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ON THE QUADRATIC FRACTIONAL OPTIMIZATION WITH A STRICTLY CONVEX QUADRATIC CONSTRAINT

Maziar Salahi, Saeed Fallahi

In this paper, we have studied the problem of minimizing the ratio of two indefinite quadratic functions subject to a strictly convex quadratic constraint. First utilizing the relationship between fractional and parametric programming problems due to Dinkelbach, we reformulate the fractional problem as a univariate equation. To find the root of the univariate equation, the generalized Newton method is utilized that requires solving a nonconvex quadratic optimization problem at each iteration. A key difficulty with this problem is its nonconvexity. Using Lagrange duality, we show that this problem can be solved by solving a convex univariate minimization problem. Attainment of the global optimality conditions is discussed. Our preliminary numerical experiments on several randomly generated test problems show that, the new approach is much faster in finding the global optimal solution than the known semidefinite relaxation approach, especially when solving large scale problems.

Keywords: fractional optimization, indefinite quadratic optimization, semidefinite relaxation, diagonalization, generalized Newton method

Classification: 90C32, 90C26, 90C22

1. INTRODUCTION

Fractional optimization problems has attracted much attention in the last decades due to the wide range of applications in different areas including signal processing, communications, financial analysis, location theory and etc. [13, 14]. They are in general nonconvex, however in most recent works convex optimization approach has been successfully applied to solve various classes of them [3, 4, 5, 6]. In [4] authors have derived an efficient global optimization algorithm for the regularized total least squares problem and applied it on problems arising from the inverse Laplace transform and image processing. Tuy et al. [18] have proposed a new approach for optimizing polynomial fractional functions under polynomial constraints. Their approach is based on reformulation into a monotone optimization problem. Benson [7, 8] has studied fractional programs that involve ratios of convex terms and presented a new branch and bound algorithm that requires solving a sequence of convex optimization problems. He also has focused on linear sum-of-ratios problems and used a simplicial branch and bound duality-bounds algorithm to globally solve them [9]. Shen and Yuan [16] have proposed

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a branch and bound approach for the global optimization problem of the sum of generalized polynomial fractional functions under generalized polynomial constraints. In [1] Amaral et al. have developed an RLT-based algorithm for the global optimization of a nonconvex problem that arises in total least squares with inequality constraints, and in the correction of infeasible linear systems of equalities. A well known and old approach to tackle these problems, goes back to Dinkelbach, who has proved an interesting and useful relationship between fractional and parametric optimization problems [10]. His idea has been applied by several authors [3, 21]. Yamamoto and Konno [19] have proposed an efficient algorithm for solving convex-convex quadratic fractional programs that combines the classical Dinkelbach approach, the integer programming approach for solving nonconvex quadratic programs, and a standard nonlinear programming solver. Most recently, in [21] authors have applied Dinkelbach's idea to the following quadratic fractional optimization problem with two convex quadratic constraints:

$$\begin{aligned} \min \quad & \frac{x^T A_1 x + b_1^T x + c_1}{x^T A_2 x + b_2^T x + c_2} \\ & \|P^T x + q\| \leq \xi, \\ & \|x\| \leq \Delta. \end{aligned}$$

They have developed a generalized Newton based iterative algorithm to solve the problem with no global convergence guarantee. In this paper, we consider the following problem with a strictly convex quadratic constraint

$$\begin{aligned} \min \quad & \frac{x^T A_1 x + f_1^T x + c_1}{x^T A_2 x + f_2^T x + c_2} \\ & x^T A_3 x + f_3^T x + c_3 \leq 0, \qquad \text{(QCQFO)} \end{aligned}$$

where $A_i^T = A_i \in R^{n\times n}$, $f_i \in R^n$, $c_i \in R$, $i = 1, 2, 3$, $x^T A_2 x + f_2^T x + c_2 > 0$ in the feasible region and A_3 is assumed to be positive definite. Special case of regularized total least squares (RTLS) problem is in the form of (QCQFO) which appears in disciplines such as signal processing, automatic control, economic, biology and medicine. Although the problem can be cast as a semidefinite optimization (SDO) problem which is polynomially solvable, but such an approach is computationally expensive specially when solving large scale problems [5]. In this paper, a new diagonalization approach is introduced to solve the underlying problem and compared with a knwon SDO relaxation approach. Our preliminary numerical experiments show the efficiency of the diagonalization approach to the SDO relaxation on finding the global optimal solution.

