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BOUNDEDNESS IN A QUASILINEAR PARABOLIC-PARABOLIC CHEMOTAXIS SYSTEM WITH NONLINEAR LOGISTIC SOURCE

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Abstract. We study a quasilinear parabolic-parabolic chemotaxis system with nonlinear logistic source, under homogeneous Neumann boundary conditions in a smooth bounded domain. By establishing proper a priori estimates we prove that, with both the diffusion function and the chemotaxis sensitivity function being positive, the corresponding initial boundary value problem admits a unique global classical solution which is uniformly bounded. The result of this paper is a generalization of that of Cao (2014).

Keywords: boundedness; chemotaxis; nonlinear logistic source

MSC 2010: 35K59, 92C17

1. Introduction

Chemotaxis is a kind of strictly oriented movement or partially oriented and partially tumbling movement of mobile species. This kind of movement is caused by the chemical substances in the environment. In 1970, Keller and Segel [7] introduced the well-known chemotaxis model

$$\begin{cases} u_t = \nabla(\mathcal{A}\nabla u) - \nabla(\mathcal{B}u\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f(u) = 0$, $u(x,t)$ is the cell density, $v(x,t)$ denotes the concentration of chemoattractant, $\mathcal{A} > 0$ represents the diffusion rate of cells and $\mathcal{B} > 0$ stands for the chemotactic sensitivity.

During the past four decades, (1.1) has already been investigated successfully and the main issue of the investigation was whether the solutions are bounded or blow up. With $n = 1$ and $f(u) = 0$, Osaki and Yagi [9] showed that all solutions of (1.1) are global in time and bounded. If $n = 2$ and $\|u_0\|_{L^1(\Omega)} < 4\pi$, all solutions of (1.1) exist globally (cf. [8]). Nevertheless, if almost every $\|u_0\|_{L^1(\Omega)} > 4\pi$, Horstmann and Wang [5] proved that the corresponding solutions of (1.1) blow up either in finite or in infinite time. Moreover, for $n = 2$ and Ω being a ball, radially symmetric solutions of (1.1) blow up in finite time (cf. [4]). For $n \geqslant 3$, Winkler [13] showed that there exists an $\eta_0 > 0$ such that if $\|u_0\|_{L^q(\Omega)} < \eta_0$ and $\|\nabla v_0\|_{L^p(\Omega)} < \eta_0$ with $q > n/2$ and $p > n$, then the solution of (1.1) is global in time and bounded. If Ω is a ball in $\mathbb{R}^n$ with $n \geqslant 3$, then the radially symmetric solution of (1.1) blows up in finite time (cf. [12]). In addition, under the effect of the logistic source $f(u)$ which satisfies $f(u) \leqslant a - bu^2$ with $a \geqslant 0$ and $b > 0$, there exists a constant $b_0 > 0$ such that (1.1) possesses a uniquely determined global solution if $b > b_0$ (cf. [14]).

If we consider filling-volume effect in the chemotaxis model, that is, the movement of cells is inhibited near points where the cells are densely packed (cf. [10]), then (1.1) transforms into the system

$$(1.2)\qquad \begin{cases} u_t = \nabla(\mathcal{A}(u)\nabla u) - \nabla(\mathcal{B}(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of u. For $n \geqslant 1$, Tao and Winkler [11] proved that the classical solutions to (1.2) are uniformly-in-time bounded if an appropriate relation between $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ holds. Also, by modifying a well-established iterative technique, the authors of [11] proved a general boundedness result for quasilinear non-uniformly parabolic equations, which we use to get the $\|\cdot\|_{L^\infty(\Omega)}$ estimates of $u(\cdot,t)$ in Section 4. However, for $n = 2$, the radially symmetric solutions of (1.2) blow up in finite time if $\mathcal{B}(u)$ satisfies some an appropriate non-decay condition (cf. [2]). For $n \geqslant 3$, there exists an optimal blow-up time for the radially symmetric solutions provided one assumes some an appropriate non-decay of $\mathcal{B}(u)$ (cf. [3]). Furthermore, assuming that the logistic source $f(\cdot) \in C^\infty([0,\infty))$ satisfies

$$(1.3)\qquad f(0) \geqslant 0 \quad \text{and} \quad f(s) \leqslant \alpha s - \nu s^2, \quad s > 0,$$

with $\alpha \geqslant 0$, $\nu > 0$ and $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$ fulfil

$$(1.4)\qquad \mathcal{A}(\cdot), \mathcal{B}(\cdot) \in C^2([0,\infty)),\ \mathcal{A}(s) > 0, \quad s \geqslant 0,$$

$$c_1 s^p \leqslant \mathcal{A}(s), \qquad s \geqslant \iota, \tag{1.5}$$

$$c_1 s^q \leqslant \mathcal{B}(s) \leqslant c_2 s^q, \qquad s \geqslant \iota, \tag{1.6}$$

with $c_2 > c_1 > 0, \iota > 1$ and $p, q \in \mathbb{R}$, Cao [1] proved that if $q < 1$, then the classical solution of (1.2) is global in time and bounded.

Motivated by the above results, with (1.4)–(1.6) and $f(\cdot) \in C^1([0,\infty))$, we modify (1.3) into

$$f(0) = 0 \quad \text{and} \quad f(s) \leqslant \alpha s - \nu s^\beta, \quad s > 0, \tag{1.7}$$

where $\alpha \geqslant 0$, $\nu > 0$ and $\beta > 1$. Moreover, we prove that all the classical solutions of (1.2) globally exist for any $q < \beta - 1$. One can see that Cao's result [1] is the case $\beta = 2$ in this paper. The main difficulty of this paper is that we cannot get the desired a priori estimates of the solutions to (1.2) for all $\beta \in (1,\infty)$ at a time. It needs to depart $\beta \in (1,\infty)$ into $\beta \in (1,2)$ and $\beta \in [2,\infty)$, and then establish a priori estimates, respectively, for either of them (for the details, see the proof of Lemma 3.3 and Lemma 4.1). Now, we state the main results of this paper.

Theorem 1.1. *Assume that the nonnegative initial data satisfy $u_0 \in C^\mu(\overline{\Omega})$ with $0 < \mu < 1$ and $v_0 \in W^{1,\theta}(\Omega)$ with $\theta > n$. Let $\mathcal{A}(\cdot), \mathcal{B}(\cdot)$ satisfy (1.4)–(1.6) with some $q < \beta - 1$ and let $f(\cdot) \in C^1([0,\infty))$ fulfil (1.7). Then there exists a unique nonnegative classical solution $(u,v) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times [0,\infty)))^2$ which is globally bounded.*

The plan of this paper is as follows. In Section 2, we prove local existence and uniqueness of the classical solution to (1.2). In Section 3, a priori estimates of the solutions to (1.2) are established. In Section 4, basing on a priori estimates established in Section 3, we prove the main results of this paper.

2. Local existence and uniqueness of classical solutions

In this section, we give the results concerning local existence and uniqueness of the classical solutions to (1.2).

