Applications of Mathematics

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Applications of Mathematics, Vol. 60 (2015), No. 4, 355–382

Persistent URL: `http://dml.cz/dmlcz/144313`

ON A CAGINALP PHASE-FIELD SYSTEM WITH A LOGARITHMIC NONLINEARITY

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(Received May 19, 2014)

Abstract. We consider a phase field system based on the Maxwell Cattaneo heat conduction law, with a logarithmic nonlinearity, associated with Dirichlet boundary conditions. In particular, we prove, in one and two space dimensions, the existence of a solution which is strictly separated from the singularities of the nonlinear term and that the problem possesses a finite-dimensional global attractor in terms of exponential attractors.

Keywords: Caginalp phase-field system; Dirichlet boundary conditions; well-posedness; long time behavior of solution; global attractor; exponential attractor; Maxwell-Cattaneo law; logarithmic potential

MSC 2010: 35B40, 35B41, 35K51, 80A22, 80A20, 35Q53, 45K05, 35K55, 35G30, 92D50

1. Introduction

The Caginalp phase-field model

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = \theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{1.2}$$

was proposed to model phase transition phenomena, for example melting-solidification phenomena, in certain classes of materials. Caginalp considered the (total) Ginzburg-Landau free energy and the classical Fourier law to derive his system (see [13] and [5]). Here, u denotes the order parameter and θ the (relative) temperature. Furthermore, all physical constants have been set equal to one. For more details and references we refer the reader to [5], [16], and [6]. This model has been extensively studied (see, for example, [18], and the references therein). Now, a drawback of the Fourier law is the so-called "paradox of heat conduction", namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see e.g. [18]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo law. We refer the reader to [16], [18], and [17] for more discussions on the subject.

In this paper, we consider the model

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = \frac{\partial \alpha}{\partial t}, \tag{1.3}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} - u, \tag{1.4}$$

which is a generalization of the original Caginalp system (see [5]). In this context α is the thermal displacement variable, defined by

$$\alpha = \int_0^t \theta \, \mathrm{d}\tau + \alpha_0. \tag{1.5}$$

As mentioned before the Caginalp system can be obtained by considering the Landau free energy

$$\Psi(u, \theta) = \int_\Omega \Big(\frac{1}{2} |\nabla u|^2 + G(u) - \theta u \Big) \, \mathrm{d}x, \tag{1.6}$$

the enthalpy $H = u + \theta$, and by writing

$$\frac{1}{d} \frac{\partial u}{\partial t} = - \partial_u \Psi, \tag{1.7}$$

$$\frac{\partial H}{\partial t} = - \operatorname{div} q, \tag{1.8}$$

where $d > 0$ is a relaxation parameter, ∂_u denotes a variational derivative and q is the thermal flux vector. Setting $d = 1$ and taking the usual Fourier law

$$q = -\nabla \theta, \tag{1.9}$$

we find (1.1)–(1.2).

The Maxwell-Cattaneo law reads

$$\Big(1 + \eta \frac{\partial}{\partial t} \Big) q = -\nabla \theta, \tag{1.10}$$

where η is a relaxation parameter; when $\eta = 0$, one recovers the Fourier law. Taking for simplicity $\eta = 1$, it follows from (1.8) that

$$\Big(1 + \frac{\partial}{\partial t} \Big) \frac{\partial H}{\partial t} - \Delta \theta = 0,$$

which yields the second-order (in time) equation for the relative temperature

$$\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2}. \tag{1.11}$$

Integrating finally (1.11) between 0 and t, we obtain the equation

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} - u + f, \tag{1.12}$$

where f depends on the initial data (for u and θ), which reduces to (1.4) when f vanishes. Furthermore, noting that $\theta = \partial\alpha/\partial t$, (1.1) can be rewritten in the equivalent form (1.3).

We endow this model with Dirichlet boundary conditions and initial conditions. Then we are led to the initial and boundary value problem (P)

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = \frac{\partial\alpha}{\partial t} \quad \text{in } \Omega, \tag{1.13}$$

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} - u \quad \text{in } \Omega, \tag{1.14}$$

$$u = \alpha = 0 \quad \text{on } \partial\Omega, \tag{1.15}$$

$$u(0) = u_0, \quad \alpha(0) = \alpha_0, \quad \frac{\partial\alpha}{\partial t}(0) = \alpha_1, \tag{1.16}$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^n$ (n is to be specified later), with boundary $\partial\Omega$.

We assume here that $g = G'$, where

$$G(s) = -\kappa_0 s^2 + \kappa_1[(1+s)\ln(1+s) + (1-s)\ln(1-s)], \quad s \in (-1,1),\ 0 < \kappa_1 < \kappa_0, \tag{1.17}$$

i.e.,

$$g(s) = -2\kappa_0 s + \kappa_1 \ln\frac{1+s}{1-s}, \quad s \in (-1,1). \tag{1.18}$$

In particular, it follows from (1.18) that

$$g'(s) \geqslant -2\kappa_0, \quad s \in (-1,1), \tag{1.19}$$

and

$$-c_0 \leqslant G(s) \leqslant g(s)s + c_0, \quad s \in (-1,1). \tag{1.20}$$

Concerning the mathematical setting, we adopt the notation

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad W = H^2(\Omega),$$

and then introduce the Hilbert spaces

$$E = V \times V \times H, \quad E_1 = (W \cap V)^2 \times V,$$
$$E_2 = (H^3(\Omega) \cap V)^2 \times (W \cap V), \quad E_3 = (H^4(\Omega) \cap V)^2 \times H^3(\Omega),$$

and denote by $\|\cdot\|_Y$ the norm on the Banach space Y. Throughout this paper, the inner product and the norm of the $L^2(\Omega)$ space will be denoted by $((\cdot,\cdot))$ and $\|\cdot\|$ respectively.

This paper is organized as follows.

Section 2 is devoted to the well-posedness of problem (P), after proving a strict separation property, in one and two space dimension. In Section 3, we prove the existence of a global attractor of problem (P). Finally, Section 4 deals with the finite dimensionality of the global attractor obtained in the previous section, by proving the existence of exponential attractors.

2. Well-posedness based on a separation property

Our aim in this section is to prove the well-posedness of problem (P) with the logarithmic nonlinearity g. The main difficulty is to prove that the order parameter is separated from the singularities of g. In particular, we are only able to prove such a property in one and two space dimensions. To do so, we assume that

$$u_0 \in H_0^1(\Omega) \cap H^3(\Omega), \quad \alpha_0 \in H_0^1(\Omega) \cap H^3(\Omega), \quad \alpha_1 \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.1}$$
$$\|u_0\|_{L^\infty(\Omega)} < 1, \tag{2.2}$$
$$\Delta u_0 = \Delta \alpha_0 = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

and begin with the following result.

Proposition 2.1. *The phase field u satisfies the strict separation property, namely*

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant 1 - \delta \quad \forall\, t \geqslant 0, \tag{2.4}$$

where $\delta > 0$ is to be specified later.

P r o o f. The singularities of the potential g lead us to define the quantity

$$D(v)=\frac{1}{1-\|v\|_{L^\infty}},\quad v\in L^\infty(\Omega),\ \|v\|_{L^\infty}\neq 1.$$

We a priori assume that $\|u\|_{L^\infty((t,t+1)\times\Omega)}<1$.

We multiply (1.13) by $u+\partial u/\partial t$ and (1.14) by $\partial\alpha/\partial t$, and add the two resulting inequalities to obtain

$$\begin{aligned}(2.5)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|u\|^2+\|\nabla u\|^2+2\int_\Omega G(u)\,\mathrm{d}x+\|\nabla\alpha\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big\}\\ &\qquad+\int_\Omega G(u)\,\mathrm{d}x+\|\nabla u\|^2+\Big\|\frac{\partial u}{\partial t}\Big\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\leqslant c_0|\Omega|.\end{aligned}$$

We multiply now (1.14) by α. We have

$$(2.6)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(2\Big(\Big(\frac{\partial\alpha}{\partial t},\alpha\Big)\Big)+\|\alpha\|^2\Big)+\|\nabla\alpha\|^2=-\Big(\Big(\frac{\partial u}{\partial t},\alpha\Big)\Big)-((u,\alpha))+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2.$$

Summing finally (2.5) and ε(2.6), where $\varepsilon>0$ is small enough, using Hölder's and Poincaré inequalities, we end up with

$$(2.7)\qquad \frac{\mathrm{d}F}{\mathrm{d}t}+cF+\Big\|\frac{\partial u}{\partial t}\Big\|^2\leqslant c',\quad c>0,$$

where

$$(2.8)\ \ F=\|u\|^2+\|\nabla u\|^2+2\int_\Omega G(u)\,\mathrm{d}x+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2+\|\nabla\alpha\|^2+2\varepsilon\Big(\Big(\frac{\partial\alpha}{\partial t},\alpha\Big)\Big)+\varepsilon\|\alpha\|^2$$

satisfies

$$\begin{aligned}(2.9)\qquad &c\Big(\|u\|_V^2+\|\alpha\|_V^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big)+c''\leqslant F\\ &\qquad\leqslant c'\Big(\|u\|_V^2+\|\alpha\|_V^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big)+c''',\quad c,c'>0.\end{aligned}$$