2. PARAMETRIC PROGRAMMING APPROACH

The following theorem gives the relation between (QCQFO) and a parametric programming problem [10].

Theorem 2.1. The following two statements are equivalent:

- $$\begin{aligned} \alpha = \min \quad & \frac{x^T A_1 x + f_1^T x + c_1}{x^T A_2 x + f_2^T x + c_2} \\ & x^T A_3 x + f_3^T x + c_3 \leq 0. \end{aligned} \tag{1}$$

- $F(\alpha) = 0$, where

$$F(\alpha) := \min\Big\{x^T A_1 x + f_1^T x + c_1 - \alpha(x^T A_2 x + f_2^T x + c_2)\Big\} \qquad (2)$$
$$x^T A_3 x + f_3^T x + c_3 \leq 0.$$

The function F in (2) is continuous, concave and strictly decreasing with a unique root [21]. Thus by Theorem (2.1) the root of F is also an optimal solution of (1). Therefore, in the sequel we focus on (2). A subgradient of F at iteration k is given by $-(x_k^T A_2 x_k + f_2^T x_k + c_2)$ [21], where

$$x_k \in \arg\min\Big\{x^T A_1 x + f_1^T x + c_1 - \alpha_k(x^T A_2 x + f_2^T x + c_2)\Big\}$$
$$x^T A_3 x + f_3^T x + c_3 \leq 0.$$

Thus the iterations of the generalized Newton method to find the root of F is

$$\begin{aligned}
\alpha_{k+1} &:= \alpha_k - \frac{F(\alpha_k)}{-(x_k^T A_2 x_k + f_2^T x_k + c_2)} \\
&= \alpha_k - \frac{(x_k^T A_1 x_k + f_1^T x_k + c_1) - \alpha_k(x_k^T A_2 x_k + f_2^T x_k + c_2)}{-(x_k^T A_2 x_k + f_2^T x_k + c_2)} \\
&= \frac{x_k^T A_1 x_k + f_1^T x_k + c_1}{x_k^T A_2 x_k + f_2^T x_k + c_2}.
\end{aligned}$$

Now the generalized Newton algorithm can be outlined as follows.

Parametric generalized Newton method

Inputs: $A_1, A_2, A_3 \in \mathbb{R}^{n\times n}, A_3 \succ 0, f_1, f_2, f_3 \in \mathbb{R}^n, c_1, c_2, c_3 \in \mathbb{R}$, $k = 0$, starting point $\alpha_0 \in R$, and an accuracy parameter $\epsilon > 0$.
begin
while $|F(\alpha_k)| \geq \epsilon$
Solve the following minimization subproblem to obtain a global optimum x_k:

$$\min\Big\{x^T A_1 x + f_1^T x + c_1 - \alpha_k(x^T A_2 x + f_2^T x + c_2)\Big\}$$
$$x^T A_3 x + f_3^T x + c_3 \leq 0. \qquad (3)$$

Calculate $F(\alpha_k) = \Big\{x^T A_1 x + f_1^T x + c_1 - \alpha_k(x^T A_2 x + f_2^T x + c_2)\Big\}$.
Set

$$\alpha_{k+1} := \frac{x_k^T A_1 x_k + b_1^T x_k + c_1}{x_k^T A_2 x_k + b_2^T x_k + c_2},$$
$$k := k + 1.$$

end
end

The main computational effort at each iteration is to solve the (3). One may use semidefinite optimization (SDO) relaxation to solve these subproblems, which is expensive even for medium scale problems [20]. In the sequel we present an efficient algorithm to solve the nonconvex problem in which the problem reduces to a univariate minimization using a diagonalization method.

3. DIAGONALIZATION AND UNIVARIATE MINIMIZATION

In this section, first we present some preliminary results from linear algebra.