Lemma 2.1. *Assume that $\mathcal{A}(\cdot), \mathcal{B}(\cdot)$ satisfy (1.4)–(1.6), $f(\cdot)$ fulfils (1.7) and the nonnegative functions satisfy $u_0(x) \in C^\mu(\overline{\Omega}), v_0(x) \in W^{1,\theta}(\Omega)$ with $\mu \in (0,1)$ and $\theta > n$. Then there exist a maximal existence time $T_{\max} \in (0,\infty)$ and a unique pair of nonnegative functions $(u,v) \in (C^0(\overline{\Omega} \times [0,T_{\max})) \bigcap C^{2,1}(\overline{\Omega} \times (0,T_{\max})))^2$ such that (u,v) is a classical solution of (1.2) in $\Omega \times (0,T_{\max})$. Finally, if $T_{\max} < \infty$, then*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

P r o o f. The detailed proof of local existence and (2.1) can be referred to that of ([1], Lemma 2.1 and Lemma 2.2). Here we only prove uniqueness. The proof of uniqueness proceeds precisely along the lines of the arguments of Theorem 3.1 in [6]. Let $T \in (0, T_{\max})$. Suppose that both (u_1, v_1) and (u_2, v_2) solve (1.2) in the classical sense and fix $T_0 \in (0, T)$. From the proof of Lemma 2.1 in [1] and the semigroup estimates, we know that $u_i \in C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ for $i = 1, 2$ and

$$\|\nabla v_i(\cdot, t)\|_{L^\theta(\Omega)} \leqslant C_v(\|u_i\|_{L^\infty(\Omega\times(0,T))}, T, \|v_{i0}\|_{W^{1,\theta}(\Omega)}) \tag{2.2}$$

for $i = 1, 2$. Due to $u_i \in C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ for $i = 1, 2$, there exists a positive constant denoted by C_u which depends on T such that

$$\|\nabla u_i(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_u \quad \text{for } t \in (0, T) \quad \text{and } i = 1, 2. \tag{2.3}$$

Combining this with (1.2), in $\Omega \times [0, T_0]$ we have

$$\begin{cases} (u_1 - u_2)_t = \nabla[\mathcal{A}(u_1)\nabla u_1 - \mathcal{A}(u_2)\nabla u_2] - \nabla[\mathcal{B}(u_1)\nabla v_1 - \mathcal{B}(u_2)\nabla v_2] \\ \qquad\qquad\qquad + f(u_1) - f(u_2), \\ (v_1 - v_2)_t = \Delta(v_1 - v_2) - (v_1 - v_2) + u_1 - u_2. \end{cases} \tag{2.4}$$

Multiplying the first equation in (2.4) by $u_1 - u_2$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_1 - u_2|^2 = & -\int_\Omega [\mathcal{A}(u_1)\nabla u_1 - \mathcal{A}(u_2)\nabla u_2] \cdot \nabla(u_1 - u_2) \\ & + \int_\Omega [\mathcal{B}(u_1)\nabla v_1 - \mathcal{B}(u_2)\nabla v_2] \cdot \nabla(u_1 - u_2) \\ & + \int_\Omega [f(u_1) - f(u_2)](u_1 - u_2). \end{aligned} \tag{2.5}$$

It follows from (1.4) and the boundedness of u_i $(i = 1, 2)$ on $\Omega \times (0, T)$ that there exist positive constants c_3, c_4 and c_5 such that $|\mathcal{A}(u_1) - \mathcal{A}(u_2)| \leqslant c_3|u_1 - u_2|$, $|\mathcal{B}(u_1) - \mathcal{B}(u_2)| \leqslant c_4|u_1 - u_2|$ and $|\mathcal{B}(u_i)| \leqslant c_5$ $(i = 1, 2)$. Therefore, applying (1.5) and the Young inequality, we have

$$\begin{aligned} & -\int_\Omega [\mathcal{A}(u_1)\nabla u_1 - \mathcal{A}(u_2)\nabla u_2]\nabla(u_1 - u_2) \\ & \quad = -\int_\Omega \mathcal{A}(u_1)|\nabla(u_1 - u_2)|^2 - \int_\Omega [\mathcal{A}(u_1) - \mathcal{A}(u_2)]\nabla u_2 \nabla(u_1 - u_2) \\ & \quad \leqslant -\int_\Omega \mathcal{A}(u_1)|\nabla(u_1 - u_2)|^2 + \int_\Omega |\mathcal{A}(u_1) - \mathcal{A}(u_2)||\nabla u_2||\nabla(u_1 - u_2)| \\ & \quad \leqslant -c_1\iota^p \int_\Omega |\nabla(u_1 - u_2)|^2 + c_3 C_u \int_\Omega |u_1 - u_2||\nabla(u_1 - u_2)| \\ & \quad \leqslant -\frac{3c_1\iota^p}{4}\int_\Omega |\nabla(u_1 - u_2)|^2 + \frac{c_3^2 C_u^2}{c_1\iota^p}\int_\Omega |u_1 - u_2|^2. \end{aligned} \tag{2.6}$$

Also,

$$\begin{aligned}(2.7)\quad &\int_\Omega [\mathcal{B}(u_1)\nabla v_1 - \mathcal{B}(u_2)\nabla v_2]\nabla(u_1-u_2) \\ &= \int_\Omega [\mathcal{B}(u_1)-\mathcal{B}(u_2)]\nabla v_1 \nabla(u_1-u_2) + \int_\Omega \mathcal{B}(u_2)\nabla(v_1-v_2)\nabla(u_1-u_2) \\ &\leqslant \int_\Omega |\mathcal{B}(u_1)-\mathcal{B}(u_2)||\nabla v_1||\nabla(u_1-u_2)| + \int_\Omega |\mathcal{B}(u_2)||\nabla(v_1-v_2)||\nabla(u_1-u_2)| \\ &\leqslant c_4 \int_\Omega |u_1-u_2||\nabla v_1||\nabla(u_1-u_2)| + c_5\int_\Omega |\nabla(v_1-v_2)||\nabla(u_1-u_2)|.\end{aligned}$$

It follows from the Hölder, the Young, the Gagliardo-Nirenberg inequalities and (2.2) that

$$\begin{aligned}(2.8)\quad c_4\int_\Omega |u_1-u_2||\nabla v_1||\nabla(u_1-u_2)| &\leqslant c_4\Big(\int_\Omega |\nabla(u_1-u_2)|^2\Big)^{1/2} \\ &\quad\times \Big(\int_\Omega |\nabla v_1|^\theta\Big)^{1/\theta}\Big(\int_\Omega |u_1-u_2|^{2\theta/(\theta-2)}\Big)^{(\theta-2)/(2\theta)} \\ &\leqslant c_4 C_{GN}\Big(\int_\Omega |\nabla(u_1-u_2)|^2\Big)^{1/2+n/(2\theta)} \\ &\quad\times \Big(\int_\Omega |\nabla v_1|^\theta\Big)^{1/\theta}\Big(\int_\Omega |u_1-u_2|^2\Big)^{(\theta-n)/(2\theta)} \\ &\leqslant c_4 C_{GN} C_v\Big(\int_\Omega |\nabla(u_1-u_2)|^2\Big)^{1/2+n/(2\theta)}\Big(\int_\Omega |u_1-u_2|^2\Big)^{(\theta-n)/(2\theta)} \\ &\leqslant \frac{c_1\iota^p}{2}\int_\Omega |\nabla(u_1-u_2)|^2 + M\int_\Omega |u_1-u_2|^2\end{aligned}$$

and

$$(2.9)\quad c_5\int_\Omega |\nabla(v_1-v_2)||\nabla(u_1-u_2)| \leqslant \frac{c_1\iota^p}{4}\int_\Omega |\nabla(u_1-u_2)|^2 + \frac{c_5^2}{c_1\iota^p}\int_\Omega |\nabla(v_1-v_2)|^2,$$

where C_{GN} comes from the Gagliardo-Nirenberg inequality,

$$M := \frac{\theta-n}{2\theta}\Big(\frac{\theta+n}{c_1\iota^p\theta}\Big)^{(\theta+n)/(\theta-n)}(c_4 C_{GN} C_v)^{2\theta/(\theta-n)}$$

and we have used that $\int_\Omega (u_1-u_2)=0$ by simple integration of the first equation of (2.4). By the local Lipschitz continuity of $f(\cdot)$, we have

$$(2.10)\quad \int_\Omega [f(u_1)-f(u_2)](u_1-u_2) \leqslant L_f\int_\Omega |u_1-u_2|^2,$$

where $L_f > 0$ is the Lipschitz constant. It follows from (2.5)–(2.10) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_1-u_2|^2 \leqslant \Big(\frac{c_3^2C_u^2}{c_1\iota^p}+M+L_f\Big)\int_\Omega |u_1-u_2|^2+\frac{c_5^2}{c_1\iota^p}\int_\Omega |\nabla(v_1-v_2)|^2. \tag{2.11}$$