Gronwall's lemma and (2.7) imply

$$\begin{aligned}(2.10)\quad &\|u(t)\|_V^2+\|\alpha(t)\|_V^2+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2+\int_0^t\exp\left(-c(t-s)\right)\Big\|\frac{\partial u}{\partial t}(s)\Big\|^2\,\mathrm{d}s\\ &\qquad\leqslant Q(D(u_0)+\|u_0\|_V^2+\|\alpha_0\|_V^2+\|\alpha_1\|^2)\exp\left(-ct\right)+c',\quad c>0.\end{aligned}$$

We differentiate (1.13) with respect to time to have, owing to (1.14),

$$(2.11)\qquad \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big)-\Delta\frac{\partial u}{\partial t}+g'(u)\frac{\partial u}{\partial t}=-\frac{\partial\alpha}{\partial t}+\Delta\alpha-u-\frac{\partial u}{\partial t}.$$

Multiplying (2.11) by $\partial u/\partial t$, and in view of (1.19), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\leqslant c\Big(\|u\|^2+\|\nabla\alpha\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\alpha}{\partial t}\right\|^2\Big). \tag{2.12}$$

We now multiply (1.14) by $-\Delta\partial\alpha/\partial t$ and easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2+\|\Delta\alpha\|^2\Big)+\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\leqslant c\Big(\|\nabla u\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\Big). \tag{2.13}$$

Again by Gronwall's lemma applied to (2.12) we have

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}(t)\right\|^2&+\int_0^t\exp\left(-c(t-s)\right)\left\|\nabla\frac{\partial u}{\partial t}(s)\right\|^2\mathrm{d}s\\&\leqslant c\int_0^t\exp\left(-c(t-s)\right)\Big(\|u\|^2+\|\nabla\alpha\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\alpha}{\partial t}\right\|^2\Big)\,\mathrm{d}s\\&\quad+\exp\left(-ct\right)\left\|\frac{\partial u}{\partial t}(0)\right\|^2.\end{aligned} \tag{2.14}$$

Noting that

$$\left\|\frac{\partial u}{\partial t}(0)\right\|^2\leqslant Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2), \tag{2.15}$$

by (2.10) and (2.15) we deduce from (2.14) that

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|^2&+\int_0^t\exp\left(-c(t-s)\right)\left\|\frac{\partial u}{\partial t}\right\|^2_V\mathrm{d}s\\&\leqslant Q(D(u_0)+\|u_0\|^2_W+\|\alpha_0\|^2_V+\|\alpha_1\|^2)\exp\left(-ct\right)+c',\quad c>0.\end{aligned} \tag{2.16}$$

We rewrite (1.13) as an elliptic equation for $t\geqslant 0$ fixed:

$$-\Delta u+g(u)=-\frac{\partial u}{\partial t}+\frac{\partial\alpha}{\partial t},\quad u=0\quad\text{on }\partial\Omega. \tag{2.17}$$

If we multiply (2.17) by $-\Delta u$, Hölder's inequality gives

$$\|\Delta u\|^2\leqslant c\Big(\|\nabla u\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\frac{\partial\alpha}{\partial t}\right\|^2\Big); \tag{2.18}$$

by (2.10) and (2.16), we deduce from (2.18) that

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)}\leqslant Q(D(u_0)&+\|u_0\|^2_{H^2(\Omega)}\\&+\|\alpha_0\|^2_{H^1_0(\Omega)}+\|\alpha_1\|^2)\exp\left(-ct\right)+c',\quad c>0.\end{aligned} \tag{2.19}$$

We multiply (1.14) by $-\Delta\alpha$, and Hölder's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla\alpha\|^2+2\Big(\Big(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha\Big)\Big)\Big)+2\|\Delta\alpha\|^2 \leqslant 2\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2+\frac{1}{\varepsilon}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2+\|u\|^2\Big)+2\varepsilon\|\Delta\alpha\|^2. \tag{2.20}$$

Summing up (2.13) and $\varepsilon_1(2.20)$, where $\varepsilon_1>0$ is small enough, we obtain the inequality

$$\frac{\mathrm{d}F_1}{\mathrm{d}t}+cF_1(t)\leqslant c'\Big(\|\nabla u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2\Big), \tag{2.21}$$

where

$$F_1(t)=\|\Delta\alpha(t)\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}(t)\Big\|^2+2\varepsilon_1\Big(\Big(\nabla\frac{\partial\alpha}{\partial t}(t),\nabla\alpha(t)\Big)\Big)+\varepsilon_1\|\nabla\alpha(t)\|^2 \tag{2.22}$$

satisfies

$$c\Big(\|\Delta\alpha(t)\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}(t)\Big\|^2\Big)\leqslant F_1(t)\leqslant c'\Big(\|\Delta\alpha(t)\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}(t)\Big\|^2\Big). \tag{2.23}$$

Applying Gronwall's lemma to (2.21), by (2.23), (2.10), and (2.16) we obtain

$$\|\alpha(t)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2_{H^1_0(\Omega)} \leqslant Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})\exp(-ct)+c'.$$

In one space dimension: We have, owing to the embedding $H^1(\Omega)\subset L^\infty(\Omega)$, an estimate on $\partial\alpha/\partial t$ in L^∞, namely

$$\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_{L^\infty}\leqslant Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)} +\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})\exp(-ct)+c',\quad c>0. \tag{2.24}$$

It is not difficult to prove the separation property (2.4) for solutions to the parabolic equation $\partial u/\partial t-\Delta u+g(u)=f$, with the right-hand side $f\in L^\infty((t,t+1)\times\Omega)$, by using the comparison principle (see [11]). We deduce

$$D(u(t))\leqslant Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})\exp(-ct)+c', \tag{2.25}$$

and from the above estimates we conclude that

$$D(u(t))+\|u\|^2_W+\|\alpha\|^2_W+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_V+\Big\|\frac{\partial u}{\partial t}\Big\|^2+\int_0^t\exp(-c(t-s))\Big\|\frac{\partial u}{\partial t}\Big\|^2_V\,\mathrm{d}s \leqslant Q(D(u_0)+\|u_0\|^2_{H^2(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})\exp(-ct)+c',\quad c>0, \tag{2.26}$$

which, in particular, implies that

$$\|u(t)\|_{L^\infty} \leqslant 1-\delta \quad \forall\, t \geqslant 0, \tag{2.27}$$

where $\delta > 0$ depends on $D(u_0)$, $\|u_0\|_{H^2(\Omega)}$, $\|\alpha_0\|_{H^2(\Omega)}$, and $\|\alpha_1\|_{H_0^1(\Omega)}$.

In two space dimensions: We first prove

Lemma 2.1. *We have, for every $M > 0$,*

$$\begin{aligned}\int_{(t,t+1)\times\Omega} &\exp\left(M|g(u(x,t))|\right) \mathrm{d}x\,\mathrm{d}t \\ &\leqslant Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H_0^1(\Omega)}) \exp\left(-ct\right) + c',\end{aligned} \tag{2.28}$$

where c' depends on M.

Proof. We proceed as in [19]. We rewrite (1.13) in the form

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = f, \tag{2.29}$$

where

$$\|f(t)\|_{H_0^1(\Omega)} \leqslant Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H_0^1(\Omega)}) \exp\left(-ct\right) + c'. \tag{2.30}$$

We can also assume, without loss of generality, that

$$g'(s) \geqslant 0 \quad \forall s \in (-1,1). \tag{2.31}$$

We fix $M > 0$ and multiply (2.29) by $g(u)\exp\left(M|g(u)|\right)$ to have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G_M(u)\,\mathrm{d}x &+ \int_\Omega |\nabla u|^2 g'(u)(1+M|g(u)|)\exp\left(M|g(u)|\right)\mathrm{d}x \\ &+ \int_\Omega |g(u)|^2 \exp\left(M|g(u)|\right)\mathrm{d}x = \int_\Omega f\cdot g(u)\exp\left(M|g(u)|\right)\mathrm{d}x,\end{aligned}$$

where $G_M(s) = \int_0^s \tau \exp\left(M|\tau|\right)\mathrm{d}\tau$, which yields, by integrating between t and $t+1$,

$$\begin{aligned}\int_\Omega G_M(u(t+1))\,\mathrm{d}x &+ \int_{(t,t+1)\times\Omega} |\nabla u|^2 g'(u)(1+M|g(u)|)\exp\left(M|g(u)|\right)\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{(t,t+1)\times\Omega} |g(u)|^2 \exp\left(M|g(u)|\right)\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega G_M(u(t))\,\mathrm{d}x + \int_{(t,t+1)\times\Omega} f\cdot g(u)\exp\left(M|g(u)|\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.32}$$

We thus deduce that

$$
\int_{(t,t+1)\times\Omega} |g(u)|^2 \exp(M|g(u)|)\,\mathrm{d}x\,\mathrm{d}t \leqslant Q(D(u_0)+\|u_0\|_W^2+\|\alpha_0\|_W^2+\|\alpha_1\|_V^2)\exp(-ct)+c' + \int_{(t,t+1)\times\Omega} |f|\cdot|g(u)|\exp(M|g(u)|)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.33}
$$

In order to estimate the second term on the right-hand side of (2.33), we use Young's inequality (see [15])

$$
ab \leqslant \varphi(a)+\psi(b), \quad a,b\geqslant 0, \tag{2.34}
$$

where

$$
\varphi(s)=\exp(s)-s-1, \quad \psi(s)=(1+s)\ln(1+s)-s, \quad s\geqslant 0. \tag{2.35}
$$