Lemma 3.1. Let $A, B \in R^{n\times n}$ with $A = A^T$ and $B = B^T \succ 0$. Then there exists a nonsingular matrix Q and diagonal matrix D such that

$$Q^T AQ = D, \quad Q^T BQ = I.$$

Proof. Let $B = LL^T$ be the Cholesky factorization of B and set $C = L^{-1}A(L^{-1})^T$. Since C is symmetric, there exists an orthogonal matrix P such that $P^T CP = D$, where D is diagonal. Let $Q = (L^{-1})^T P$, then

$$Q^T AQ = P^T L^{-1}A(L^{-1})^T P = P^T CP = D,$$

and

$$Q^T BQ = P^T L^{-1}(LL^T)(L^{-1})^T P = P^T P = I.$$

□

Since by our assumption A_3 is positive definite, then by Lemma 3.1 there exists a nonsingular matrix Q and a diagonal matrix D such that $Q^T(A_1 - \alpha_k A_2)Q = D = \mathrm{diag}(d_1, \ldots, d_n)$ and $Q^T A_3 Q = I$. By the change of variables $x := Qx$, $b := Q^T(f_1 - \alpha_k f_2)$, $c := (c_1-\alpha_k c_2)$ and $b_3 := Q^T f_3$, problem(3) is equivalent to the following problem:

$$\begin{array}{ll} \min & x^T Dx + b^T x + c \\ & \|x\|^2 + b_3^T x + c_3 \le 0. \end{array} \tag{4}$$

The associated Lagrangian of (4) is

$$L(x, \mu) = x^T Dx + b^T x + c + \mu\Big(\|x\|^2 + b_3^T x + c_3\Big),$$

and thus the Lagrange dual of (4) becomes

$$\max_{\mu\ge 0} g(\mu)$$

where

$$g(\mu) = \min_x \Big\{x^T(D + \mu I)x + (b + \mu b_3)^T x\Big\} + \mu c_3 + c.$$

Since

$$x^T(D + \mu I)x + (b + \mu b_3)^T x = \sum_{i=1}^n (d_i + \mu)x_i^2 + (b + \mu b_3)_i x_i,$$

if there exist j such that $d_j + \mu < 0$ or $d_j + \mu = 0$ when $(b + \mu b_3)_j \neq 0$, then the inner minimization gives the value $-\infty$. Let

$$T := \Big\{1, \ldots, n\Big\}, \qquad \mu_1 = \max\Big\{0, \max_{i \in T}\{-d_i\}\Big\},$$

$$J_1 = \Big\{i \in T \Big| \; -d_i = \mu_1\Big\}, \qquad J_2 = \Big\{i \in J_1 \Big| \; (b + \mu_1 b_3)_i = 0\Big\},$$

and for any finite set A, $n(A)$ denotes the number of elements in A. We have the following cases:

1. $D \succ 0$
   In this case we have $d_j + \mu > 0$ for all j. Therefore the inner minimization is the following convex minimization problem

$$g(\mu) = \min_x \sum_{i=1}^{n} (d_i + \mu) x_i^2 + (b + \mu b_3)_i x_i.$$

2. $D \not\succ 0$, $n(J_1) = n(J_2) \neq 0$
   In this case μ might be μ_1, therefore the inner minimization is the following convex minimization problem

$$g(\mu) = \min \Big\{g_1(\mu), g_2(\mu_1)\Big\},$$

   where, for $\mu > \mu_1$

$$g_1(\mu) = \min_x \sum_{i=1}^{n} (d_i + \mu) x_i^2 + (b + \mu b_3)_i x_i,$$

   and

$$g_2(\mu_1) = \min_x \sum_{i \in T \setminus J_2} (d_i + \mu_1) x_i^2 + (b + \mu_1 b_3)_i x_i.$$

3. $n(J_1) > n(J_2)$
   In this case μ can not be μ_1, therefore the inner minimization is the following convex minimization problem for $\mu > \mu_1$

$$g(\mu) = \min_x \sum_{i=1}^{n} (\mu + d_i) x_i^2 + (b + \mu b_3)_i x_i.$$

As one can see, in all cases $g(\mu)$ is the optimal value of separable strictly convex minimization problem, thus to solve it, it is sufficient to solve the following single variables problems for each j

$$\min_{x_j \in R} \Big\{(d_j + \mu) x_j^2 + (b + \mu b_3)_j x_j\Big\}. \tag{5}$$

Now since the objective in (5) is separable and strictly convex, thus the minimum is the root of

$$2x_j(d_j+\mu)+(b+\mu b_3)_j=0 \qquad \forall j,$$

i. e.,

$$x_j=-\frac{(b+\mu b_3)_j}{2(d_j+\mu)}.$$

Therefore

$$\min_x \sum_i (d_i+\mu_1)x_i^2+(b+\mu_1 b_3)_i x_i=-\frac{1}{4}\sum_{i\in S(\mu)}\frac{(b+\mu b_3)_i^2}{(d_i+\mu)},$$

where

$$S(\mu)=\begin{cases} T, & \text{if} \quad \Big(\mu\ge\mu_1 \bigwedge J_1=\phi\Big)\bigvee\Big(\mu>\mu_1, n(J_1)>n(J_2)\Big),\\ T\setminus J_2, & \text{if} \quad \mu=\mu_1, n(J_1)=n(J_2)\neq 0,\end{cases}$$

and the Lagrange dual of (4) is equivalent to the following one-dimensional problem:

$$\min_{\mu\ge 0} f(\mu), \tag{6}$$

where

$$f(\mu)=\frac{1}{4}\sum_{i\in S(\mu)}\frac{(b+\mu b_3)_i^2}{(d_i+\mu)}-\mu c_3.$$

