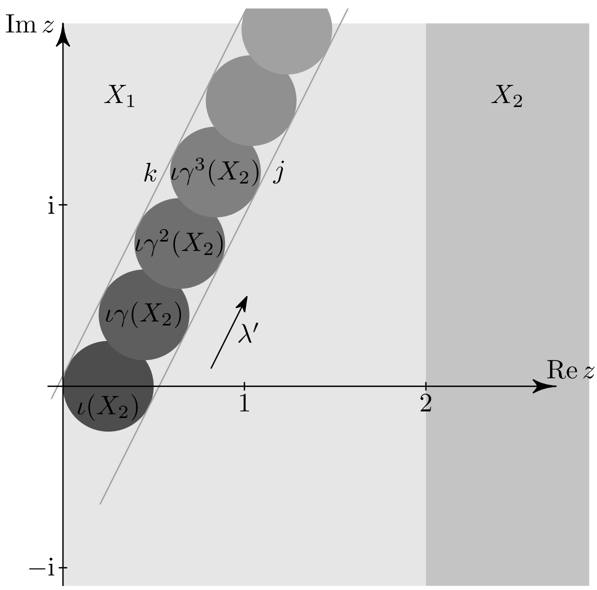

Figure 2.

Let $\mathrm{Re}(\lambda) = \lambda_1$, $\mathrm{Im}(\lambda) = \lambda_2$. By Proposition 2.2, $k' = \iota(k)$ is the circle defined by

$$\left| z + \frac{4(\lambda_2 + \lambda_1 \mathrm{i})}{1 - 2\lambda_2} \right| < \frac{2}{1 - 2\lambda_2}$$

and $j' = \iota(j)$ is a circle, too.

By Proposition 2.3, for any natural n, the images $\iota\gamma^n_{\lambda'}\iota(X_2) = \beta^n_{\lambda'}(X_2)$ are circles tangent to the circles k' and j'. Because $\varepsilon < 1$, $\beta(X_2)_{\lambda'} \subset X_1$ fails.

We define $\overline{\beta_{\lambda'}(X_2)}$ as the circle obtained from $\beta_{\lambda'}(X_2)$ by translation by the vector $[-2, 0]$ and let

$$X_1' = \frac{X_1 \cup \beta_{\lambda'}(X_2)}{\overline{\beta_{\lambda'}(X_2)}}, \quad X_2' = \frac{X_2}{\beta_{\lambda'}(X_2)}$$

(see Figure 3).

Now for any natural n, we have $\alpha^n(X_1' \cup X_2') \subset X_2'$. To use Lemma 2.5, we need only $\beta^n_{\lambda'}(X_2') \subset X_1'$ to be satisfied for any natural n. To prove that, we can observe first that this inclusion is satisfied if the circle $\overline{\beta_{\lambda'}(X_2)}$ is inside the circle k', that means, if

$$r(k') - r(\overline{\beta_{\lambda'}(X_2)}) > |s(k') - s(\overline{\beta_{\lambda'}(X_2)})|$$

where $s(C)$ is the center and $r(C)$ is the radius of the circle C.