Taking $a=N|f|$ and $b=N^{-1}|g(u)|\exp(M|g(u)|)$, where $N>0$ is to be fixed later, in (2.34), we obtain

$$
|f|\cdot|g(u)|\exp(M|g(u)|) \leqslant \exp(N|f|) + (1+N^{-1}|g(u)|\exp(M|g(u)|))\ln(1+N^{-1}|g(u)|\exp(M|g(u)|)).
$$

Choosing finally $N=N(M)$ large enough, we find

$$
|f|\cdot|g(u)|\exp(M|g(u)|) \leqslant \exp(N|f|)+\frac{1}{2}|g(u)|^2\exp(M|g(u)|)+c, \tag{2.36}
$$

where c depends only on M. We thus deduce from (2.33) and (2.36) the inequality

$$
\int_{(t,t+1)\times\Omega} |g(u)|^2\exp(M|g(u)|)\,\mathrm{d}x\,\mathrm{d}t \leqslant Q(D(u_0)+\|u_0\|_{H^2(\Omega)}^2+\|\alpha_0\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H_0^1(\Omega)}^2)\exp(-ct) + c' + 2\int_{(t,t+1)\times\Omega}\exp(N|f|)\,\mathrm{d}x\,\mathrm{d}t, \tag{2.37}
$$

where c' depends only on M. Using a proper Orlicz's inequality (see [10] and [12]), we deduce

$$
\int_{(t,t+1)\times\Omega} |g(u)|^2\exp(M|g(u)|)\,\mathrm{d}x\,\mathrm{d}t \leqslant Q(D(u_0)+\|u_0\|_{H^2(\Omega)}^2+\|\alpha_0\|_{H^2(\Omega)}^2+\|\alpha_1\|_{H_0^1(\Omega)}^2)\exp(-ct)+c'. \tag{2.38}
$$

Noting finally that

$$\int_{(t,t+1)\times\Omega} \exp(M|g(u)|)\,\mathrm{d}x \leqslant \int_{|g(u)|\leqslant 1} \exp(M|g(u)|)\,\mathrm{d}x$$
$$+ \int_{|g(u)|\geqslant 1} \exp(M|g(u)|)\,\mathrm{d}x \leqslant c + \int_{|g(u)|\geqslant 1} |g(u)|^2 \exp(M|gu)|)\,\mathrm{d}x \leqslant c$$
$$+ \int_{(t,t+1)\times\Omega} |g(u)|^2 \exp(M|g(u)|)\,\mathrm{d}x,$$

where c depends only on M, (2.38) yields the desired inequality (2.28). $\square$

It is not difficult to show, by comparing growths, that the logarithmic function g satisfies

$$|g'(s)| \leqslant \exp(c|g(s)| + c'), \quad s \in (-1,+1),\ c, c' \geqslant 0. \tag{2.39}$$

Therefore,

$$\int_{(t,t+1)\times\Omega} |g'(s)|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{(t,t+1)\times\Omega} \exp(cp|g(u)| + c'p)\,\mathrm{d}x\,\mathrm{d}t,$$

whence, owing to (2.28), we have

$$\|g'(u)\|_{L^p((t,t+1)\times\Omega)} \leqslant Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^1_0(\Omega)}) \exp(-ct) + c'. \tag{2.40}$$

We then rewrite (1.13) in the form $\partial u/\partial t - \Delta u = \partial\alpha/\partial t - g(u)$ and have, differentiating with respect to time,

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) - \Delta\frac{\partial u}{\partial t} = h, \tag{2.41}$$

where

$$h = -\frac{\partial\alpha}{\partial t} + \Delta\alpha - u - \frac{\partial u}{\partial t} - g'(u)\frac{\partial u}{\partial t} \tag{2.42}$$

satisfies, owing to (2.40) (for $p = 4$) and the above a priori estimates (which imply that $\partial u/\partial t \in L^\infty(t, t+1, L^2(\Omega)) \cap L^2(t, t+1, H^1_0(\Omega)) \subset L^4(t, t+1, H^{1/2}(\Omega)) \subset L^4((t, t+1) \times \Omega))$,

$$\|h\|_{L^2((t,t+1)\times\Omega)} \leqslant Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} + \|\alpha_1\|^2_{H^1_0(\Omega)}) \exp(-ct) + c'. \tag{2.43}$$

Multiplying (2.41) by $-\Delta\partial u/\partial t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\leqslant\|h\|^2, \tag{2.44}$$

which, owing to Gronwall's lemma, gives

$$\begin{aligned}\Big\|\frac{\partial u}{\partial t}\Big\|_V^2&+\int_0^t\exp\left(-c(t-s)\right)\Big\|\frac{\partial u}{\partial t}\Big\|_W^2\,\mathrm{d}s\\&\leqslant c'\int_0^t\exp\left(-c(t-s)\right)\Big(\|h\|^2+\Big\|\frac{\partial u}{\partial t}\Big\|_V^2\Big)\,\mathrm{d}s+\exp\left(-ct\right)\Big\|\nabla\frac{\partial u}{\partial t}(0)\Big\|^2.\end{aligned} \tag{2.45}$$

Noting that $\|\nabla(\partial u/\partial t)(0)\|^2\leqslant Q(D(u_0)+\|u_0\|^2_{H^3(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})$ and using (2.43), we deduce from (2.45) that

$$\begin{aligned}\int_0^t&\exp\left(-c(t-s)\right)\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\,\mathrm{d}s\\&\leqslant Q(D(u_0)+\|u_0\|^2_{H^3(\Omega)}+\|\alpha_0\|^2_{H^2(\Omega)}+\|\alpha_1\|^2_{H^1_0(\Omega)})\exp\left(-ct\right)+c'.\end{aligned} \tag{2.46}$$

Multiplying (1.14) by $\Delta^2\partial\alpha/\partial t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2+\|\nabla\Delta\alpha\|^2\Big)+2\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2\leqslant\frac{1}{\varepsilon}\Big(\|\Delta u\|^2+\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\Big)+2\varepsilon\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2. \tag{2.47}$$

Multiplying now (1.14) by $\Delta^2\alpha$, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(2\Big(\Big(\Delta\frac{\partial\alpha}{\partial t},\Delta\alpha\Big)\Big)&+\|\Delta\alpha\|^2\Big)+2\|\nabla\Delta\alpha\|^2\\&\leqslant\frac{1}{\varepsilon_3}\Big(\|\nabla u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2\Big)+2\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2+2\varepsilon_3\|\nabla\Delta\alpha\|^2.\end{aligned} \tag{2.48}$$

Summing up (2.47) and ε_2(2.48), with $\varepsilon_2>0$ small enough, we have

$$\frac{\mathrm{d}F_2(t)}{\mathrm{d}t}+cF_2(t)\leqslant c'\Big(\|\Delta u\|^2+\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\Big), \tag{2.49}$$

where

$$F_2(t)=\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2+\|\nabla\Delta\alpha\|^2+\varepsilon_2\Big(2\Big(\Big(\Delta\frac{\partial\alpha}{\partial t},\Delta\alpha\Big)\Big)+\|\Delta\alpha\|^2\Big) \tag{2.50}$$

satisfies

$$c\Big(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2\Big)\leqslant F_2(t)\leqslant c'\Big(\|\Delta\alpha\|^2+\|\nabla\Delta\alpha\|^2+\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2\Big). \tag{2.51}$$

We deduce from (2.49) and Gronwall's lemma, together with (2.46) and (2.51), an estimate on $\partial\alpha/\partial t$ in $L^\infty(t,t+1,W\cap V)$. Summarizing, we have

$$
\begin{aligned}
(2.52)\quad & D(u(t)) + \|u\|^2_{H^3(\Omega)} + \|\alpha\|^2_{H^3(\Omega)} + \left\|\frac{\partial\alpha}{\partial t}\right\|^2_W + \left\|\frac{\partial u}{\partial t}\right\|^2_V \\
&+ \int_0^t \exp\left(-c(t-s)\right)\left\|\frac{\partial u}{\partial t}\right\|^2_W \mathrm{d}s \\
&\leqslant Q(D(u_0) + \|u_0\|^2_{H^3(\Omega)} + \|\alpha_0\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_W)\exp\left(-ct\right) + c', \quad c>0.
\end{aligned}
$$

Rewriting again (1.13) in the form $\partial u/\partial t - \Delta u + g(u) = f$, we have, owing to the above estimates, that $f \in L^\infty((t,t+1)\times\Omega)$, and the separation property follows as in the one-dimensional case. We deduce in particular

$$
(2.53)\qquad \|u(t)\|_{L^\infty} \leqslant 1-\delta \quad \forall\, t \geqslant 0;
$$

here $\delta > 0$ depends on $D(u_0)$, $\|u_0\|_{H^3(\Omega)}$, $\|\alpha_0\|_{H^3(\Omega)}$, and $\|\alpha_1\|_{H^2(\Omega)}$, which completes the proof of the proposition. □

Consequently, every solution $(u,\alpha,\partial\alpha/\partial t)$ of problem (1.13)–(1.16) is a priori strictly separated from the singular points ± 1 of the nonlinear term g. Thus we have