Theorem 3.2. If μ^* is an optimal solution of (6), then

$$x^{**}=\begin{cases} x^*, & \text{if} \quad \Big(\mu^*>\mu_1\Big)\bigvee\Big(\mu^*=\mu_1=0\Big)\bigvee\Big(\mu^*=\mu_1>0, \lim\limits_{\mu\searrow\mu^*} f^{'}(\mu)=0\Big),\\ x^*+\alpha^* e_k, & \text{if} \quad \mu^*=\mu_1=-d_k>0, k\in J_2, \lim\limits_{\mu\searrow\mu^*} f^{'}(\mu)>0,\end{cases}$$

is optimal for (4), where

$$x_j^*=\lim_{\mu\to\mu^*}-\frac{(b+\mu b_3)_j}{2(d_j+\mu)}, \qquad j\in T, \tag{7}$$

e_k is the standard unit vector in $\mathbb{R}^n$ and α^* is the root of following quadratic equation of variable α:

$$\alpha^2+\alpha\Big(2e_k^T x^*+b_3^T e_k\Big)+\Big(\|x^*\|^2+b_3^T x^*+c_3\Big)=0.$$

Proof. For (4), suppose that there exists $x_0\in\mathbb{R}^n$ with $\|x_0\|^2+b_3^T x_0+c_3<0$. If x be a feasible point of (4), then, it is a global minimizer of (4) if and only if there exists scalar $\mu\ge 0$ such that the following condition holds: (see Lemma 4.4.1 of Chapter 4 in [2])

(i) $2(D + \mu I)x = -(b + \mu b_3)$ (KKT Condition)

(ii) $\mu\Big(\|x\|^2 + b_3^T x + c_3\Big) = 0$ (Complementary Slackness)

(iii) $(D + \mu I) \succeq 0$ (Second Order Condition).

We have the following cases:

1. $\mu^* > \mu_1$:
   In this case, it is necessary that
   $$f^{'}(\mu^*) = 0.$$
   By the definition of μ_1 and x^*, it is obvious that conditions (i) and (iii) hold for x^* and μ^*. For the complementary slackness condition (ii), we have
   $$\Big(\|x^*\|^2 + b_3^T x^* + c_3\Big) = -f^{'}(\mu^*) = 0,$$
   which implies that $\mu^*\Big(\|x^*\|^2 + b_3^T x^* + c_3\Big) = 0$.

2. $\mu^* = \mu_1 = 0$:
   In this case $D \succeq 0$, so condition (iii) hold for μ^*. Moreover, by (7) and $\mu^* = 0$ it is obvious that conditions (i) and (ii) hold for x^* and μ^*.

3. $\mu^* = \mu_1 = -d_k > 0, k \in J_2$ and $\lim\limits_{\mu \searrow \mu^*} f^{'}(\mu) > 0$:
   Since $D + \mu^* I$ is positive semi definite but not positive definite, then we have
   $$(D + \mu^* I)e_k = 0.$$
   Moreover, for $\mu > \mu^*$ we have
   $$f^{'}(\mu) = \frac{1}{4}\sum_{i=1}^{n} \frac{2b_{3i}(b + \mu b_3)_i}{(d_i + \mu)} - \frac{1}{4}\sum_{i=1}^{n} \frac{(b + \mu b_3)_i^2}{(d_i + \mu)^2} - c_3,$$
   and we have
   $$\begin{aligned}\lim_{\mu \searrow \mu^*} f^{'}(\mu) &= \lim_{\mu \searrow \mu^*} \frac{1}{4}\sum_{i=1}^{n} \frac{2b_{3i}(b + \mu b_3)_i}{(d_i + \mu)} - \lim_{\mu \searrow \mu^*} \frac{1}{4}\sum_{i=1}^{n} \frac{(b + \mu b_3)_i^2}{(d_i + \mu)^2} - c_3 \\ &= -b_3^T x^* - \|x^*\|^2 - c_3,\end{aligned}$$
   which $\lim\limits_{\mu \searrow \mu^*} f^{'}(\mu) > 0$ implies that $\|x^*\|^2 + b_3^T x^* + c_3 < 0$. Now we consider the following quadratic equation of variable α:
   $$\alpha^2 + \alpha\Big(2e_k^T x^* + b_3^T e_k\Big) + \Big(\|x^*\|^2 + b_3^T x^* + c_3\Big) = 0.$$
   Thus
   $$\triangle = \Big(2e_k^T x^* + b_3^T e_k\Big)^2 - 4\Big(\|x^*\|^2 + b_3^T x^* + c_3\Big) > 0,$$