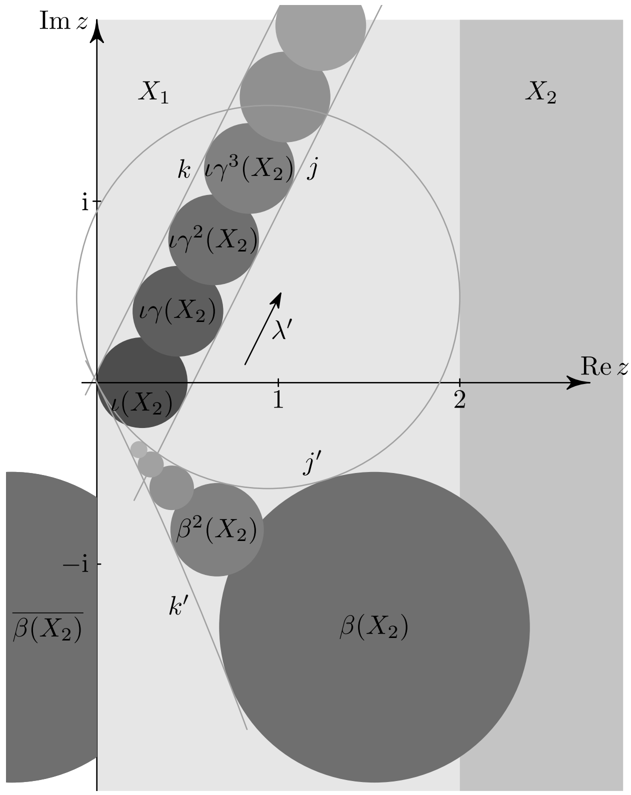

Figure 3.

This inequality leads to

$$\frac{2}{1-2\lambda_2} - \frac{4}{|1+4\lambda\varepsilon|^2-1} > \left|-\frac{4(\lambda_2+\lambda_1\mathrm{i})}{1-2\lambda_2} - \frac{4(1+4\overline{\lambda}\varepsilon)}{|1+4\lambda\varepsilon|^2-1} + 2\right|.$$

Multiplying by the common denominator and using

$$|1+4\lambda\varepsilon|^2-1=|1+4\overline{\lambda}\varepsilon|^2-1=4\varepsilon^2+8\lambda_1\varepsilon$$

we have

$$\begin{aligned}2(4\varepsilon^2+8\lambda_1\varepsilon)-4(1-2\lambda_2)>|&-4(\lambda_2+\lambda_1\mathrm{i})(4\varepsilon^2+8\lambda_1\varepsilon)\\&-4(1+4\overline{\lambda}\varepsilon)(1-2\lambda_2)+2(1-2\lambda_2)(4\varepsilon^2+8\lambda_1\varepsilon)|\end{aligned}$$

and finally

$$\begin{aligned}2\varepsilon^2+4\lambda_1\varepsilon-1+2\lambda_2>|&-8\lambda_2\varepsilon^2-8\lambda_1\lambda_2\varepsilon-1+2\lambda_2+2\varepsilon^2\\&+(-4\lambda_1\varepsilon^2-8\lambda_1^2\varepsilon-8\lambda_2^2\varepsilon+4\lambda_2\varepsilon)\mathrm{i}|.\end{aligned}$$

Applying $\lambda_1^2+\lambda_2^2=\frac{1}{4}$ yields

$$\begin{aligned}&(-64\lambda_2^2+32\lambda_2-16\lambda_1^2)\varepsilon^4+(-128\lambda_1\lambda_2^2+64\lambda_1\lambda_2)\varepsilon^3\\&\quad+(-64\lambda_1^2\lambda_2^2+16\lambda_1^2+16\lambda_2^2-4)\varepsilon^2+(-8\lambda_1+32\lambda_1\lambda_2^2)\varepsilon>0,\end{aligned}$$

and finally we get

$$(3(\lambda_2-\tfrac{1}{6})\varepsilon^3+8\lambda_1\lambda_2\varepsilon^2-2\lambda_2^2(2\lambda_2+1)\varepsilon-\lambda_1(2\lambda_2+1))(1-2\lambda_2)\varepsilon>0, \tag{3.1}$$

where $\lambda=\lambda_1+\lambda_2\mathrm{i}$, $\lambda_1,\lambda_2\in\mathbb{R}$.