Multiplying the second equation of (2.4) by $\Delta(v_1 - v_2)$ we get that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla(v_1-v_2)|^2 + \int_\Omega |\Delta(v_1-v_2)|^2 + \int_\Omega |\nabla(v_1-v_2)|^2 \\ = -\int_\Omega (u_1-u_2)\Delta(v_1-v_2) \\ \leqslant \frac{1}{4}\int_\Omega |u_1-u_2|^2 + \int_\Omega |\Delta(v_1-v_2)|^2,\end{aligned} \tag{2.12}$$

i.e.,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla(v_1-v_2)|^2 + \int_\Omega |\nabla(v_1-v_2)|^2 \leqslant \frac{1}{4}\int_\Omega |u_1-u_2|^2. \tag{2.13}$$

Adding (2.11) and (2.13), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_\Omega |u_1-u_2|^2 + \int_\Omega |\nabla(v_1-v_2)|^2\Big] \\ \leqslant c_6\Big[\int_\Omega |u_1-u_2|^2 + \int_\Omega |\nabla(v_1-v_2)|^2\Big], \quad t\in(0,T_0),\end{aligned} \tag{2.14}$$

where $c_6 := \max\{c_3^2C_u^2/c_1\iota^p + M + L_f + 1/4,\ c_5^2/c_1\iota^p\}$. The Gronwall inequality and (2.14) imply that $u_1 = u_2$ and $v_1 = v_2$ in $\Omega\times(0,T)$, because $T_0 \in (0,T)$ is arbitrary. □

3. A priori estimates

First, let us state a lemma which we will use in the establishment of a priori estimates.

Lemma 3.1. *Let $r \in (1,\infty)$, $T \in (0, T_{\max})$ and let (u, v) be the solution of (1.2). Then for $s \in (0,T)$ there exists $C_r > 0$ such that*

$$\begin{aligned}\int_s^T \|v(\cdot,t)\|^r_{L^r(\Omega)}\,\mathrm{d}t + \int_s^T \|v_t(\cdot,t)\|^r_{L^r(\Omega)}\,\mathrm{d}t + \int_s^T \|\Delta v(\cdot,t)\|^r_{L^r(\Omega)}\,\mathrm{d}t \\ \leqslant C_r\int_s^T \|u(\cdot,t)\|^r_{L^r(\Omega)}\,\mathrm{d}t + C_r\int_s^T \|v(\cdot,t)\|^r_{L^r(\Omega)}\,\mathrm{d}t \\ + C_r\|v(\cdot,s)\|^r_{L^r(\Omega)} + C_r\|\Delta v(\cdot,s)\|^r_{L^r(\Omega)}.\end{aligned} \tag{3.1}$$

Proof. From (1.2) and Lemma 2.1 we know that $v(s) \in W^{2,r}(\Omega)$ with $(\partial v/\partial n)(s) = 0$ on $\partial\Omega$ for $s \in (0,T)$. Therefore, (3.1) directly follows from Lemma 3.1 in [1]. Thus, we complete the proof. □

In addition, by Lemma 2.1 we know that for any $s \in (0, T_{\max})$, $(u(\cdot,s), v(\cdot,s)) \in (C^2(\overline{\Omega}))^2$ and $\partial v(\cdot,s)/\partial n = 0$ on $\partial\Omega$. Therefore, there exists a constant $K > 0$ such that

$$\sup_{0\leqslant\tau\leqslant s} \|u(\tau)\|_{L^\infty(\Omega)} \leqslant K, \quad \sup_{0\leqslant\tau\leqslant s} \|v(\tau)\|_{L^\infty(\Omega)} \leqslant K \quad \text{and} \quad \|\Delta v(s)\|_{L^\infty(\Omega)} \leqslant K. \tag{3.2}$$

In the following two lemmas, we establish a priori estimates for the solutions to (1.2).

Lemma 3.2. *Assume $s \in (0, T_{\max})$, (u,v) is the solution of (1.2) and f satisfies (1.7) with $\beta \in (1,\infty)$. Then for any $T \in (s, T_{\max})$ there exists $C > 0$ such that*

$$\int_\Omega u \leqslant C, \quad t \in (s,T), \quad \int_s^T \int_\Omega u^\beta \leqslant C(T+1). \tag{3.3}$$

Moreover, if $\beta \in (1,2)$, then

$$\int_\Omega u^{\beta-1} \leqslant C \tag{3.4}$$

also holds for all $t \in (s,T)$ and some $C > 0$.

Proof. With help of the Hölder inequality, we integrate the first equation in (1.2) and then get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \leqslant \alpha\int_\Omega u - \nu\int_\Omega u^\beta \leqslant \alpha\int_\Omega u - \frac{\nu}{|\Omega|^{\beta-1}}\Big(\int_\Omega u\Big)^\beta, \quad t \in (s,T). \tag{3.5}$$

It follows from (3.5) that

$$\int_\Omega u \leqslant \max\Big\{K|\Omega|, \Big(\frac{\alpha}{\nu}\Big)^{1/(\beta-1)}|\Omega|\Big\}, \quad t \in (s,T). \tag{3.6}$$

Integrating (3.5) over (s,T) with respect to t and using (3.2), we have

$$\int_s^T\int_\Omega u^\beta \leqslant \frac{\alpha}{\nu}\int_s^T\int_\Omega u + \frac{1}{\nu}\int_\Omega u(s) \leqslant \frac{\alpha}{\nu}\int_s^T\int_\Omega u + \frac{K}{\nu}|\Omega|, \quad t \in (s,T). \tag{3.7}$$

Combining (3.6) with (3.7), we obtain

$$\int_s^T\int_\Omega u^\beta \leqslant \max\Big\{\frac{K|\Omega|}{\nu}(\alpha+1), \Big[\Big(\frac{\alpha}{\nu}\Big)^{\beta/(\beta-1)} + \frac{K}{\nu}\Big]|\Omega|\Big\}(T+1). \tag{3.8}$$

It follows from (3.6) and the Hölder inequality that if $\beta \in (1,2)$, then

$$\int_\Omega u^{\beta-1} \leqslant \Big(\int_\Omega u\Big)^{\beta-1}|\Omega|^{2-\beta} \leqslant \max\Big\{K^{\beta-1}|\Omega|, \frac{\alpha}{\nu}|\Omega|\Big\}. \tag{3.9}$$