therefore the quadratic equation has a root α^*. Now it is easy to verify that $x^* + \alpha^* e_k$ is a required boundary global optimal solution:

$$(x^* + \alpha^* e_k)^T (x^* + \alpha^* e_k) + b_3^T (x^* + \alpha^* e_k) + c_3 = 0,$$

$$\begin{aligned}(D + \mu^* I)(x^* + \alpha^* e_k) &= (D + \mu^* I)x^* + \alpha^*(D + \mu^* I)e_k \\ &= (D + \mu^* I)x^* = -\tfrac{1}{2}(b + \mu^* b_3).\end{aligned}$$

4. $\mu^* = \mu_1 > 0$ and $\lim_{\mu \searrow \mu^*} f'(\mu) = 0$:
   By the definition of x^* and μ_1, it is obvious that conditions (i) and (iii) hold for x^* and μ^*. For the complementary slackness condition (ii), we have

$$\left(\|x^*\|^2 + b_3^T x^* + c_3\right) = -\lim_{\mu \searrow \mu^*} f'(\mu) = 0,$$

   which implies that $\mu^* \left(\|x^*\|^2 + b_3^T x^* + c_3\right) = 0.$

□

4. SDO RELAXATION

In this section, we present the known SDO relaxation approach to solve (3) globally under specific conditions, see [20]. The homogenized version of (3) is

$$\begin{aligned}\min \quad & x^T(A_1 - \alpha_k A_2)x + (f_1 - \alpha_k f_2)^T xt + (c_1 - \alpha_k c_2)t^2 \\ & x^T A_3 x + f_3^T xt + c_3 t^2 \le 0.\end{aligned} \tag{8}$$

Clearly, if $(t, x^T)^T$ with $t \neq 0$ solves (8), then $\frac{x}{t}$ solves (3). Problem (8) in matrix form, including the condition $t^2 = 1$, can be written as

$$\begin{aligned}\min \quad & Q_0 \bullet \widehat{X} \\ & Q_1 \bullet \widehat{X} \le 0, \\ & Q_2 \bullet \widehat{X} = 1,\end{aligned} \tag{9}$$

where

$$A \bullet B = Tr(A^T B), \qquad \widehat{X} = \begin{bmatrix} t^2 & tx^T \\ tx & xx^T \end{bmatrix},$$

and

$$Q_0 = \begin{bmatrix} c_1 - \alpha_k c_2 & \frac{(f_1 - \alpha_k f_2)^T}{2} \\ \frac{(f_1 - \alpha_k f_2)}{2} & A_1 - \alpha_k A_2 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} c_3 & \frac{f_3^T}{2} \\ \frac{f_3}{2} & A_3 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{bmatrix}.$$

The SDO relaxation of (9) is given by

$$\begin{aligned}\min \quad & Q_0 \bullet X \\ & Q_1 \bullet X \le 0, \\ & Q_2 \bullet X = 1, \\ & X \succeq 0_{(n+1)\times(n+1)},\end{aligned} \tag{10}$$

where

$$X = \begin{bmatrix} X_{00} & x_0^T \\ x_0 & \overline{X} \end{bmatrix},$$

and its dual problem is

$$\begin{aligned} \max \quad & y_1 \\ & Z = Q_0 - y_1 Q_1 - y_2 Q_2, \\ & Z \succeq 0_{(n+1)\times(n+1)}, \\ & y_2 \leq 0. \end{aligned} \tag{11}$$

Various sufficient conditions have been derived for ensuring the strong duality of different nonconvex QP problems. In [22] the authors have established new sufficient conditions for verifying zero duality gap between (10) and (11). They have demonstrated that the duality gap is zero if and only if there exists an optimal solution X to (10) satisfying $\overline{X} = x_0 x_0^T$. Now, we show in Lemma 4.1 that, under a simple condition, both (10) and (11) are strictly feasible, thus both are solvable with zero duality gap.