In the table below we present positive approximate solutions of inequality (3.1) for some values of λ:

$\arg(\lambda)$	20°	30°	45°	60°	75°	85°
ε	$(1.043;\infty)$	$(0.866;\infty)$	$(0.808;\infty)$	$(0.827;\infty)$	$(0.891;\infty)$	$(0.960;\infty)$

As we can see, not for every λ ($\mathrm{Re}(\lambda)>0$, $|\lambda|>\frac{1}{2}$) there exists $\varepsilon\in(0;1)$ such that $\lambda'\geqslant\varepsilon\lambda$ is a free point. To evaluate "boundary" values of ε, we substitute 1 for ε in (3.1) and then get

$$\Big(\lambda_1+\frac{1}{2}\Big)(2\lambda_2-1+\lambda_1+2\lambda_1\lambda_2)>0$$

and hence

$$\lambda_2>\frac{1}{1+\lambda_1}-\frac{1}{2}.$$

We are looking now for $\lambda=\lambda_1+\lambda_2\mathrm{i}$ satisfying

$$\lambda_2>\frac{1}{1+\lambda_1}-\frac{1}{2},\qquad\text{and}\qquad\lambda_1^2+\lambda_2^2=\frac{1}{4}.$$

It follows that $\lambda_1(\lambda_1^3 + 2\lambda_1^2 + \lambda_1 - 1) < 0$ and

$$0 < \lambda_1 < \frac{1}{3}\left(-2 + \sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} + \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}\right).$$

This condition is equivalent to $\arg(\lambda) \in (21.41°; 90°)$.

If we solve inequality (3.1) for any other values λ satisfying our assumptions, we get new free λ's (see Figure 4).

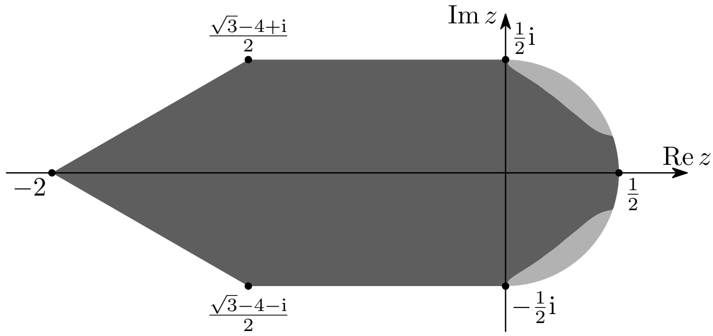

Figure 4.

Theorem 3.1. *Light grey points in Figure* 4 *are semigroup free.*

For any group G, the set $\Omega(G) \subset \mathbb{C}$ on which the elements form a normal family is called the *regular set of* G. The set of λ's for which $\Omega(\mathrm{gp}(\alpha, \beta_\lambda))/\,\mathrm{gp}(\alpha, \beta_\lambda)$ is a four times punctured sphere is called the *Riley slice of Schottky space* and consists only of free points [10].

In Figure 5 (due to David Wright, see for example [10]), the Riley slice of Schottky space is the set outside the dark area.

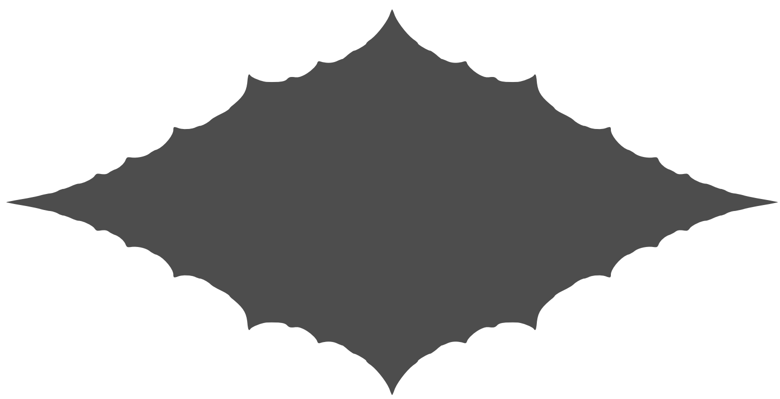

Figure 5.

If we combine the figure from [1] and Figures 4 and 5 we obtain Figure 6.