Theorem 2.1. (i) *In one space dimension, we assume that*

$$
(2.54)\qquad D(u_0) + \|u_0\|^2_W + \|\alpha_0\|^2_W + \|\alpha_1\|^2_V < \infty.
$$

Then (1.13)–(1.16) *possesses a unique solution such that*

$$
\begin{aligned}
(2.55)\quad & D(u(t)) + \|u\|^2_W + \|\alpha\|^2_W + \left\|\frac{\partial\alpha}{\partial t}\right\|^2_V + \left\|\frac{\partial u}{\partial t}\right\|^2 + \int_0^t \exp\left(-c(t-s)\right)\left\|\frac{\partial u}{\partial t}\right\|^2_V \mathrm{d}s \\
&\leqslant Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)} + \|\alpha_0\|^2_{H^2(\Omega)} \\
&\quad + \|\alpha_1\|^2_{H^1_0(\Omega)})\exp\left(-ct\right) + c', \quad c>0,\ t\geqslant 0.
\end{aligned}
$$

(ii) *In two space dimensions we assume that*

$$
(2.56)\qquad D(u_0) + \|u_0\|^2_{H^3} + \|\alpha_0\|^2_{H^3} + \|\alpha_1\|^2_W < \infty.
$$

Then (1.13)–(1.16) *possesses a unique solution such that*

$$
\begin{aligned}
(2.57)\quad & D(u(t)) + \|u\|^2_{H^3} + \|\alpha\|^2_{H^3} + \left\|\frac{\partial\alpha}{\partial t}\right\|^2_W + \left\|\frac{\partial u}{\partial t}\right\|^2_V + \int_0^t \exp\left(-c(t-s)\right)\left\|\frac{\partial u}{\partial t}\right\|^2_W \mathrm{d}s \\
&\leqslant Q(D(u_0) + \|u_0\|^2_{H^3(\Omega)} + \|\alpha_0\|^2_{H^3(\Omega)} \\
&\quad + \|\alpha_1\|^2_{H^2(\Omega)})\exp\left(-ct\right) + c', \quad c>0,\ t\geqslant 0.
\end{aligned}
$$

P r o o f. *Existence*. The proof of existence is standard, once we have the separation property (2.4), since the problem then reduces to one with a regular nonlinearity. Indeed, we consider the same problem, in which the logarithmic function g is replaced by the C^1 function

$$g_\delta(s) = \begin{cases} g(-\delta) + g'(-\delta)(s+\delta), & s \in (-\infty, -\delta[, \\ g(s), & s \in [-\delta, \delta], \\ g(\delta) + g'(\delta)(s-\delta), & s \in]\delta, \infty), \end{cases}$$

where δ is the same constant as in (2.4).

This function meets all the requirements of [17] for the existence of a regular solution $(u_\delta, \alpha_\delta)$.

Furthermore, it is not difficult to see that g and g_δ satisfy (1.19), (1.20), and (2.39), for the same constants (taking, if necessary, δ close enough to 1 so that g and g' are nonnegative on $[\delta, 1)$ and $|g_\delta| \leqslant |g|$). We can thus derive the same estimates as above, with the very same constants. Indeed, we can note that the bounds on $\partial\alpha/\partial t$ obtained there depend only on g through the constants in (1.19), (1.20), and (2.39). Since g and g_δ coincide on $[-\delta, \delta]$, we finally deduce that u_δ is a solution to the original problem.

Uniqueness. We suppose the existence of two solutions $(u^{(1)}, \alpha^{(1)}, \partial\alpha^{(1)}/\partial t)$ and $(u^{(2)}, \alpha^{(2)}, \partial\alpha^{(2)}/\partial t)$ to the problem (1.13)–(1.16) associated with the initial conditions $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, respectively. Setting

$$\Big(u, \alpha, \frac{\partial\alpha}{\partial t}\Big) = \Big(u^{(1)} - u^{(2)}, \alpha^{(1)} - \alpha^{(2)}, \frac{\partial\alpha^{(1)}}{\partial t} - \frac{\partial\alpha^{(2)}}{\partial t}\Big),$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)} - u_0^{(2)}, \alpha_0^{(1)} - \alpha_0^{(2)}, \alpha_1^{(1)} - \alpha_1^{(2)}),$$

we then have

$$\frac{\partial u}{\partial t} - \Delta u + l(t)u = \frac{\partial\alpha}{\partial t}, \tag{2.58}$$

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} - u, \tag{2.59}$$

$$u = \alpha = 0, \tag{2.60}$$

$$u(0) = u_0, \quad \alpha(0) = \alpha_0, \quad \frac{\partial\alpha}{\partial t}(0) = \alpha_1, \tag{2.61}$$

where $l(t) = \int_0^1 g'(su^{(1)}(t) + (1-s)u^{(2)}(t))\,\mathrm{d}s$. Moreover, according to (2.55), we have, for every $t \geqslant 0$

$$\|u^{(i)}(t)\|_{L^\infty} \leqslant 1 - \delta_i, \quad \delta_i = \delta_i(D(u_0^{(i)}), \|u_0^{(i)}\|_W, \|\alpha_0^{(i)}\|_W, \|\alpha_1^{(i)}\|_V), \quad i = 1, 2.$$

Then, setting $\delta_0 = \min(\delta_1, \delta_2)$, we deduce that

$$\|su^{(1)}(t) + (1-s)u^{(2)}(t)\|_{L^\infty} \leqslant 1 - \delta_0 \quad \forall\, 0 \leqslant s \leqslant 1,$$

and, consequently,

$$\|l(t)\|_{L^\infty} \leqslant C \ (= C(\delta_0)). \tag{2.62}$$

We multiply (2.58) by $u + \partial u/\partial t$ and (2.59) by $\partial\alpha/\partial t$; summing up the resulting equations, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2 + \|\nabla u\|^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2 + \|\nabla\alpha\|^2\Big) + \|\nabla u\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2 \\ = -((l(t)u, u)) - \Big(\Big(l(t)u, \frac{\partial u}{\partial t}\Big)\Big).\end{aligned} \tag{2.63}$$

We have

$$-((l(t)u, u)) \leqslant (\mathrm{par}\, l(t) \geqslant -2\kappa_0) \leqslant 2\kappa_0\|u\|^2, \tag{2.64}$$

and Hölder's inequality and (2.62) imply

$$\Big|\Big(\Big(l(t)u, \frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant c\|u\|^2 + \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2. \tag{2.65}$$

Hence, inserting (2.64) and (2.65) into (2.63), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^2 + \|\nabla u\|^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2 + \|\nabla\alpha\|^2\Big) + \|\nabla u\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2 \leqslant c\|u\|^2,$$

in particular

$$\frac{\mathrm{d}F_3}{\mathrm{d}t} \leqslant cF_3, \tag{2.66}$$

where

$$F_3 = \|u\|^2 + \|\nabla u\|^2 + \|\nabla\alpha\|^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2,$$

satisfies

$$c\Big(\|u\|_V^2 + \|\alpha\|_V^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big) \leqslant F_3 \leqslant c'\Big(\|u\|_V^2 + \|\alpha\|_V^2 + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big),$$

which yields, using Gronwall's lemma,

$$\|u(t)\|_V^2 + \|\alpha(t)\|_V^2 + \Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2 \leqslant \exp(ct)(\|u_0\|_V^2 + \|\alpha_0\|_V^2 + \|\alpha_1\|^2), \tag{2.67}$$

hence the uniqueness (for $u_0^{(1)} = u_0^{(2)}$, $\alpha_0^{(1)} = \alpha_0^{(2)}$ and $\alpha_1^{(1)} = \alpha_1^{(2)}$) as well as the continuity with respect to the initial data. □

Thanks to Theorem 2.1 (i), we can define the semigroup $S(t)$ of problem (1.13)–(1.16) on the phase space X, where

$$X = \left\{ \left(u, \alpha, \frac{\partial \alpha}{\partial t} \right) \in E_1,\ \|u\|_{L^\infty} < 1 \right\}.$$

Taking

$$D(u_0) + \|u_0\|_W^2 + \|\alpha_0\|_W^2 + \|\alpha_1\|_V^2 \leqslant R, \quad R > 0,$$

we obtain that B_R^1 is a bounded absorbing set for $S(t)$, where

$$B_R^1 = \left\{ \left(u, \alpha, \frac{\partial \alpha}{\partial t} \right) \in E_1, \quad D(u(t)) + \|u\|_W^2 + \|\alpha\|_W^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_V^2 \leqslant R \right\};$$

indeed, by (2.55) we have

$$D(u(t)) + \|u(t)\|_W^2 + \|\alpha(t)\|_W^2 + \left\| \frac{\partial \alpha}{\partial t}(t) \right\|_V^2 \leqslant R \quad \forall\, t \geqslant t_0. \tag{2.68}$$

Concerning the two dimensional case, we define again the space X', where

$$X' = \left\{ \left(u, \alpha, \frac{\partial \alpha}{\partial t} \right) \in E_2,\ \|u\|_{L^\infty} < 1 \right\},$$

and

$$B_R^2 = \left\{ \left(u, \alpha, \frac{\partial \alpha}{\partial t} \right) \in E_2,\ D(u(t)) + \|u\|_{H^3}^2 + \|\alpha\|_{H^3}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_W^2 \leqslant R \right\},$$

is a bounded absorbing set for $S(t)$ in X'. Indeed, we have

$$D(u(t)) + \|u(t)\|_{H^3}^2 + \|\alpha(t)\|_{H^3}^2 + \left\| \frac{\partial \alpha}{\partial t}(t) \right\|_{H^2}^2 \leqslant R \quad \forall\, t \geqslant t_1. \tag{2.69}$$

3. Global attractor for problem (P)

We have the following result.