Lemma 4.1. If $c_3 < 0$, then both problems (10) and (11) satisfy the Slater regularity conditions. Hence both problems attain their optimal values and the duality gap is zero.

Proof. Let

$$X = \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & \lambda I_n \end{bmatrix},$$

where $\lambda > 0$ such that $c_3 + \lambda Tr(A_3) \leq 0$. Then $X \succ 0, Q_1 \bullet X \leq 0$ and $Q_2 \bullet X = 0$. Therefore it is strictly feasible for (10). For the dual problem (11), by choosing $y_2 < \lambda_{min}(A_1 - \alpha A_2, A_3)$ and y_1 sufficiently small negative number, the Schur complement theorem gives

$$Z = \begin{bmatrix} c_1 - \alpha c_2 - y_1 - y_2 c_3 & (b_1 - \alpha b_2 - y_2 b_3)^T/2 \\ (b_1 - \alpha b_2 - y_2 b_3)/2 & A_1 - \alpha A_2 - y_2 A_3 \end{bmatrix} \succ 0.$$

Thus Z is a strictly feasible solution for (11). Consequently, the strong duality theorem [20] imply that both (10) and (11) have optimal solutions with zero duality gap. □

In the next theorem, it is shown that the global optimal solution of (3) can be derived from an optimal solution of (10) in polynomial time.

Theorem 4.2. If (10) has an optimal solution X^* and strong duality holds, then it has a rank one optimal solution which further gives us a global optimal solution of (3).

Proof. Suppose X^* is an optimal solution of rank r for (10) and (Z^*, y_1^*, y_2^*) is an optimal solution for (11). Then we have the following cases:

1. $Q_1 \bullet X^* < 0$:
   From the complementary condition in this case, obviously $y_1^* = 0$. Now Suppose

$$X^* = \sum_{i=1}^{r} x_i^* \left(x_i^*\right)^T$$

be a rank one decomposition [17] of X^* for which $\left(x_i^*\right)^T Q_1 x_i^* \leq 0$ for all $i = 1,\ldots,r$. By the second constraint of (10), for at least one $k,\ 1 \leq k \leq r$, we have $x_k^* = \left(t_k^*, (\bar{x}_k^*)^T\right)^T$, with $t_k^* \neq 0$.

2. $Q_1 \bullet X^* = 0$:
Suppose
$$X^* = \sum_{i=1}^{r} x_i^* \left(x_i^*\right)^T$$
be a rank one decomposition of X^* for which $\left(x_i^*\right)^T Q_1 x_i^* = 0$ for all $i = 1,\ldots,r$. By the second constraint of (10), for at least one $k,\ 1 \leq k \leq r$, we have $x_k^* = \left(t_k^*, (\bar{x}_k^*)^T\right)^T$, with $t_k^* \neq 0$.

Now for both cases by denoting $z_k^* = \left[1, (\bar{z}_k^*)^T\right]^T$ with $\bar{z}_k^* = \frac{\bar{x}_k^*}{t_k^*}$, and $Y^* = z_k^*(z_k^*)^T$, one can easily check that
$$Q_1 \bullet Y^* \leq 0, \quad Q_2 \bullet Y^* = 1, \quad y_1^* \left(Q_1 \bullet Y^*\right) = 0.$$
Therefore Y^* is a rank one optimal solution for (10) and thus $\bar{z}_k^*$ is an optimal solution for (3). □