Therefore, combining (3.6), (3.8) with (3.9), we can choose

$$\begin{aligned} C := \max\Big\{&K|\Omega|, \Big(\frac{\alpha}{\nu}\Big)^{1/(\beta-1)}|\Omega|, \frac{K|\Omega|}{\nu}(\alpha+1), \Big[\Big(\frac{\alpha}{\nu}\Big)^{\beta/(\beta-1)} + \frac{K}{\nu}\Big]|\Omega|, \\ &K^{\beta-1}|\Omega|, \frac{\alpha}{\nu}|\Omega|, |\Omega|\Big\}, \end{aligned} \tag{3.10}$$

which together with (3.6), (3.8) and (3.9) implies (3.3) and (3.4). □

In order to improve the regularity of u in a higher L^p space, we give the following lemma with help of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. (i) *Let $s \in (0, T_{\max})$, $T \in (s, T_{\max})$, $\delta \geqslant \beta - 1$, $\beta \in (1,2)$ and $\gamma = (\beta - q)\delta + \beta - 1 - q$. If*

$$\int_\Omega u^\delta \leqslant C_1(T+1) \quad \text{for any } t \in (s,T) \quad \text{and} \quad \int_s^T\int_\Omega u^{\delta+1} \leqslant C_1(T+1) \tag{3.11}$$

hold for some constant $C_1 > 0$, then there exist $C_2(\gamma,\nu,q,\alpha,K,|\Omega|) > 0$ and $Q(\nu,q,\alpha,K,|\Omega|) > 0$ such that

$$\int_\Omega u^\gamma \leqslant C_2Q^\gamma C_1(T+1) \text{ for any } t \in (s,T) \text{ and } \int_s^T\int_\Omega u^{\gamma+1} \leqslant C_2Q^\gamma C_1(T+1). \tag{3.12}$$

(ii) *Let $s \in (0, T_{\max})$, $T \in (s, T_{\max})$, $\delta \geqslant 1$, $\beta \in [2,\infty)$ and $\gamma = (\beta - q)\delta + \beta - 1 - q$. If*

$$\int_\Omega u^\delta \leqslant C_1'(T+1) \quad \text{for any } t \in (s,T) \quad \text{and} \quad \int_s^T\int_\Omega u^{\delta+\beta-1} \leqslant C_1'(T+1) \tag{3.13}$$

hold for some constant $C_1' > 0$, then there exist $C_2'(\gamma,\nu,q,\alpha,K,|\Omega|) > 0$ and $Q'(\nu,q,\alpha,K,|\Omega|) > 0$ such that

$$\begin{aligned} &\int_\Omega u^\gamma \leqslant C_2'Q'^\gamma C_1'(T+1) \quad \text{for any } t \in (s,T) \text{ and} \\ &\int_s^T\int_\Omega u^{\gamma+\beta-1} \leqslant C_2'Q'^\gamma C_1'(T+1). \end{aligned} \tag{3.14}$$

P r o o f. First, we prove part (i). Combining (1.5) with (1.6) and multiplying the first equation in (1.2) by $(\gamma-\beta+2)u^{\gamma-\beta+1}$, we integrate it by parts over Ω and then get that

$$
\begin{aligned}
(3.15)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\gamma-\beta+2} &\leqslant -(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \mathcal{A}(u)u^{\gamma-\beta}|\nabla u|^2\\
&\quad+(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \mathcal{B}(u)u^{\gamma-\beta}\nabla u\nabla v\\
&\quad+\alpha(\gamma-\beta+2)\int_\Omega u^{\gamma-\beta+2}-\nu(\gamma-\beta+2)\int_\Omega u^{\gamma+1}\\
&\leqslant (\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \nabla\mathcal{F}(u)\nabla v\\
&\quad+\alpha(\gamma-\beta+2)\int_\Omega u^{\gamma-\beta+2}-\nu(\gamma-\beta+2)\int_\Omega u^{\gamma+1},\quad t\in(s,T),
\end{aligned}
$$

where $\mathcal{F}(u)=\int_0^u \mathcal{B}(\sigma)\sigma^{\gamma-\beta}\,\mathrm{d}\sigma$. It follows from the second equation in (1.2) and from (1.6) that

$$
\begin{aligned}
(3.16)\quad &(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \nabla\mathcal{F}(u)\nabla v\\
&= -(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \mathcal{F}(u)(v_t+v-u)\\
&\leqslant -(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \mathcal{F}(u)v_t+(\gamma-\beta+2)(\gamma-\beta+1)\int_\Omega \mathcal{F}(u)u\\
&\leqslant \frac{c_2(\gamma-\beta+2)(\gamma-\beta+1)}{\gamma-\beta+q+1}\Big(\int_\Omega u^{\gamma-\beta+q+1}|v_t|+\int_\Omega u^{\gamma-\beta+q+2}\Big)\\
&\leqslant c_7(\gamma-\beta+2)\Big(\int_\Omega u^{\gamma-\beta+q+1}|v_t|+\int_\Omega u^{\gamma-\beta+q+2}\Big),\quad t\in(s,T),
\end{aligned}
$$

where $c_7:=\sup\limits_{\gamma\geqslant\beta^2-q\beta-1} c_2(\gamma-\beta+1)/(\gamma-\beta+q+1)>0$. By the Young inequality, we obtain

$$
(3.17)\qquad \int_\Omega u^{\gamma-\beta+q+1}|v_t|\leqslant\frac{\nu}{4c_7}\int_\Omega u^{(\gamma-\beta+q+1)l_1}+C(l_1,\nu)\int_\Omega |v_t|^{l_1/(l_1-1)},
$$

where

$$
(3.18)\qquad \begin{cases}
l_1=\dfrac{\gamma+1}{\gamma-\beta+q+1},\\
C(l_1,\nu)=\dfrac{\beta-q}{\gamma-\beta+q+1}\Big(1+\dfrac{\beta-q}{\gamma-\beta+q+1}\Big)^{-(\gamma+1)/(\beta-q)}\\
\qquad\qquad\times\Big(\dfrac{\nu}{4c_7}\Big)^{-\gamma/(\beta-q)-(q-\beta+1)/(\beta-q)}.
\end{cases}
$$

Since $\delta \geqslant \beta - 1$ implies $\gamma \geqslant (\beta - q)(\beta - 1) + \beta - 1 - q$, it can be deduced that

$$\frac{\beta - q}{\gamma - \beta + q + 1} \leqslant \frac{(\beta - q)(\beta - 1) + \beta - 1 - q}{(\beta - 1)\gamma} \leqslant \frac{(\beta - q)(\beta - 1) + 1 - q}{(\beta - 1)(\gamma - \beta + 2)}. \tag{3.19}$$

Therefore, we can choose

$$\begin{cases} Q_1 > \max\Big\{\Big(\dfrac{4c_7}{\nu}\Big)^{1/(\beta - q)}, 1\Big\}, \\ c_8 > \sup\limits_{\gamma > \beta^2 - q\beta - 1} \dfrac{(\beta - q)(\beta - 1) + 1 - q}{\beta - 1} \\ \qquad \times\Big(1 + \dfrac{\beta - q}{\gamma - \beta + q + 1}\Big)^{-(\gamma+1)/(\beta - q)} \Big(\dfrac{\nu}{4c_7}\Big)^{-(q - \beta + 1)/(\beta - q)}. \end{cases} \tag{3.20}$$

