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A SHORT NOTE ON L^q THEORY FOR STOKES PROBLEM WITH A PRESSURE-DEPENDENT VISCOSITY

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Abstract. We study higher local integrability of a weak solution to the steady Stokes problem. We consider the case of a pressure- and shear-rate-dependent viscosity, i.e., the elliptic part of the Stokes problem is assumed to be nonlinear and it depends on p and on the symmetric part of a gradient of u, namely, it is represented by a stress tensor $T(Du,p) := \nu(p,|D|^2)D$ which satisfies r-growth condition with $r \in (1,2]$. In order to get the main result, we use Calderón-Zygmund theory and the method which was presented for example in the paper Caffarelli, Peral (1998).

Keywords: Stokes problem; L^q theory; pressure-dependent viscosity

MSC 2010: 35B65, 35Q35, 76D03

1. Introduction

In the celebrated Navier-Stokes system, which deals with the flow of Newtonian incompressible fluids, viscosity is assumed to be constant. However, the constant viscosity cannot explain many interesting physical phenomena such as shear-thinning, shear-thickening, die-swell, etc. Viscosity of non-Newtonian fluids is not generally constant but depends on shear rate and, as many experimental works show, there are fluids whose viscosity depends also on pressure. On the other hand, changes in the density of these liquids are negligible as the pressure grows and thus these fluids can be still treated as incompressible, see [2]. Also in some situations the pressure grows tremendously and its influence on viscosity cannot be neglected. For example, in works written by Knauf et al. and Lanzendörfer ([14], [15]) the authors provide numerical simulation of the flow of a lubricant through a ball bearing (or, journal

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bearing). The governing equation for a steady flow of a fluid with a pressure- and shear rate-dependent viscosity has the form

$$\begin{aligned} -\operatorname{div} T(Du,p) + \operatorname{div}(u\otimes u) + \nabla p &= \operatorname{div} T(F,0) \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a domain, $T\colon \mathbb{R}^{d^2}\times\mathbb{R}\to\mathbb{R}^{d^2}$ and $F\colon \Omega\to\mathbb{R}^{d^2}$ are given functions, $u\colon \Omega\to\mathbb{R}^d$ and $p\colon \Omega\to\mathbb{R}$ are unknowns and Du stands for the symmetric part of a gradient of u, i.e., $2Du = \nabla u + (\nabla u)^T$.

A plenty of works studying this system have been published sofar. Regarding the steady case, Gazzola and Secchi in [11] consider a viscosity depending only on the pressure. The existence of a solution for more general viscosities is discussed by Bulíček and Fišerová in [3], by Lanzendörfer and Stebel in [17] and also by Franta et al. in [10]. $C^{1,\alpha}$ interior regularity in two dimensions is solved by Bulíček and Kaplický in [4], Hölder regularity in the three-dimensional case was announced by Mingione, Málek and Stará in [21]. Further, $C^{1,\alpha}$ regularity under Dirichlet boundary conditions in three dimensions is provided in [18]. The unsteady case was investigated, for example, by Bulíček, Málek and Rajagopal in [5].

This article is devoted to the interior L^q regularity for a simplified problem which has the following form:

$$\begin{aligned} -\operatorname{div} T(Du,p) + \nabla p &= \operatorname{div} T(F,0) \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega. \end{aligned} \tag{P}$$

Since this article concerns only local properties of the solution, we do not care about any boundary conditions.

The stress tensor T is considered to be in the form $T(Du,p) = \nu(p,|Du|^2)Du$. Moreover, we suppose that following growth conditions are fulfilled.

$$\gamma_1(1+|D|^2)^{(r-2)/2}|B|^2 \leqslant \frac{\partial(\nu(p,|D|^2)D_{ij})}{\partial D_{kl}} B_{ij}B_{kl} \leqslant \gamma_2(1+|D|^2)^{(r-2)/2}|B|^2, \tag{A1}$$

$$\Big|\frac{\partial\nu(p,|D|^2)}{\partial p} D\Big| \leqslant \gamma_3(1+|D|^2)^{(r-2)/4}. \tag{A2}$$

Furthermore, γ_3 is assumed to be small, specifically

$$\gamma_3 < \frac{\gamma_1}{c_{\mathrm{div}}(\gamma_1+\gamma_2)}, \tag{A3}$$

where the constant c_{div} comes from Bogovskii inequality—see Lemma 4.

For physically relevant viscosities fulfilling (A1), (A2) and (A3) we refer the reader to [15], [20] or [22].