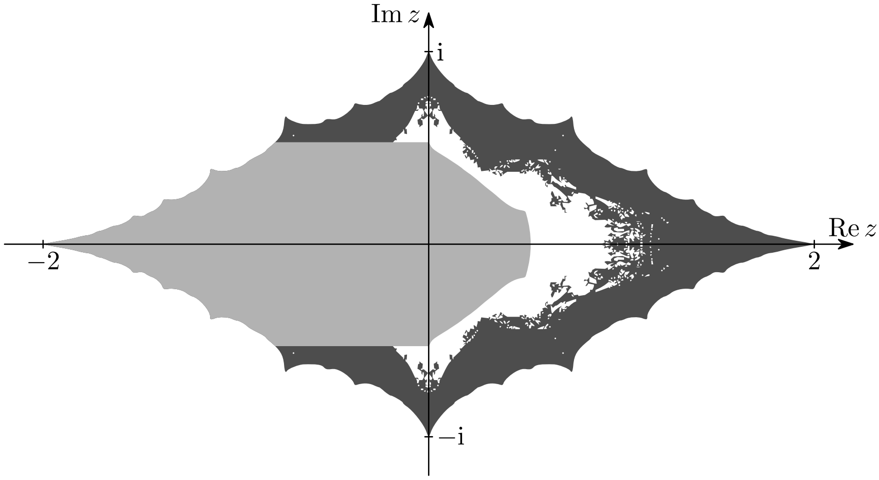

Figure 6.

Corollary 3.2. *In Figure* 6*:*

(i) *White points outside the grey area and dark area are free.*

(ii) *White points inside the dark area mark the area where the set of nonfree points is "almost dense" and this points are semigroup free at the same time. The set of nonfree points is "almost dense" means that in every pixel in this area there is a nonfree point* [1].

(iii) *All points outside the grey area are semigroup free.*

References

[1] *J. Bamberg*: Non-free points for groups generated by a pair of 2×2 matrices. J. Lond. Math. Soc., II. Ser. *62* (2000), 795–801.

[2] *J. L. Brenner*: Quelques groupes libres de matrices. C. R. Acad. Sci., Paris *241* (1955), 1689–1691. (In French.)

[3] *J. L. Brenner, A. Charnow*: Free semigroups of 2×2 matrices. Pac. J. Math. *77* (1978), 57–69.

[4] *B. Chang, S. A. Jennings, R. Ree*: On certain pairs of matrices which generate free groups. Can. J. Math. *10* (1958), 279–284.

[5] *P. de la Harpe*: Topics in Geometric Group Theory. Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 2000.

[6] *D. J. Fuchs-Rabinowitsch*: On a certain representation of a free group. Leningrad State Univ. Annals Math. Ser. *10* (1940), 154–157. (In Russian.)

[7] *J. A. Ignatov*: Free and nonfree subgroups of $\mathrm{PSL}_2(\mathbb{C})$ that are generated by two parabolic elements. Mat. Sb., Nov. Ser. *106* (1978), 372–379. (In Russian.)

[8] *Y. A. Ignatov, A. V. Evtikhova*: Free groups of linear-fractional transformations. Chebyshevskiĭ Sb. *3* (2002), 78–81. (In Russian.)

[9] *Y. A. Ignatov, T. N. Gruzdeva, I. A. Sviridova*: Free groups of linear-fractional transformations. Izv. Tul. Gos. Univ. Ser. Mat. Mekh. Inform. *5* (1999), 116–120.
[10] *L. Keen, C. Series*: The Riley slice of Schottky space. Proc. Lond. Math. Soc. (3) *69* (1994), 72–90.
[11] *R. C. Lyndon, J. L. Ullman*: Groups generated by two parabolic linear fractional transformations. Can. J. Math. *21* (1969), 1388–1403.
[12] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80, Springer, New York, 1996.
[13] *S. Saks, A. Zygmund*: Analytic Functions. PWN-Polish Scientific Publishers, Warsaw, 1965.
[14] *I. N. Sanov*: A property of a representation of a free group. Dokl. Akad. Nauk SSSR, N. Ser. *57* (1947), 657–659. (In Russian.)
[15] *P. Słanina*: On some free groups, generated by matrices. Demonstr. Math. (electronic only) *37* (2004), 55–61.

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