Theorem 3.1. (i) *If $n = 1$, the semigroup $S(t)$, $t \geqslant 0$, defined from X to itself possesses a connected global attractor $\mathcal{A}_1$ in X.*

(ii) *If $n = 2$, $S(t)$ defined from X' to itself possesses a connected global attractor $\mathcal{A}_2$.*

P r o o f. The absence of regularizing effects related to the presence of the term $\partial^2\alpha/\partial t^2$, does not allow us to prove the existence of a global attractor by using standard methods (see, for example, [21], [7], and [20]). For the proof we use a semigroup decomposition argument (see, for example, [17]) consisting in splitting the semigroup $S(t)$, $t \geqslant 0$, into the sum of two operators families: $S(t) = S_1(t) + S_2(t)$, where the operators $S_1(t)$ go to zero as t tends to infinity while the operators $S_2(t)$ are compact.

This corresponds to the solution decomposition

$$\Big(u, \alpha, \frac{\partial \alpha}{\partial t}\Big) = \Big(v, a, \frac{\partial a}{\partial t}\Big) + \Big(w, b, \frac{\partial b}{\partial t}\Big),$$

where $(v, a, \partial a/\partial t)$ is a solution to

$$\frac{\partial v}{\partial t} - \Delta v = \frac{\partial a}{\partial t}, \tag{3.1}$$

$$\frac{\partial^2 a}{\partial t^2} + \frac{\partial a}{\partial t} - \Delta a = -\frac{\partial v}{\partial t} - v, \tag{3.2}$$

$$v = a = 0, \tag{3.3}$$

$$v(0) = u_0, \quad a(0) = \alpha_0, \quad \frac{\partial a}{\partial t}(0) = \alpha_1, \tag{3.4}$$

and $(w, b, \partial b/\partial t)$ solves

$$\frac{\partial w}{\partial t} - \Delta w + g(u) = \frac{\partial b}{\partial t}, \tag{3.5}$$

$$\frac{\partial^2 b}{\partial t^2} + \frac{\partial b}{\partial t} - \Delta b = -\frac{\partial w}{\partial t} - w, \tag{3.6}$$

$$w = b = 0, \tag{3.7}$$

$$w(0) = 0, \quad b(0) = 0, \quad \frac{\partial b}{\partial t}(0) = 0, \tag{3.8}$$

with initial data belonging to B^1_R. Multiplying (3.1) by $-\Delta v - \Delta\partial v/\partial t$, (3.2) by $-\Delta\partial a/\partial t$, and summing the resulting equations, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla v\|^2 + \|\Delta v\|^2 + \|\Delta a\|^2 + \Big\|\nabla\frac{\partial a}{\partial t}\Big\|^2\Big] + \|\Delta v\|^2 + \Big\|\nabla\frac{\partial a}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 = 0. \tag{3.9}$$

Now, multiplying (3.2) by $-\Delta a$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\nabla\frac{\partial a}{\partial t}, \nabla a\Big)\Big) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla a\|^2 + \|\Delta a\|^2 \\ = -\Big(\Big(\nabla\frac{\partial v}{\partial t}, \nabla a\Big)\Big) - ((\nabla a, \nabla v)) + \Big\|\nabla\frac{\partial a}{\partial t}\Big\|^2.\end{aligned} \tag{3.10}$$

Summing up (3.9) and ε_3(3.10), where $\varepsilon_3 > 0$ is small enough, we deduce the inequality

$$\frac{\mathrm{d}F_4(t)}{\mathrm{d}t} + cF_4(t) + \Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \leqslant 0, \tag{3.11}$$

where F_4 satisfies

$$F_4(t) \geqslant c\Big(\|a\|_W^2 + \|v\|_W^2 + \Big\|\frac{\partial a}{\partial t}\Big\|_V^2\Big). \tag{3.12}$$

Applying Gronwall's lemma to (3.11), we write

$$\begin{aligned}\|v(t)\|_W^2 + \|a(t)\|_W^2 + \Big\|\frac{\partial a}{\partial t}(t)\Big\|_V^2 \\ \leqslant \exp(-ct)(\|u_0\|_W^2 + \|\alpha_0\|_W^2 + \|\alpha_1\|_V^2) \quad \forall\, t \geqslant 0.\end{aligned} \tag{3.13}$$

We can see that $S_1(t)(u_0, \alpha_0, \alpha_1) = (v(t), a(t), \partial a/\partial t(t))$ tends to zero when t tends to infinity.

Now, we consider the system (3.5)–(3.8). One multiplies (3.5) by $\Delta^2 w + \Delta^2 \partial w/\partial t$ and (3.6) by $\Delta^2 \partial b/\partial t$. Summing up the two resulting equations, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla\Delta w\|^2 + \|\Delta w\|^2 + \|\nabla\Delta b\|^2 + \Big\|\Delta\frac{\partial b}{\partial t}\Big\|^2\Big) \\ &\qquad + \Big\|\Delta\frac{\partial w}{\partial t}\Big\|^2 + \Big\|\Delta\frac{\partial b}{\partial t}\Big\|^2 + \|\nabla\Delta w\|^2 \\ &\quad = -\Big(\Big(\Delta g(u), \Delta\frac{\partial w}{\partial t}\Big)\Big) - ((\Delta g(u), \Delta w)).\end{aligned} \tag{3.14}$$

From Hölder's inequality, we write

$$\Big|\Big(\Big(\Delta g(u), \Delta\frac{\partial w}{\partial t}\Big)\Big)\Big| \leqslant \frac{1}{2\varepsilon_4}\|\Delta g(u)\|^2 + \frac{\varepsilon_4}{2}\Big\|\Delta\frac{\partial w}{\partial t}\Big\|^2. \tag{3.15}$$

Moreover, one has, using Hölder's inequality and the continuous embedding $H^3(\Omega) \subset H^2(\Omega)$,

$$|((\Delta g(u), \Delta w))| \leqslant \frac{1}{2\varepsilon_4}\|\Delta g(u)\|^2 + c\varepsilon_4\|\nabla\Delta w\|^2. \tag{3.16}$$

Inserting (3.15) and (3.16) into (3.14), choosing $\varepsilon_4 > 0$ small enough, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla\Delta w\|^2 + \|\Delta w\|^2 + \|\nabla\Delta b\|^2 + \Big\|\Delta\frac{\partial b}{\partial t}\Big\|^2\Big] \\ &\qquad + c\Big(\Big\|\Delta\frac{\partial w}{\partial t}\Big\|^2 + \Big\|\Delta\frac{\partial b}{\partial t}\Big\|^2 + \|\nabla\Delta w\|^2\Big) \leqslant c'\|\Delta g(u)\|^2.\end{aligned} \tag{3.17}$$

Integrating (3.17) over $(0,t)$ and using (3.8), we get

$$\|\nabla\Delta w(t)\|^2+\|\Delta w(t)\|^2+\|\nabla\Delta b(t)\|^2+\Big\|\Delta\frac{\partial b}{\partial t}(t)\Big\|^2\leqslant c'\int_0^t\|\Delta g(u)\|^2\,\mathrm{d}s. \tag{3.18}$$

By (2.55), we have

$$\int_0^t\|\Delta g(u)\|^2\,\mathrm{d}s\leqslant C_{T,\|(u_0,\alpha_0,\alpha_1)\|_{E_1},B^1_R}. \tag{3.19}$$

Finally, inserting (3.19) into (3.18), we have

$$\Big\|\Big(w(t),b(t),\frac{\partial b}{\partial t}(t)\Big)\Big\|^2_{E_2}\leqslant C_{T,\|(u_0,\alpha_0,\alpha_1)\|_{E_1},B^1_R}. \tag{3.20}$$

Hence, the operator $S_2(t)(u_0,\alpha_0,\alpha_1):=(w(t),b(t),\partial b/\partial t(t))$ is asymptotically compact in the sense of the Kuratowski measure of noncompactness, which proves the existence part of Theorem 3.1 (i).