We point out that the form of the right-hand side of $(\mathrm{P})_1$ is not restrictive since for every $G\colon \Omega \to \mathbb{R}^{d^2}$ there exists F such that $G = T(F,0)$. The chosen form of the right-hand side allows us to write the main result in a clear way.

In what follows, usual Lebesque spaces of functions defined on a set Ω are denoted by $L^s(\Omega)$, Sobolev spaces are denoted by $W^{1,s}(\Omega)$. Lebesque (or, Sobolev) spaces of vector-valued functions are denoted by $L^s(\Omega)^d$ (or, $W^{1,s}(\Omega)^d$). However, we will use $L^s(\Omega)$ (or, $W^{1,s}(\Omega)$) for vector-valued functions in cases where there is no risk of misunderstanding. Norms in these spaces will be denoted by $\|\cdot\|_s$ (or, $\|\cdot\|_{1,s}$). In case we want to emphasize the supporting set, we write $\|\cdot\|_{s,\Omega}$ (or, $\|\cdot\|_{1,s,\Omega}$), i.e., $\|f\|_{s,\Omega} = (\int_\Omega |f|^s)^{1/s}$. Further, $L^s_{\mathrm{loc}}(\Omega)$ (or, $W^{1,s}_{\mathrm{loc}}(\Omega)$) stands for a local version of Lebesque (or, Sobolev) space. A space of all functions from $W^{1,s}(\Omega)$ with compact support is denoted by $W^{1,s}_c(\Omega)$. We define also an integral average as

$$\fint_\Omega f := \frac{1}{|\Omega|}\int_\Omega f.$$

For $\alpha > 0$ and a cube $Q \subset \mathbb{R}^d$ centered at $a \in \mathbb{R}^d$ we define a cube αQ as

$$\alpha Q := \Big\{x \in \mathbb{R}^d\colon\ \frac{x-a}{\alpha} \in Q\Big\}.$$

A weak solution of (P) is a pair $(u,p) \in W^{1,r}_{\mathrm{loc}}(\Omega) \times L^{r'}_{\mathrm{loc}}(\Omega)$, $\operatorname{div} u = 0$ fulfilling

$$\int_\Omega T(Du,p)\nabla\varphi - \int_\Omega p \operatorname{div}\varphi = \int_\Omega T(F,0)\nabla\varphi$$

for every $\varphi \in W^{1,r}_c(\Omega)$. The existence of a weak solution to (P) is solved by Theorem 3.11 in [16]. We also refer to Theorem 2 in [15] and references given there. In the rest of this paper we always assume that (A1), (A2) and (A3) hold.

The main result of this paper is summed up in the following theorem.

Theorem 1. *Let $r \in (1,2]$, $q \in (1, d/(d-2))$ (or $q \in (1,\infty)$ in the case $d = 2$) and let assumptions* (A1)–(A3) *be fulfilled. Let $F \in L^{qr}(\Omega)$, (u,p) be a weak solution to* (P) *and let $Q \subset 4Q \subset \Omega$ be a sufficiently small cube. Then*[1]

$$\begin{aligned}
&\fint_Q ((1+|Du|^2)^{(r-2)/2}|Du|^2)^q \\
&\leqslant c\bigg(1 + \fint_{4Q} F^{qr} + \Big(\fint_{4Q}(1+|Du|^2)^{(r-2)/2}|Du|^2\Big)^q\bigg),
\end{aligned} \tag{1.1}$$

[1] Hereinafter we use the letter c for a constant which may vary from line to line, however, it is always independent of the solution and right-hand side.

and, moreover,

$$(1.2)\qquad \fint_Q \Big| p - \fint_Q p \Big|^{qr'} \leqslant c\Big(1 + \fint_{4Q} F^{qr} + \Big(\fint_{4Q} (1+|Du|^2)^{(r-2)/2}|Du|^2\Big)^q\Big),$$

where c *is a constant independent of* f, u, p *and* Q.

The method of the proof is based on a local comparison with the solution to the problem with zero right-hand side. Among lots of papers based on the comparison technique we would like to mention [13], where Iwaniec showed a L^q theory result for the linear problem, [6], where Caffarelli and Peral proved L^q estimates for elliptic equations in divergence form, and [8], where Diening and Kaplický used the method to derive L^q estimates of Stokes system with general growth. Our approach is based on [6], Theorem A. However, since our system is slightly more complicated than the one in [6], we have to use a more general version which is presented here as Lemma 5.