Then (3.17) can be rewritten as

$$\int_\Omega u^{\gamma - \beta + q + 1}|v_t| \leqslant \frac{\nu}{4c_7}\int_\Omega u^{\gamma+1} + \frac{c_8}{\gamma - \beta + 2} Q_1^\gamma \int_\Omega |v_t|^{\delta+1}, \tag{3.21}$$

where $\delta + 1 = l_1/(l_1 - 1)$. Similarly, we can also get that

$$\int_\Omega u^{\gamma - \beta + q + 2} \leqslant \frac{\nu}{4c_7}\int_\Omega u^{(\gamma - \beta + q + 2)l_2} + C(l_2, \nu)|\Omega| \tag{3.22}$$

with

$$\begin{cases} l_2 = \dfrac{\gamma + 1}{\gamma - \beta + q + 2}, \\ C(l_2, \nu) = \dfrac{\beta - q - 1}{\gamma - \beta + q + 2}\Big(1 + \dfrac{\beta - q - 1}{\gamma - \beta + q + 2}\Big)^{-(\gamma+1)/(\beta - q - 1)} \\ \qquad \times\Big(\dfrac{\nu}{4c_7}\Big)^{-\gamma/(\beta - q - 1) - (q - \beta + 2)/(\beta - q - 1)}. \end{cases} \tag{3.23}$$

Since

$$\frac{\beta - q - 1}{\gamma - \beta + q + 2} \leqslant \frac{\beta - q}{\gamma - \beta + q + 1}, \tag{3.24}$$

thus applying (3.19) we choose

$$\begin{cases} Q_2 > \max\Big\{\Big(\dfrac{4c_7}{\nu}\Big)^{1/(\beta - q - 1)}, 1\Big\}, \\ c_9 > \sup\limits_{\gamma > \beta^2 - q\beta - 1} \dfrac{(\beta - q)(\beta - 1) + 1 - q}{\beta - 1} \\ \qquad \times\Big(1 + \dfrac{\beta - q - 1}{\gamma - \beta + q + 2}\Big)^{-(\gamma+1)/(\beta - q - 1)} \Big(\dfrac{\nu}{4c_7}\Big)^{-(q - \beta + 2)/(\beta - q - 1)}. \end{cases} \tag{3.25}$$

Therefore, we rewrite (3.22) as

$$\int_\Omega u^{\gamma-\beta+q+2} \leqslant \frac{\nu}{4c_7}\int_\Omega u^{\gamma+1} + \frac{c_9 Q_2^\gamma}{\gamma-\beta+2}|\Omega|. \tag{3.26}$$

Analogously,

$$\int_\Omega u^{\gamma-\beta+2} \leqslant \frac{\nu}{4\alpha}\int_\Omega u^{(\gamma-\beta+2)l_3} + C(l_3,\nu,\alpha)|\Omega|, \tag{3.27}$$

where

$$\begin{cases} l_3 = \dfrac{\gamma+1}{\gamma-\beta+2}, \\ C(l_3,\nu,\alpha) = \dfrac{\beta-1}{\gamma-\beta+2} \\ \qquad\qquad \times\Big(1+\dfrac{\beta-1}{\gamma-\beta+2}\Big)^{-(\gamma+1)/(\beta-1)}\Big(\dfrac{\nu}{4\alpha}\Big)^{-\gamma/(\beta-1)-(2-\beta)/(\beta-1)}. \end{cases} \tag{3.28}$$

Choosing

$$\begin{cases} Q_3 > \max\Big\{\Big(\dfrac{4\alpha}{\nu}\Big)^{1/(\beta-1)}, 1\Big\}, \\ c_{10} > \sup\limits_{\gamma>\beta^2-q\beta-1}(\beta-1)\Big(1+\dfrac{\beta-1}{\gamma-\beta+2}\Big)^{-(\gamma+1)/(\beta-1)}\Big(\dfrac{\nu}{4\alpha}\Big)^{-(2-\beta)/(\beta-1)}, \end{cases} \tag{3.29}$$

we have

$$\int_\Omega u^{\gamma-\beta+2} \leqslant \frac{\nu}{4\alpha}\int_\Omega u^{\gamma+1} + \frac{c_{10} Q_3^\gamma}{\gamma-\beta+2}|\Omega|. \tag{3.30}$$

Let $Q_4 := \max\{Q_1, Q_2, Q_3\}$ and $c_{11} := \max\{c_8, c_9, c_{10}\}$. It follows from (3.15), (3.16), (3.21), (3.26) and (3.30) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\gamma-\beta+2} &\leqslant -\frac{\nu(\gamma-\beta+2)}{4}\int_\Omega u^{\gamma+1} + c_7 c_{11} Q_4^\gamma \int_\Omega |v_t|^{\delta+1} \\ &\quad + c_7 c_{11} Q_4^\gamma|\Omega| + \alpha c_{11} Q_4^\gamma |\Omega| \\ &\leqslant -\frac{\nu(\gamma-\beta+2)}{4}\int_\Omega u^{\gamma+1} + c_{12} Q_4^\gamma \int_\Omega |v_t|^{\delta+1} \\ &\quad + c_{12} Q_4^\gamma |\Omega|, \quad t\in(s,T), \end{aligned} \tag{3.31}$$

where $c_{12} := c_7 c_{11} + \alpha c_{11}$. We integrate (3.31) over (s,T) and (s,t) for any $t\in(s,T)$ and get, respectively, that

$$\frac{\nu(\gamma-\beta+2)}{4}\int_s^T\!\!\int_\Omega u^{\gamma+1} \leqslant \int_\Omega u^\gamma(s) + c_{12} Q_4^\gamma \int_s^T\!\!\int_\Omega |v_t|^{\delta+1} + c_{12} Q_4^\gamma |\Omega|(T+1) \tag{3.32}$$

and

$$\int_\Omega u^{\gamma-\beta+2}(t) \leqslant \int_\Omega u^{\gamma-\beta+2}(s) + c_{12}Q_4^\gamma \int_s^T\int_\Omega |v_t|^{\delta+1} + c_{12}Q_4^\gamma|\Omega|(T+1). \tag{3.33}$$

Now, with help of the Young inequality, we multiply the second equation of (1.2) by $(\delta+1)v^\delta$ and integrate by parts to get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^{\delta+1} &= -\delta(\delta+1)\int_\Omega v^{\delta-1}|\nabla v|^2 - (\delta+1)\int_\Omega v^{\delta+1} + (\delta+1)\int_\Omega uv^\delta \\ &\leqslant -\int_\Omega v^{\delta+1} + \int_\Omega u^{\delta+1}, \quad t\in(s,T).\end{aligned} \tag{3.34}$$