In order to prove part (ii) of Theorem 3.1, we now take the initial data in B^2_R, then multiply (3.1) by $\Delta^2 v+\Delta^2\partial v/\partial t$ and (3.2) by $\Delta^2\partial a/\partial t$. Summing up the two resulting equations, we end up with

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Delta v\|^2+\|\nabla\Delta v\|^2+\|\nabla\Delta a\|^2+\Big\|\Delta\frac{\partial a}{\partial t}\Big\|^2\Big)&\\+\|\nabla\Delta v\|^2+\Big\|\Delta\frac{\partial a}{\partial t}\Big\|^2+\Big\|\Delta\frac{\partial v}{\partial t}\Big\|^2&=0.\end{aligned} \tag{3.21}$$

Now, multiplying (3.2) by $\Delta^2 a$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big(\Delta\frac{\partial a}{\partial t},\Delta a\Big)\Big)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta a\|^2+\|\nabla\Delta a\|^2\\&\quad=-\Big(\Big(\Delta\frac{\partial v}{\partial t},\Delta a\Big)\Big)-((\Delta a,\Delta v))+\Big\|\Delta\frac{\partial a}{\partial t}\Big\|^2.\end{aligned} \tag{3.22}$$

Summing up (3.21) and ε_5(3.22), where $\varepsilon_5>0$ is small enough, and using the Cauchy-Schwarz inequality we end up with

$$\frac{\mathrm{d}F_5(t)}{\mathrm{d}t}+cF_5(t)+\Big\|\Delta\frac{\partial v}{\partial t}\Big\|^2\leqslant 0,\quad c>0, \tag{3.23}$$

where F_5 satisfies

$$F_5(t)\geqslant c\Big(\|a\|^2_{H^3(\Omega)}+\|v\|^2_{H^3(\Omega)}+\Big\|\frac{\partial v}{\partial t}\Big\|^2_{H^2(\Omega)}\Big). \tag{3.24}$$

Applying Gronwall's lemma to (3.23), we write

$$\|v(t)\|^2_{H^3(\Omega)} + \|a(t)\|^2_{H^3(\Omega)} + \left\|\frac{\partial a}{\partial t}(t)\right\|^2_{H^2(\Omega)} \leqslant \exp(-ct)(\|u_0\|^2_{H^3(\Omega)} + \|\alpha_0\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}). \tag{3.25}$$

Concerning the system (3.5)–(3.8), we multiply (3.5) by $\Delta^3 w + \Delta^3 \partial w/\partial t$ and (3.6) by $\Delta^3 \partial b/\partial t$. Summing up the two resulting equations, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla\Delta w\|^2 + \|\Delta^2 w\|^2 + \|\Delta^2 b\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2\Big] \\ + \left\|\nabla\Delta\frac{\partial w}{\partial t}\right\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2 + \|\Delta^2 w\|^2 \\ = -\Big(\Big(\nabla\Delta g(u), \nabla\Delta\frac{\partial w}{\partial t}\Big)\Big) - ((\nabla\Delta g(u), \nabla\Delta w)).\end{aligned} \tag{3.26}$$

Having this in mind, using Hölder's inequality and the continuous embedding $H^4(\Omega) \subset H^3(\Omega)$, we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla\Delta w\|^2 + \|\Delta^2 w\|^2 + \|\Delta^2 b\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2\Big] \\ + c\Big(\left\|\nabla\Delta\frac{\partial w}{\partial t}\right\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}\right\|^2 + \|\Delta^2 w\|^2\Big) \leqslant c'\|\nabla\Delta g(u)\|^2.\end{aligned} \tag{3.27}$$

Integrating (3.27) over $(0,t)$, by (3.8) we get

$$\|\nabla\Delta w(t)\|^2 + \|\Delta^2 w(t)\|^2 + \left\|\nabla\Delta\frac{\partial b}{\partial t}(t)\right\|^2 + \|\Delta^2 b(t)\|^2 \leqslant c'\int_0^t \|\nabla\Delta g(u)\|^2\,\mathrm{d}s. \tag{3.28}$$

Furthermore, we have

$$\int_0^t \|\nabla\Delta g(u)\|^2\,\mathrm{d}s \leqslant C_{T,\|(u_0,\alpha_0,\alpha_1)\|_{E_2},B_R^2}. \tag{3.29}$$

Inserting (3.29) into (3.28), we deduce

$$\left\|\Big(w(t), b(t), \frac{\partial b}{\partial t}(t)\Big)\right\|^2_{E_3} \leqslant C_{T,\|(u_0,\alpha_0,\alpha_1)\|_{E_2},B_R^2}, \tag{3.30}$$

which completes the proof of the theorem. □

We define, for the sequel, the following invariant sets: in one space dimension, $\mathcal{X}_1 = \overline{\bigcup_{t\geqslant t_0} S(t)B_R^1}$, where B_R^1 is the bounded absorbing set for $S(t)$ in X, and in two space dimensions, $\mathcal{X}_2 = \overline{\bigcup_{t\geqslant t_1} S(t)B_R^2}$, where B_R^2 is the bounded absorbing set for $S(t)$

in X'. In what follows, we will work in these two subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$, which are positively invariant for $S(t)$, $t \geqslant 0$.

Now that the existence of the global attractor is proved, one natural question is to know if this attractor has finite dimension in terms of the fractal or Hausdorff dimension. That is the aim of the last section.

4. Exponential attractors

The aim of this section is to prove the existence of exponential attractors for the semigroup $S(t)$, $t \geqslant 0$, associated with problem (P), in one and two space dimensions using the separation property (2.4). To do so, we need the semigroup to be Lipschitz continuous and have the smoothing property, but we also have to verify the Hölder condition in time. This is enough to establish the existence of exponential attractors. Due to the lack of regularizing effects on the initial data, methods applied with success in, for example, [3], [4], [2], and [20], do not work here. To overcome this difficulty, we will make use of the so-called decomposition method which has been successfully applied by many authors (see [14], [9], [1], and [8]). This method consists in decomposing the difference of two trajectories of the problem into two parts; one tending to zero as time goes to infinity and the other one continuous. But before going further, let us recall the definition of an exponential attractor which is also called an inertial set.

Definition 4.1. A compact set $\mathcal{M}$ is called an *exponential attractor* for $(\{S(t)\}_{t\geqslant 0}, \mathcal{X})$ if

(i) $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$, where $\mathcal{A}$ is the global attractor,
(ii) $\mathcal{M}$ is positively invariant for $S(t)$, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \geqslant 0$,
(iii) $\mathcal{M}$ has finite fractal dimension,
(iv) it attracts exponentially the bounded subsets of $\mathcal{X}$ in the following sense:

$$\forall\, B \subset \mathcal{X} \quad \text{bounded,} \ \operatorname{dist}(S(t)B, \mathcal{M}) \leqslant Q(\|B\|_{\mathcal{X}}) \exp(-\alpha t), \ t \geqslant 0,$$

where the positive constant α and the monotonic function Q are independent of B, and dist stands for the Hausdorff semi-distance between sets in $\mathcal{X}$, defined by

$$\operatorname{dist}(A, B) = \max_{a\in A} \inf_{b\in B} \|a - b\|_{\mathcal{X}}.$$

We start by stating an abstract result that will be useful in the sequel (see [20]).

Theorem 4.1. *Let Ψ and Ψ_1 be two Banach spaces such that Ψ_1 is compactly embedded into Ψ and $S(t)\colon Y \to Y$ is a semigroup acting on a closed subset Y of Ψ. We assume that*

(i) $\forall\, x_1, x_2 \in Y$, $\forall\, t \geqslant 0$, $S(t)x_1 - S(t)x_2 = S_1(t, x_1, x_2) + S_2(t, x_1, x_2)$, *where*

$$\|S_1(t, x_1, x_2)\|_\Psi \leqslant d(t)\|x_1 - x_2\|_\Psi,$$

d is continuous, $t \geqslant 0$, $d(t) \to 0$ as $t \to \infty$, and

$$\|S_2(t, x_1, x_2)\|_{\Psi_1} \leqslant h(t)\|x_1 - x_2\|_\Psi, \quad t > 0, \ h \text{ continuous},$$

(ii) *$(t, x) \mapsto S(t)x$ is Lipschitz on $[0, T] \times B$ for all $T > 0$ and for all $B \subset Y$ bounded. Then $S(t)$ possesses an exponential attractor $\mathcal{M}$ on Y.*

In order to get the existence of exponential attractors in our case, we will base on Theorem 4.1. We have the following result:

Theorem 4.2. (i) *In one space dimension, the semigroup $S(t)$, $t \geqslant 0$, corresponding to equations* (1.13)–(1.16) *defined from $\mathcal{X}_1$ to itself satisfies a decomposition as in Theorem* 4.1.

(ii) *In two space dimensions, $S(t)$, $t \geqslant 0$, defined from $\mathcal{X}_2$ to itself also satisfies such a decomposition.*

P r o o f. Let $\big(u^{(1)}, \alpha^{(1)}, \partial\alpha^{(1)}/\partial t\big)$ and $\big(u^{(2)}, \alpha^{(2)}, \partial\alpha^{(2)}/\partial t\big)$ be two solutions to the problem (1.13)–(1.16) and let $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$ be their respective initial data. Set

$$\Big(u, \alpha, \frac{\partial \alpha}{\partial t}\Big) = \Big(u^{(1)} - u^{(2)}, \alpha^{(1)} - \alpha^{(2)}, \frac{\partial \alpha^{(1)}}{\partial t} - \frac{\partial \alpha^{(2)}}{\partial t}\Big),$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)} - u_0^{(2)}, \alpha_0^{(1)} - \alpha_0^{(2)}, \alpha_1^{(1)} - \alpha_1^{(2)}).$$

Thus $\big(u, \alpha, \partial\alpha/\partial t\big)$ is a solution to

$$\frac{\partial u}{\partial t} - \Delta u + l(t)u = \frac{\partial \alpha}{\partial t}, \tag{4.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} - u, \tag{4.2}$$

$$u = \alpha = 0, \tag{4.3}$$

$$u(0) = u_0, \quad \alpha(0) = \alpha_0, \quad \frac{\partial \alpha}{\partial t}(0) = \alpha_1. \tag{4.4}$$