5. HOMOGENEOUS CASE

In this section, we consider the following homogenous nonconvex quadratic optimization problem:
$$\begin{aligned} \min \quad & \frac{x^T A_1 x + c_1}{x^T A_2 x + c_2} \\ & x^T A_3 x + c_3 \leq 0, \end{aligned} \tag{12}$$
where where $A_i^T = A_i \in R^{n\times n}$, $c_i \in R$, $i = 1,2,3$, $x^T A_2 x + c_2 > 0$ in the feasible region and A_3 is assumed to be positive definite. According to the Theorem 3.2, we can construct the primal solution of non-homogenous case via a dual approach, but in the homogenous case (12), it is more simple and we do not need to solve the dual problem in (6). Instead, we solve directly the equivalent linear optimization problem to get the global optimal solution. First using the Dinkelbach's idea for (12), one requires solving the following nonconvex quadratic optimization problem at each iteration of the generalized Newton method.
$$\begin{aligned} \min \quad & x^T \Big(A_1 - \alpha_k A_2\Big) x + c_1 - \alpha_k c_2 \\ & x^T A_3 x + c_3 \leq 0. \end{aligned} \tag{13}$$
Since $A_3 \succ 0$, then by Lemma 3.1 there exists a nonsingular matrix Q and a diagonal matrix D such that $Q^T(A_1 - \alpha_k A_2)Q = D = \mathrm{diag}(d_1,\ldots,d_n)$ and $Q^T A_3 Q = I$. Now by

the change of variables $x := Qx$ and $c := (c_1 - \alpha_k c_2)$, problem (13) is equivalent to the following problem:

$$\begin{aligned} \min \quad & x^T Dx + c \\ & \|x\|^2 + c_3 \le 0. \end{aligned} \tag{14}$$

By setting $y_i := x_i^2$, problem (14) is equivalent to the following linear optimization problem.

$$\begin{aligned} \min \quad & d^T y + c \\ & \sum_{i=1}^{n} y_i + c_3 \le 0, \\ & y_i \ge 0, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{15}$$

Therefore, by utilizing the generalized Newton algorithm, which requires solving a linear optimization problem at each iteration, one can get the global optimal solution of (12).

6. COMPUTATIONAL EXPERIMENTS

In this section we give comparison of the new approach with the known SDO relaxation on several randomly generated test problems. Computations are performed in MATLAB 7.13.0 on a 2.3GHz laptop with 4 GB of RAM. To solve the SDO problems we have used SeDuMi 1.21 and to solve the one dimensional problem (6) we have used the fminbnd command in MATLAB. Both solvers are used with their default tolerance setting. For the generalized Newton method, the initial point is set to be $\alpha_0 = 1$ and we choose $\epsilon = 10^{-6}$ as the tolerance of the optimality. The test problems are generated using the following MATLAB code:

Random test problems generator

- n=input(′ enter the size of the problem = ′);
- $A_1 = \mathrm{rand(n); A_1 = A_1 + A_1'; f_1 = rand(n, 1); c_1 = rand;}$
- $A_2 = \mathrm{rand(n); A_2 = A_2 * A_2'; f_2 = rand(n, 1); c_2 = 10 * rand;}$
- $A_3 = \mathrm{rand(n); A_3 = (A_3 * A_3') + eye(n); f_3 = 10 * rand(n, 1); c_3 = -10 * rand;}$

We then use the following MATLAB code to compute matrix Q for diagonalization.

Diagonalization scheme for A indefinie and B $\succ 0$

— L = chol(B);

— $\mathrm{C = L' \setminus (A/L);}$

— $[\mathrm{U}, \sim, \sim] = \mathrm{svd}(\mathrm{C} * \mathrm{C}')$;

— $\mathrm{Q} = \mathrm{L} \setminus \mathrm{U}$;

The computational results are reported in Tables 1 and 2, where we report the dimension of problem (size), minimum CPU time in seconds (min), maximum CPU time in seconds (max) and average CPU time in seconds (mean) at termination.

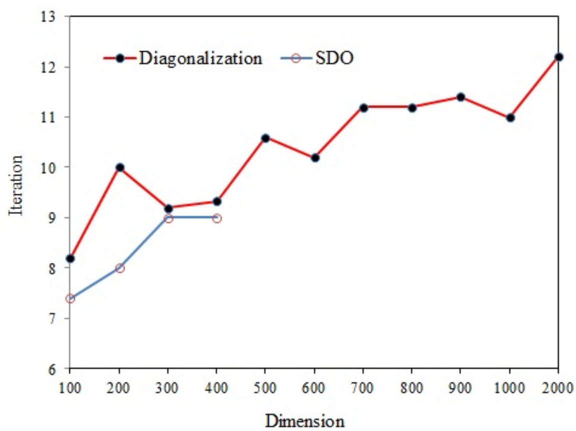

Fig. 1. Average iterations number versus the dimensions of the problems for the generalized Newton algorithm.