Since (3.10) implies $C_1 > |\Omega|$, it follows from (3.11) and (3.34) that

$$\begin{aligned}\int_s^T\int_\Omega v^{\delta+1} &\leqslant \int_\Omega v^{\delta+1}(s) + \int_s^T\int_\Omega u^{\delta+1} \\ &\leqslant K^{\delta+1}|\Omega| + C_1(T+1) \leqslant c_{13}Q_5^\delta C_1(T+1),\end{aligned} \tag{3.35}$$

where $Q_5 := \max\{K,1\}$ and $c_{13} := K+1$. Lemma 3.1 ensures that

$$\begin{aligned}&\int_s^T\int_\Omega |v_t|^{\delta+1} \\ &\leqslant C_\delta\int_s^T\int_\Omega u^{\delta+1} + C_\delta\int_s^T\int_\Omega v^{\delta+1} + C_\delta\int_\Omega|\Delta v(s)|^{\delta+1} + C_\delta\int_\Omega v^{\delta+1}(s) \\ &\leqslant C_1C_\delta(T+1) + c_{13}Q_5^\delta C_1C_\delta(T+1) + 2K^{\delta+1}|\Omega|C_\delta(T+1) \\ &\leqslant c_{14}Q_5^\delta C_1(T+1)\end{aligned} \tag{3.36}$$

with C_δ being a constant derived from Lemma 3.1 and $c_{14} := C_\delta(1+c_{13}+2Q_5)$. Combining (3.36) with (3.32)–(3.33), we obtain

$$\begin{aligned}\int_\Omega u^{\gamma-\beta+2} &\leqslant Q_5^\gamma K^{2-\beta}|\Omega| + c_{12}Q_4^\gamma c_{14}Q_5^\delta C_1(T+1) + c_{12}Q_4^\gamma|\Omega|(T+1) \\ &\leqslant c_{15}Q^\gamma C_1(T+1), \quad t\in(s,T)\end{aligned} \tag{3.37}$$

and

$$\int_s^T\int_\Omega u^{\gamma+1} \leqslant \frac{4c_{15}Q^\gamma}{\nu(\gamma-\beta+2)}C_1(T+1) \leqslant \frac{4c_{15}Q^\gamma}{\nu}C_1(T+1), \tag{3.38}$$

where $c_{15} := c_{13}^{2-\beta} + c_{12}c_{14} + c_{12}$, $Q := Q_4Q_5$. It follows from the Hölder inequality and (3.37) that

$$\begin{aligned}\int_\Omega u^\gamma &\leqslant \left(\int_\Omega u^{\gamma-\beta+2}\right)^{\gamma/(\gamma-\beta+2)}|\Omega|^{(2-\beta)/(\gamma-\beta+2)} \\ &\leqslant [c_{15}Q^\gamma C_1(T+1)]^{\gamma/(\gamma-\beta+2)}|\Omega|^{(2-\beta)/(\gamma-\beta+2)}, \quad t\in(s,T).\end{aligned} \tag{3.39}$$

By (3.10), we know that $C_1 > |\Omega|$. Combining it with $c_{15} > 1$ and $Q > 1$, we have $|\Omega| < c_{15}Q^\gamma C_1(T+1)$. Therefore, (3.39) implies that

$$\int_\Omega u^\gamma \leqslant c_{15}Q^\gamma C_1(T+1), \quad t \in (s,T). \tag{3.40}$$

Thus, by virtue of (3.38), if we choose $C_2 := \max\{c_{15}, 4c_{15}/\nu\}$, we obtain (3.12).

Now, we prove part (ii). Since the case $\beta = 2$ has been studied in [1], we only consider the case $\beta \in (2,\infty)$. Multiplying the first equation of (1.2) by $\gamma u^{\gamma-1}$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^\gamma \leqslant \gamma(\gamma-1)\int_\Omega \nabla\mathcal{F}^*(u)\nabla v + \alpha\gamma\int_\Omega u^\gamma - \nu\gamma\int_\Omega u^{\gamma+\beta-1}, \quad t \in (s,T), \tag{3.41}$$

where $\mathcal{F}^*(u) = \int_0^u \mathcal{B}(\sigma)\sigma^{\gamma-2}\,\mathrm{d}\sigma$. Similarly to (3.16), it follows from the second equation of (1.2) and from (1.6) that

$$\begin{aligned}\gamma(\gamma-1)\int_\Omega \nabla\mathcal{F}^*(u)\nabla v &\leqslant -\gamma(\gamma-1)\int_\Omega \mathcal{F}^*(u)v_t + \gamma(\gamma-1)\int_\Omega \mathcal{F}^*(u)u \\ &\leqslant \gamma(\gamma-1)\Big[\int_\Omega \mathcal{F}^*(u)|v_t| + \int_\Omega \mathcal{F}^*(u)u\Big] \\ &\leqslant c_7'\gamma\Big(\int_\Omega u^{\gamma+q-1}|v_t| + \int_\Omega u^{\gamma+q}\Big), \quad t \in (s,T)\end{aligned} \tag{3.42}$$

with $c_7' := \sup\limits_{\gamma\geqslant 2\beta-2q-1} c_2(\gamma-1)/(\gamma+q-1)$, thus by the Young inequality, we have

$$\int_\Omega u^{\gamma+q-1}|v_t| \leqslant \frac{\nu}{4c_7'}\int_\Omega u^{(\gamma+q-1)l_1'} + C(l_1',\nu)\int_\Omega |v_t|^{l_1'/(l_1'-1)}, \tag{3.43}$$

where

$$\begin{cases} l_1' = \dfrac{\gamma+\beta-1}{\gamma+q-1}, \\ C(l_1',\nu) = \dfrac{\beta-q}{\gamma+q-1}\Big(1+\dfrac{\beta-q}{\gamma+q-1}\Big)^{-(\gamma+\beta-1)/(\beta-q)}\Big(\dfrac{\nu}{4c_7'}\Big)^{-(\gamma+q-1)/(\beta-q)}. \end{cases} \tag{3.44}$$

Considering that

$$\frac{2\beta-2q-1}{\gamma} \geqslant \frac{\beta-q}{\gamma-\beta+q+1} \geqslant \frac{\beta-q}{\gamma+q-1}, \tag{3.45}$$

we choose

$$\begin{cases} Q_1' > \max\Big\{\Big(\dfrac{4c_7'}{\nu}\Big)^{1/(\beta-q)}, 1\Big\}, \\ c_8' > \sup\limits_{\gamma>2\beta-2q-1}(2\beta-2q-1) \\ \qquad \times\Big(1+\dfrac{\beta-q}{\gamma+q-1}\Big)^{-(\gamma+\beta-1)/(\beta-q)}\Big(\dfrac{\nu}{4c_7'}\Big)^{-(q-1)/(\beta-q)}. \end{cases} \tag{3.46}$$

It follows from (3.43)–(3.46) that

$$\int_\Omega u^{\gamma+q-1}|v_t| \leqslant \frac{\nu}{4c_7'}\int_\Omega u^{\gamma+\beta-1} + \frac{c_8' {Q_1'}^{\gamma}}{\gamma}\int_\Omega |v_t|^{(\gamma+\beta-1)/(\beta-q)}. \tag{3.47}$$

By a simple computation, we can obtain that

$$\frac{\gamma+\beta-1}{\beta-q} < \delta+\beta-1 = \frac{\gamma+1+(\beta-q)(\beta-2)}{\beta-q}. \tag{3.48}$$

Thus, the Young inequality entails that

$$\begin{aligned}\int_\Omega |v_t|^{(\gamma+\beta-1)/(\beta-q)} &\leqslant \frac{\gamma+\beta-1}{\gamma+1+(\beta-q)(\beta-2)}\int_\Omega |v_t|^{\delta+\beta-1} + \frac{(\beta-q-1)(\beta-2)|\Omega|}{\gamma+1+(\beta-q)(\beta-2)} \\ &\leqslant \int_\Omega |v_t|^{\delta+\beta-1} + |\Omega|.\end{aligned} \tag{3.49}$$

With help of (3.47) and (3.49), we have

$$\int_\Omega u^{\gamma+q-1}|v_t| \leqslant \frac{\nu}{4c_7'}\int_\Omega u^{\gamma+\beta-1} + \frac{c_8' {Q_1'}^{\gamma}}{\gamma}\int_\Omega |v_t|^{\delta+\beta-1} + \frac{c_8' {Q_1'}^{\gamma}}{\gamma}|\Omega|. \tag{3.50}$$