Now decompose the solution $(u, \alpha, \partial\alpha/\partial t)$ as follows:

$$\Big(u, \alpha, \frac{\partial\alpha}{\partial t}\Big) = \Big(u_1, \alpha_1, \frac{\partial\alpha_1}{\partial t}\Big) + \Big(u_2, \alpha_2, \frac{\partial\alpha_2}{\partial t}\Big),$$

where $\big(u_1, \alpha_1, \partial\alpha_1/\partial t\big)$ and $\big(u_2, \alpha_2, \partial\alpha_2/\partial t\big)$ are solutions to

$$\frac{\partial u_1}{\partial t} - \Delta u_1 = \frac{\partial\alpha_1}{\partial t}, \tag{4.5}$$

$$\frac{\partial^2\alpha_1}{\partial t^2} + \frac{\partial\alpha_1}{\partial t} - \Delta\alpha_1 = -\frac{\partial u_1}{\partial t} - u_1, \tag{4.6}$$

$$u_1 = \alpha_1 = 0, \tag{4.7}$$

$$u_1(0) = u_0, \quad \alpha_1(0) = \alpha_0, \quad \frac{\partial\alpha_1}{\partial t}(0) = \alpha_1, \tag{4.8}$$

and

$$\frac{\partial u_2}{\partial t} - \Delta u_2 + l(t)u = \frac{\partial\alpha_2}{\partial t}, \tag{4.9}$$

$$\frac{\partial^2\alpha_2}{\partial t^2} + \frac{\partial\alpha_2}{\partial t} - \Delta\alpha_2 = -\frac{\partial u_2}{\partial t} - u_2, \tag{4.10}$$

$$u_2 = \alpha_2 = 0, \tag{4.11}$$

$$u_2(0) = 0, \quad \alpha_2(0) = 0, \quad \frac{\partial\alpha_2}{\partial t}(0) = 0, \tag{4.12}$$

respectively. We start the proof of (i). In this case the initial conditions belong to $\mathcal{X}_1$, and by repeating for (4.5)–(4.8) the estimates which led us to (3.11), we then write

$$\frac{\mathrm{d}F_6(t)}{\mathrm{d}t} + cF_6(t) + \Big\|\nabla\frac{\partial u_1}{\partial t}\Big\|^2 \leqslant 0, \quad c > 0, \tag{4.13}$$

where

$$F_6(t) = 2\varepsilon_3\Big(\Big(\nabla\frac{\partial\alpha_1}{\partial t}, \nabla\alpha_1\Big)\Big) + \varepsilon_3\|\nabla\alpha_1\|^2 + \|\nabla u_1\|^2 + \|\Delta u_1\|^2 + \|\Delta\alpha_1\|^2 + \Big\|\nabla\frac{\partial\alpha_1}{\partial t}\Big\|^2$$

satisfies

$$c'\Big(\|u_1\|_W^2 + \|\alpha_1\|_W^2 + \Big\|\frac{\partial\alpha_1}{\partial t}\Big\|_V^2\Big) \leqslant F_6(t) \leqslant c''\Big(\|u_1\|_W^2 + \|\alpha_1\|_W^2 + \Big\|\frac{\partial\alpha_1}{\partial t}\Big\|_V^2\Big), \tag{4.14}$$

in particular,

$$\frac{\mathrm{d}F_6(t)}{\mathrm{d}t} + cF_6(t) \leqslant 0. \tag{4.15}$$

An application of Gronwall's lemma yields

$$\Big\|\Big(u_1(t),\alpha_1(t),\frac{\partial\alpha_1}{\partial t}(t)\Big)\Big\|^2_{E_1}\leqslant d(t)\|(u_0,\alpha_0,\alpha_1)\|^2_{E_1}. \tag{4.16}$$

Now we consider (4.9)–(4.12). We multiply (4.9) by $\Delta^2u_2+\Delta^2\partial u_2/\partial t$ and (4.10) by $\Delta^2\partial\alpha_2/\partial t$. Summing the resulting inequations, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Delta u_2\|^2+\|\nabla\Delta u_2\|^2+\|\nabla\Delta\alpha_2\|^2+\Big\|\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big)&\\ +\|\nabla\Delta u_2\|^2+\Big\|\Delta\frac{\partial u_2}{\partial t}\Big\|^2+\Big\|\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2&\\ =((\nabla(l(t)u),\nabla\Delta u_2))-\Big(\Big(\Delta(l(t)u),\Delta\frac{\partial u_2}{\partial t}\Big)\Big).&\end{aligned} \tag{4.17}$$

Noting that

$$|((\nabla(l(t)u),\nabla\Delta u_2))|\leqslant\frac{c}{2\varepsilon_6}\|\Delta u\|^2+\frac{c\varepsilon_6}{2}\|\nabla\Delta u_2\|^2. \tag{4.18}$$

Due to the continuous embedding $H^2(\Omega)\subset L^\infty(\Omega)$ and by (2.55), we have

$$\|l(t)\|_{H^2(\Omega)}\leqslant Q(\|(u_0^{(1)},\alpha_0^{(1)},\alpha_1^{(1)})\|_X+\|(u_0^{(2)},\alpha_0^{(2)},\alpha_1^{(2)})\|_X)\leqslant c, \tag{4.19}$$

thus,

$$\Big|\Big(\Big(\Delta(l(t)u),\Delta\frac{\partial u_2}{\partial t}\Big)\Big)\Big|\leqslant\frac{c\varepsilon_6}{2}\Big\|\Delta\frac{\partial u_2}{\partial t}\Big\|^2+\frac{c}{2\varepsilon_6}\|\Delta u\|^2. \tag{4.20}$$

Choosing $\varepsilon_6>0$ small enough and using (4.18) and (4.20), we deduce from (4.17) the using inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\Delta u_2\|^2+\|\nabla\Delta u_2\|^2+\|\nabla\Delta\alpha_2\|^2+\Big\|\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big]&\\ +c\Big(\|\nabla\Delta u_2\|^2+\Big\|\Delta\frac{\partial u_2}{\partial t}\Big\|^2+\Big\|\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big)\leqslant c'\|\Delta u\|^2,\quad c>0.&\end{aligned} \tag{4.21}$$

Integrating (4.21) over $(0,t)$, by (4.12) we have

$$\|\Delta u_2(t)\|^2+\|\nabla\Delta u_2(t)\|^2+\|\nabla\Delta\alpha_2(t)\|^2+\Big\|\Delta\frac{\partial\alpha_2}{\partial t}(t)\Big\|^2\leqslant c'\int_0^t\|\Delta u\|^2\,\mathrm{d}s. \tag{4.22}$$

It only remains to estimate $\int_0^t\|\Delta u\|^2\,\mathrm{d}s$; to do so we multiply (4.1) by $-\Delta u-\Delta\partial u/\partial t$ and (4.2) by $-\Delta\partial\alpha/\partial t$. Summing up, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla u\|^2+\|\Delta u\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2+\|\Delta\alpha\|^2\Big]&\\ +\|\Delta u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2&\\ =-((\nabla(l(t)u),\nabla u))-\Big(\Big(\nabla(l(t)u),\nabla\frac{\partial u}{\partial t}\Big)\Big).&\end{aligned} \tag{4.23}$$

Hölder's inequality, (2.55) and (2.62) yield

$$|((\nabla(l(t)u),\nabla u))| \leqslant c(\|\nabla u\|+\||u||\nabla u^{(1)}|\|+\||u||\nabla u^{(2)}|\|)\|\nabla u\| \leqslant c\|u\|^2_{H^1_0(\Omega)}. \tag{4.24}$$

Analogously, we have

$$\Big|\Big(\Big(\nabla(l(t)u),\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant c\|u\|_{H^1_0(\Omega)}\cdot\Big\|\nabla\frac{\partial u}{\partial t}\Big\| \leqslant \frac{c}{2\varepsilon_7}\|u\|^2_{H^1_0(\Omega)}+\frac{c\varepsilon_7}{2}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2. \tag{4.25}$$

Choosing $\varepsilon_7>0$ small enough and recalling (4.24) and (4.25), we obtain

$$\frac{\mathrm{d}F_7(t)}{\mathrm{d}t}+c'\Big(\|\Delta u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2\Big) \leqslant c\|u\|^2_{H^1_0(\Omega)}, \quad c'>0, \tag{4.26}$$

where

$$F_7(t)=\|\nabla u\|^2+\|\Delta u\|^2+\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2+\|\Delta\alpha\|^2, \tag{4.27}$$

in particular,

$$\frac{\mathrm{d}F_7(t)}{\mathrm{d}t}+c'\Big(\|\Delta u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2\Big) \leqslant cF_7(t), \quad c'>0. \tag{4.28}$$

We note that integrating (4.28) over $(0,t)$, we have

$$\int_0^t \|\Delta u\|^2\,\mathrm{d}s \leqslant c'\exp(ct)\|(u_0,\alpha_0,\alpha_1)\|^2_{E_1}, \tag{4.29}$$

hence (4.22) yields

$$\begin{aligned}\|u_2(t)\|^2_{H^3(\Omega)}+\|\alpha_2(t)\|^2_{H^3(\Omega)}+\Big\|\frac{\partial\alpha_2}{\partial t}(t)\Big\|^2_{H^2(\Omega)}&\\ \leqslant c'\exp(ct)(\|u_0\|^2_W+\|\alpha_0\|^2_W+\|\alpha_1\|^2_V),&\end{aligned} \tag{4.30}$$

where $h(t)=c'\exp(ct)$, with c and c' depending on $\mathcal{X}_1$. We can see that h is continuous.