To show the convergence and the speed of parametric generalized Newton method and also to examine the effect of problem size on the number of iterations for the generalized Newton algorithm, we have generated 10 test problems as described above for each dimension. As we observe, the average of iteration increase as the problems dimension increase (Figure 1). Graphs demonstrating the convergence and the speed of parametric generalized Newton method for two different randomly generated test problems appear in Figure 2. The horizontal axis in these figures is the iterations number k, while the vertical axis gives $\log_{10}(|\mathrm{F}(\alpha)|)$. In this case, the slope of the straight lines shows the speed of the parametric generalized Newton method. The righthand plot in Figure 2 shows the convergence and the speed of parametric generalized Newton method with both SDO relaxation and Diagonalization method. As the figures illustrate, the hybrid of generalized Newton algorithm with SDO relaxation converge in fewer iteration, but as in Tables 1 and 2, it is clear that SDO relaxation become so expensive as the problem dimensions increase.

7. CONCLUSIONS

In this paper, we have presented a generalized Newton algorithm to solve an indefinite fractional quadratic optimization problem with a strictly convex quadratic constraint.

Tab. 1. Numerical results

	Diagonalization			SDO relaxation		
size	Time(min)	Time(max)	Time(mean)	Time(min)	Time(max)	Time(mean)
100	0.2808	0.3432	0.3213	23.4782	28.2362	125.5435
200	0.6864	0.9984	0.7917	164.394	167.88	165.6983
300	1.2636	1.45	1.3476	525.957	540.528	535.0364
400	2.5116	2.5584	2.5324	1112.33	1250.45	1165.37
500	4.1652	4.9452	4.7361	*	*	*
600	6.786	7.332	7.0288	*	*	*
700	10.5769	12.807	11.7779	*	*	*
800	17.3785	19.5937	18.1085	*	*	*
900	21.7309	28.6418	25.2097	*	*	*
1000	30.888	36.0986	33.6243	*	*	*
2000	253.892	299.709	281.3604	*	*	*

Tab. 2. Numerical results for homogeneous case

	Diagonalization			SDO relaxation		
size	Time(min)	Time(max)	Time(mean)	Time(min)	Time(max)	Time(mean)
100	0.4056	0.5148	0.4586	11.5597	27.3	15.1757
200	0.5304	0.8892	0.7519	81.5729	100.09	90.2371
300	1.2012	1.2948	1.2402	220.429	270.974	256.1965
400	1.4196	1.7472	1.5249	440.937	544.365	474.4855
500	2.3088	2.7144	2.5168	764.717	949.781	833.2483
600	4.056	4.2432	4.1392	1484.4	1718.34	1634.27
700	6.3492	6.63	6.4857	*	*	*
800	8.736	9.6252	9.1221	*	*	*
900	12.2461	12.5893	12.3865	*	*	*
1000	13.4161	16.8325	15.9004	*	*	*
2000	116.127	120.636	119.0183	*	*	*

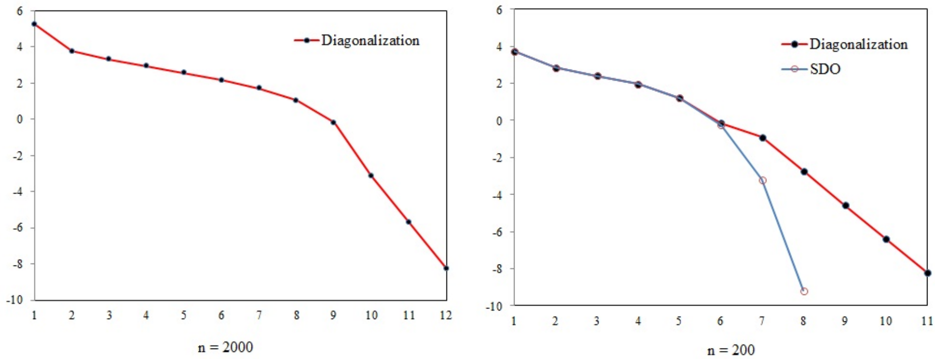

Fig. 2. Graphs of $\log_{10}(|\mathrm{F}(\alpha)|)$ versus the iteration number for two different randomly generated test problems.

To solve the underlying indefinite quadratically constraint quadratic problem at each iteration, we have used two approaches, a new diagonalization scheme which requires solving a one dimensional optimization problem and the known SDO relaxation. Our computational experiments on several randomly generated test problems with various dimensions have shown that the diagonalization approach is much faster than the SDO relaxation based approach for both homogenous and nonhomogenous case, specially when solving large scale problems. The extension of our approach to the other classes of fractional optimization problems, is left for future research.

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