Also, from the Young inequality, we obtain

$$\int_\Omega u^{\gamma+q} \leqslant \frac{\nu}{4c_7'}\int_\Omega u^{\gamma+\beta-1} + C(l_2',\nu)|\Omega| \tag{3.51}$$

with

$$\begin{cases} l_2' = \dfrac{\gamma+\beta-1}{\gamma+q}, \\ C(l_2',\nu) = \dfrac{\beta-1-q}{\gamma+q} \\ \qquad\qquad \times\Big(1+\dfrac{\beta-1-q}{\gamma+q}\Big)^{-(\gamma+\beta-1)/(\beta-1-q)}\Big(\dfrac{\nu}{4c_7'}\Big)^{-(\gamma+q)/(\beta-1-q)}. \end{cases} \tag{3.52}$$

From (3.45) we know that

$$\frac{2\beta-2q-1}{\gamma} \geqslant \frac{\beta-q}{\gamma+q-1} \geqslant \frac{\beta-1-q}{\gamma+q}. \tag{3.53}$$

Therefore, choosing

$$(3.54)\qquad \begin{cases} Q_2' > \max\Big\{\Big(\dfrac{4c_7'}{\nu}\Big)^{1/(\beta-1-q)}, 1\Big\}, \\ c_9' > \sup\limits_{\gamma>2\beta-2q-1} (2\beta-2q-1) \\ \qquad \times\Big(1+\dfrac{\beta-1-q}{\gamma+q}\Big)^{-(\gamma+\beta-1)/(\beta-1-q)}\Big(\dfrac{\nu}{4c_7'}\Big)^{-q/(\beta-1-q)}, \end{cases}$$

we can rewrite (3.51) as

$$(3.55)\qquad \int_\Omega u^{\gamma+q} \leqslant \frac{\nu}{4c_7'}\int_\Omega u^{\gamma+\beta-1} + \frac{c_9' {Q_2'}^\gamma}{\gamma}|\Omega|.$$

Similarly, we have

$$(3.56)\qquad \int_\Omega u^{\gamma} \leqslant \frac{\nu}{4\alpha}\int_\Omega u^{\gamma+\beta-1} + C(l_3', \nu, \alpha)|\Omega|,$$

where

$$(3.57)\qquad \begin{cases} l_3' = \dfrac{\gamma+\beta-1}{\gamma}, \\ C(l_3',\nu,\alpha) = \dfrac{\beta-1}{\gamma}\Big(1+\dfrac{\beta-1}{\gamma}\Big)^{-(\gamma+\beta-1)/(\beta-1)}\Big(\dfrac{\nu}{4\alpha}\Big)^{-\gamma/(\beta-1)}. \end{cases}$$

Choosing

$$(3.58)\qquad \begin{cases} Q_3' > \max\Big\{\Big(\dfrac{4\alpha}{\nu}\Big)^{1/(\beta-1)}, 1\Big\} \\ c_{10}' > \sup\limits_{\gamma>2\beta-2q-1} (\beta-1)\Big(1+\dfrac{\beta-1}{\gamma}\Big)^{-(\gamma+\beta-1)/(\beta-1)} \end{cases}$$

and combining it with (3.56) and (3.57), we get that

$$(3.59)\qquad \int_\Omega u^{\gamma} \leqslant \frac{\nu}{4\alpha}\int_\Omega u^{\gamma+\beta-1} + \frac{c_{10}' {Q_3'}^\gamma}{\gamma}|\Omega|.$$

It follows from (3.41), (3.42), (3.50), (3.55) and (3.59) that

$$(3.60)\qquad \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^\gamma \leqslant & -\frac{\nu\gamma}{4}\int_\Omega u^{\gamma+\beta-1} + c_7' c_8' {Q_1'}^\gamma \int_\Omega |v_t|^{\delta+\beta-1} \\ & + c_7' c_8' {Q_1'}^\gamma |\Omega| + c_7' c_9' {Q_2'}^\gamma |\Omega| + \alpha c_{10}' {Q_3'}^\gamma |\Omega| \\ \leqslant & -\frac{\nu\gamma}{4}\int_\Omega u^{\gamma+\beta-1} + c_{11}' {Q_4'}^\gamma \int_\Omega |v_t|^{\delta+\beta-1} + c_{11}' {Q_4'}^\gamma |\Omega|, \ t\in(s,T), \end{aligned}$$

where $Q_4' := \max\{Q_1', Q_2', Q_3'\}$ and $c_{11}' := c_7'c_8' + c_7'c_9' + \alpha c_{10}'$. After integrating (3.60) over (s, T) and (s, t) for any $t \in (s, T)$ we obtain, respectively,

$$\frac{\nu\gamma}{4}\int_s^T\int_\Omega u^{\gamma+\beta-1} \leqslant \int_\Omega u^\gamma(s) + c_{11}'{Q_4'}^\gamma\int_s^T\int_\Omega |v_t|^{\delta+\beta-1} + c_{11}'{Q_4'}^\gamma|\Omega|(T+1) \tag{3.61}$$

and

$$\int_\Omega u^\gamma(t) \leqslant \int_\Omega u^\gamma(s) + c_{11}'{Q_4'}^\gamma\int_s^T\int_\Omega |v_t|^{\delta+\beta-1} + c_{11}'{Q_4'}^\gamma|\Omega|(T+1). \tag{3.62}$$

Now, multiplying the second equation of (1.2) by $(\delta+\beta-1)v^{\delta+\beta-2}$ and integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^{\delta+\beta-1} =& -(\delta+\beta-1)(\delta+\beta-2)\int_\Omega v^{\delta+\beta-3}|\nabla v|^2\\ &-(\delta+\beta-1)\int_\Omega v^{\delta+\beta-1} + (\delta+\beta-1)\int_\Omega v^{\delta+\beta-2}u\\ \leqslant& -\int_\Omega v^{\delta+\beta-1} + \int_\Omega u^{\delta+\beta-1},\end{aligned} \tag{3.63}$$

where the second inequality sign of (3.63) comes from the Young inequality. Then we integrate (3.63) over (s, T) with respect to t and get

$$\begin{aligned}\int_s^T\int_\Omega v^{\delta+\beta-1} &\leqslant \int_\Omega v^{\delta+\beta-1}(s) + \int_s^T\int_\Omega u^{\delta+\beta-1}\\ &\leqslant K^{\delta+\beta-1}|\Omega| + C_1'(T+1)\\ &\leqslant c_{12}'{Q_5'}^\delta C_1'(T+1),\end{aligned} \tag{3.64}$$

where $c_{12}' := K^{\beta-1} + 1$ and $Q_5' := \max\{1, K\}$. Thus, from Lemma 3.1, we have

$$\begin{aligned}\int_s^T\int_\Omega |v_t|^{\delta+\beta-1} &\leqslant C_\delta'\int_s^T\int_\Omega u^{\delta+\beta-1} + C_\delta'\int_s^T\int_\Omega v^{\delta+\beta-1}\\ &\quad + C_\delta'\int_\Omega |\Delta v(s)|^{\delta+\beta-1} + C_\delta'\int_\Omega v^{\delta+\beta-1}(s)\\ &\leqslant C_1'C_\delta'(T+1) + c_{12}'Q_5'^\delta C_1'C_\delta'(T+1) + 2K^{\delta+\beta-1}|\Omega|C_\delta'(T+1)\\ &\leqslant c_{13}'Q_5'^\delta C_1'(T+1),\end{aligned} \tag{3.65}$$