We now turn to the two-dimensional case, and prove part (ii) of Theorem 4.2. To do so we take here the initial data in $\mathcal{X}_2$, and repeating for (4.5)–(4.8) the estimates which led us to (3.23), we then write

$$\frac{\mathrm{d}\psi_1(t)}{\mathrm{d}t}+c\psi_1(t)+\Big\|\Delta\frac{\partial u_1}{\partial t}\Big\|^2 \leqslant 0, \quad c>0, \tag{4.31}$$

where

$$\psi_1(t) = 2\varepsilon_1\Big(\Big(\Delta\frac{\partial\alpha_1}{\partial t}, \Delta\alpha_1\Big)\Big) + \varepsilon_1\|\Delta\alpha_1\|^2 + \|\Delta u_1\|^2 + \|\nabla\Delta u_1\|^2 + \|\nabla\Delta\alpha_1\|^2 + \Big\|\Delta\frac{\partial\alpha_1}{\partial t}\Big\|^2$$

satifies

$$c'\Big(\|u_1\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_{H^3(\Omega)} + \Big\|\frac{\partial\alpha_1}{\partial t}\Big\|^2_W\Big) \leqslant \psi_1(t) \leqslant c''\Big(\|u_1\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_{H^3(\Omega)} + \Big\|\frac{\partial\alpha_1}{\partial t}\Big\|^2_W\Big), \tag{4.32}$$

in particular

$$\frac{\mathrm{d}\psi_1(t)}{\mathrm{d}t} + c\psi_1(t) \leqslant 0. \tag{4.33}$$

An application of Gronwall's lemma yields

$$\|u_1(t)\|^2_{H^3(\Omega)} + \|\alpha_1(t)\|^2_{H^3(\Omega)} + \Big\|\frac{\partial\alpha_1}{\partial t}(t)\Big\|^2_W \leqslant c'\exp(-ct)(\|u_0\|^2_{H^3(\Omega)} + \|\alpha_0\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_W). \tag{4.34}$$

Concerning problem (4.9)–(4.12), we multiply (4.9) by $\Delta^3 u_2 + \Delta^3\partial u_2/\partial t$ and (4.10) by $\Delta^3\partial\alpha_2/\partial t$. Summing up the resulting equations, we then get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla\Delta u_2\|^2 + \|\Delta^2 u_2\|^2 + \|\Delta^2\alpha_2\|^2 + \Big\|\nabla\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big] + \|\Delta^2 u_2\|^2 + \Big\|\nabla\Delta\frac{\partial u_2}{\partial t}\Big\|^2 + \Big\|\nabla\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2 = ((\Delta(l(t)u), \Delta^2 u_2)) - \Big(\Big(\nabla\Delta(l(t)u), \nabla\Delta\frac{\partial u_2}{\partial t}\Big)\Big). \tag{4.35}$$

Analogously to (4.20), we write

$$|((\Delta(l(t)u), \Delta^2 u_2))| \leqslant \frac{c\varepsilon_8}{2}\|\Delta^2 u_2\|^2 + \frac{c}{2\varepsilon_8}\|\Delta u\|^2. \tag{4.36}$$

By (2.57) and the continuous embedding $H^3(\Omega) \subset C(\overline{\Omega})$, we have

$$\|l(t)\|_{H^3(\Omega)} \leqslant Q(\|(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})\|_{X'} + \|(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})\|_{X'}) \leqslant c, \tag{4.37}$$

thus,

$$\Big|\Big(\Big(\nabla\Delta(l(t)u), \nabla\Delta\frac{\partial u_2}{\partial t}\Big)\Big)\Big| \leqslant \frac{c\varepsilon_8}{2}\Big\|\nabla\Delta\frac{\partial u_2}{\partial t}\Big\|^2 + \frac{c}{2\varepsilon_8}\|\nabla\Delta u\|^2. \tag{4.38}$$

Choosing $\varepsilon_8 > 0$ small enough and recalling (4.36) and (4.38), we deduce from (4.35) the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla\Delta u_2\|^2 + \|\Delta^2 u_2\|^2 + \|\Delta^2\alpha_2\|^2 + \Big\|\nabla\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big] + c\Big(\|\Delta^2 u_2\|^2 + \Big\|\nabla\Delta\frac{\partial u_2}{\partial t}\Big\|^2 + \Big\|\nabla\Delta\frac{\partial\alpha_2}{\partial t}\Big\|^2\Big) \leqslant c'\|\nabla\Delta u\|^2. \tag{4.39}$$

Similarly to the above, we deduce

$$\frac{\mathrm{d}\psi_2(t)}{\mathrm{d}t} + c'\Big(\|\nabla\Delta u\|^2 + \Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2\Big) \leqslant c\|\Delta u\|^2, \quad c' > 0, \tag{4.40}$$

where

$$\psi_2(t) = \|\Delta u\|^2 + \|\nabla\Delta u\|^2 + \Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2 + \|\nabla\Delta\alpha\|^2. \tag{4.41}$$

Integrating (4.40) over $(0,t)$, we deduce

$$\int_0^t \|\nabla\Delta u\|^2\,\mathrm{d}s \leqslant c'\exp(ct)\|(u_0,\alpha_0,\alpha_1)\|^2_{E_2}. \tag{4.42}$$

Integrating again (4.39) over $(0,t)$ and using (4.42), we have

$$\|u_2(t)\|^2_{H^4(\Omega)} + \|\alpha_2(t)\|^2_{H^4(\Omega)} + \Big\|\frac{\partial\alpha_2}{\partial t}(t)\Big\|^2_{H^3(\Omega)} \leqslant c'\exp(ct)(\|u_0\|^2_{H^3(\Omega)} + \|\alpha_0\|^2_{H^3(\Omega)} + \|\alpha_1\|^2_{H^2(\Omega)}), \tag{4.43}$$

which completes the proof. $\square$

Lemma 4.1. *The semigroup $S(t)$, $t \geqslant 0$, generated by the problem* (1.13)–(1.16) *is Hölder continuous on $[0,T]\times B^i_R$, $i = 1,2$ (i depending on the space dimension).*

Proof. We treat the one dimensional case (the two dimensional case can be treated similarly). The Lipschitz continuity in space being a consequence of (2.67), it just remains to prove the continuity in time (actually, the Hölder condition in time for the semigroup $S(t)$, $t \geqslant 0$). Let the initial data belong to B^1_R.

Hence, for every $t_1 \geqslant 0$ and $t_2 \geqslant 0$, two different times, owing to the above estimates, one gets:

$$\begin{aligned}
&\|S(t_1)(u_0,\alpha_0,\alpha_1) - S(t_2)(u_0,\alpha_0,\alpha_1)\|_E \\
&\quad= \|u(t_1) - u(t_2)\|_V + \|\alpha(t_1) - \alpha(t_2)\|_V + \Big\|\frac{\partial\alpha}{\partial t}(t_1) - \frac{\partial\alpha}{\partial t}(t_2)\Big\| \\
&\quad\leqslant \int_{t_2}^{t_1}\Big\|\frac{\partial u}{\partial t}(\tau)\Big\|_V\,\mathrm{d}\tau + \int_{t_2}^{t_1}\Big\|\frac{\partial\alpha}{\partial t}(\tau)\Big\|_V\,\mathrm{d}\tau + \int_{t_2}^{t_1}\Big\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\Big\|\,\mathrm{d}\tau \\
&\quad\leqslant c|t_1 - t_2|^{1/2} + \Big(\int_{t_2}^{t_1}\Big\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\Big\|^2\,\mathrm{d}\tau\Big)^{1/2}|t_1 - t_2|^{1/2},
\end{aligned}$$

where c depends on T. We multiply (1.14) by $\partial^2\alpha/\partial t^2$ and obtain

$$(4.44)\quad \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2+2\Big(\Big(\nabla\alpha,\nabla\frac{\partial\alpha}{\partial t}\Big)\Big)\Big\}+c\Big\|\frac{\partial^2\alpha}{\partial t^2}\Big\|^2\leqslant c'\Big(\|u\|^2+\Big\|\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2\Big).$$

Integrating (4.44) between t_2 and 0, then between 0 and t_1, we deduce from the above estimates that

$$(4.45)\qquad \int_{t_2}^{t_1}\Big\|\frac{\partial^2\alpha}{\partial t^2}(\tau)\Big\|^2\,\mathrm{d}\tau\leqslant c,$$

where c depends on T and B_R^1. This concludes the proof. □

We deduce from Theorem 4.2 and Lemma 4.1 the following result.

Theorem 4.3. *The dynamical system* $(S(t),\mathcal{X}_1)$ $((S(t),\mathcal{X}_2))$ *associated to the problem* (1.13)–(1.16) *possesses, in one space dimension, an exponential attractor* $\mathcal{M}_1$ *in* $\mathcal{X}_1$ (*respectively, in two space dimensions, an exponential attractor* $\mathcal{M}_2$ *in* $\mathcal{X}_2$).

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