where C_δ' is the constant derived from Lemma 3.1 and $c_{13}' := C_\delta'(1 + c_{12}' + 2{Q_5'}^{\beta-1})$. It follows from (3.61), (3.62) and (3.65) that

$$\begin{aligned}\int_\Omega u^\gamma &\leqslant {Q_5'}^\gamma|\Omega| + c_{11}'{Q_4'}^\gamma c_{13}'{Q_5'}^\delta C_1'(T+1) + c_{11}'{Q_4'}^\gamma|\Omega|(T+1)\\ &\leqslant c_{14}'{Q'}^\gamma C_1'(T+1), \quad t \in (s, T)\end{aligned} \tag{3.66}$$

and

$$\int_s^T \int_\Omega u^{\gamma+\beta-1} \leqslant \frac{4c'_{14}}{\nu} Q'^{\gamma} C'_1(T+1), \tag{3.67}$$

where $c'_{14} := 1 + c'_{11}c'_{13} + c'_{11}$, $Q' = Q'_4 Q'_5$. With the choice of $C'_2 := \max\{c'_{14}, 4c'_{14}/\nu\}$, (3.14) follows from (3.66) and (3.67).

The proof of Lemma 3.3 is complete. □

4. Proof of the main results

In this section, we first invoke Lemma A.1 in [11] to give the $\|\cdot\|_{L^\infty(\Omega)}$ estimate of $u(\cdot,t)$ for all $t \in (0,T)$, where $T \in (0,T_{\max})$, and then prove Theorem 1.1.

Lemma 4.1. *Let $\beta \in (1,\infty)$ and $q < \beta - 1$. Then there exists a constant $C > 0$ independent of T such that with any choice of $T \in (0,T_{\max})$, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C$ for all $t \in (0,T)$.*

Proof. We will get the desired results with help of Lemma A.1 in [11]. From (1.2), (1.4)–(1.7) and Lemma 2.1, we can see that (A.1)–(A.6) in [11] are satisfied. Therefore, for the sake of applying Lemma A.1 in [11], we only need to prove that (A.7) in [11] holds, that is, for some sufficiently large $\gamma^* \geqslant 1$ which satisfies (A.8)–(A.10) in [11],

$$\|u(\cdot,t)\|_{L^\gamma(\Omega)} < \infty, \quad \gamma \geqslant \gamma^* \quad \text{and} \quad t \in (0,T). \tag{4.1}$$

If $T \in (0,s]$, where $s \in (0,T_{\max})$, (3.2) ensures that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded for any $t \in (0,T)$. So in the following, we mainly study the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ for any $t \in (0,T)$, where $T \in (s,T_{\max})$.

First, we investigate the case $\beta \in (1,2)$. Let $\gamma_0 = \beta - 1$, $\gamma_k = (\beta-q)\gamma_{k-1} + \beta - 1 - q$ with $k \geqslant 1$. Then after a simple computation, we obtain

$$\gamma_k = \beta(\beta-q)^k - 1, \tag{4.2}$$

which implies that

$$(\beta-1)(\beta-q)^k \leqslant \gamma_k \leqslant \beta(\beta-q)^k. \tag{4.3}$$

Moreover, it follows from Lemma 3.2 and Lemma 3.3 that

$$\int_\Omega u^{\gamma_k}(t) \leqslant C_2^k R^{\sum_{j=1}^k \gamma_j} C_1(T+1) \tag{4.4}$$

for all $t \in (s,T)$ and $k \in \mathbb{N}^+$. Thus, applying (4.3), we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} \leqslant\ & C_2^{k/((\beta-1)(\beta-q)^k)} C_1^{1/((\beta-1)(\beta-q)^k)} \\ & \times (T+1)^{1/(\beta-1)(\beta-q)^k} R^{\beta \sum_{j=1}^k (\beta-q)^j/(\beta-1)(\beta-q)^k}. \end{aligned} \tag{4.5}$$

Since $q < \beta - 1$ implies that there exists $k_0 \in \mathbb{N}^+$ such that $\gamma_{k_0} \geqslant \gamma^*$, therefore, by virtue of (4.5), we obtain

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k_0 \text{ and } t \in (s,T). \tag{4.6}$$

Also, it follows from (3.2) and the boundedness of Ω that

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k_0 \text{ and } t \in [0,s]. \tag{4.7}$$

From (4.6) and (4.7) we obtain

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k_0 \text{ and } t \in (0,T), \tag{4.8}$$

which, together with the boundedness of Ω and the Hölder inequality, implies (4.1) holds. Thus, by Lemma A.1 in [11], there exists $C^* > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C^*, \quad t \in (0,T). \tag{4.9}$$

Now, we deal with the case $\beta \in [2,\infty)$. In this case, we let $\gamma_0 = 1$, $\gamma_k = (\beta - q) \times \gamma_{k-1} + \beta - 1 - q$ with $k \geqslant 1$. By a computation similar to the first case, we have

$$(\beta-q)^k \leqslant \gamma_k \leqslant 2(\beta-q)^k \tag{4.10}$$

and

$$\int_\Omega u^{\gamma_k}(t) \leqslant {C'}_2^k {R'}^{\sum_{j=1}^k \gamma_j} C'_1(T+1) \tag{4.11}$$

for all $t \in (s,T)$ and $k \in \mathbb{N}^+$. Consequently,

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} \leqslant {C'_2}^{k/(\beta-q)^k} {C'_1}^{1/(\beta-q)^k} (T+1)^{1/(\beta-q)^k} {R'}^{2\sum_{j=1}^k (\beta-q)^j/(\beta-q)^k}. \tag{4.12}$$

It follows from $q < \beta - 1$ that we can find $k'_0 \in \mathbb{N}^+$ such that $\gamma_{k'_0} \geqslant \gamma^*$. Also, (4.12) implies that

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k'_0 \text{ and } t \in (s,T). \tag{4.13}$$

In addition, using (3.2) and the boundedness of Ω, we obtain

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k_0' \text{ and } t \in [0,s]. \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\|u(\cdot,t)\|_{L^{\gamma_k}(\Omega)} < \infty, \quad k \geqslant k_0' \text{ and } t \in (0,T). \tag{4.15}$$

Combining (4.15), the Hölder inequality and the boundedness of Ω, we get (4.1). Therefore, by Lemma A.1 in [11], we can find $C^{**} > 0$ such that

$$\|u(\cdot,t)\|_{L^{\infty}(\Omega)} < C^{**}, \quad t \in (0,T). \tag{4.16}$$

As a result, with the choice of $C := \max\{C^*, C^{**}\}$, we complete the proof of Lemma 4.1. □

Now, we prove Theorem 1.1.

P r o o f of Theorem 1.1. Combining Lemma 4.1 with semigroup estimates, we know that there exists a constant $C_v' > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,\theta}(\Omega)} \leqslant C_v', \quad t \in [0,T_{\max}). \tag{4.17}$$

Suppose on the contrary that $T_{\max} < \infty$. Then the results of Lemma 4.1 contradict the blow up criterion (2.1), which implies $T_{\max} = \infty$. Therefore, the desired results follow from (4.17), the embedding theorem and Lemma 2.1. The proof of Theorem 1.1 is complete. □

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