2. Preliminaries

We define a function $V\colon \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ as $V(D) := \sqrt{\nu(0,|D|^2)}D$ and, for every $s > 0$, a function $\Phi'(s) = \nu(0,|s|^2)s$. We emphasize that $\Phi(x) := \int_0^x \Phi'(s)\,\mathrm{d}s$ is an N-function satisfying the Δ_2 condition and, moreover, $\Phi'(s) \sim (1+s^2)^{(r-2)/2}s$. By Φ^* we denote an N-function which is complementary to Φ. Hereinafter, we suppose that $r \leqslant 2$.

Lemma 2. *For all cubes* Q, $u, v \in W^{1,r}(Q)$ *and* $p, \pi \in L^{r'}(Q)$ *it holds that*

$\Phi'(|Du|) \sim |T(Du,p)|$ *uniformly in* p;
$|V(Du)|^2 \sim T(Du,p)Du \sim (1+Du)^{(r-2)/2}|Du|^2 \sim \Phi(|Du|) \sim \Phi^*(\Phi'(|Du|))$, *uniformly in* p;

$$(2.1)\qquad \begin{aligned} I_{(u,v)} &:= \int_0^1 (1+|Dv+s(Du-Dv)|^2)^{(r-2)/2}\,\mathrm{d}s\,|Du-Dv|^2 \\ &\leqslant \frac{2}{\gamma_1}(T(Du,p)-T(Dv,\pi))(Du-Dv) + \frac{\gamma_3^2}{\gamma_1^2}|p-\pi|^2; \end{aligned}$$

$I_{(u,v,Q)} := \int_Q I_{u,v} \sim \int_Q |V(Du)-V(Dv)|^2$;
for all $\varphi \in W^{1,2}(Q')$ *it holds that*

$$\int_Q (T(Du,p)-T(Dv,\pi))(D\varphi) \leqslant \gamma_2\sqrt{I_{u,v,Q}}\|\nabla\varphi\|_{2,Q} + \gamma_3\|p-\pi\|_2\|\nabla\varphi\|_2.$$

P r o o f. Since $T(0,p)=0$, it holds that

$$|T(Du,p)||Du| = \int_0^1 \frac{\partial}{\partial s}|T(s(Du),p)|\,\mathrm{d}s\,|Du| \sim \int_0^1 (1+|sDu|^2)^{(r-2)/2}\,\mathrm{d}s\,|Du|^2$$
$$\sim (1+|Du|^2)^{(r-2)/2}|Du|^2 \sim \Phi'(|Du|)|Du|,$$

where we use [7], Lemma 19. The first estimate follows easily.

The second estimate is an easy consequence of the first estimate and of the following sequence of inequalities (which can be found e.g. as [7], Lemma 3):

$$|V(Du)|^2 \sim \Phi(|Du|) \sim \Phi'(|Du|)|Du| \sim \Phi^*(\Phi'(|Du|)).$$

For the third inequality, we refer the reader to [10], Lemma 3.3.

In order to prove the fourth inequality it is enough to see that

$$\begin{aligned} I_{(u,v,Q)} &\sim \int_Q \int_0^1 \frac{\partial}{\partial s} T((s(Du-Dv)+Dv),0)\,\mathrm{d}s(Du-Dv) \\ &= \int_Q (T(Du,0)-T(Dv,0))(Du-Dv) \sim \int_Q |V(Du)-V(Dv)|^2, \end{aligned}$$

where the last estimate comes from [7], Lemma 3.

It remains to prove the fifth estimate. It holds that

$$\begin{aligned} &\int_Q (T(Du,p)-T(Dv,\pi))D\varphi \\ &\quad= \int_Q \Big(\int_0^1 \frac{\partial}{\partial s} T(Dv+s(Du-Dv),\pi+s(p-\pi))\,\mathrm{d}s\Big) D\varphi \\ &\quad= \int_Q\int_0^1 \frac{\partial T}{\partial D}(Dv+s(Du-Dv),\pi+s(p-\pi))(Du-Dv)\,\mathrm{d}s\,D\varphi \\ &\qquad+ \int_Q\int_0^1 \frac{\partial T}{\partial p}(Dv+s(Du-Dv),\pi+s(p-\pi))(p-\pi)\,\mathrm{d}s\,D\varphi \\ &\quad\leqslant \gamma_2 \int_Q\int_0^1 (1+|Dv+s(Du-Dv)|^2)^{(r-2)/2}|Du-Dv|\,\mathrm{d}s\,|D\varphi| \\ &\qquad+ \gamma_3 \Big(\int_Q |p-\pi|^2\Big)^{1/2}\Big(\int_Q |\nabla\varphi|^2\Big)^{1/2} \end{aligned}$$

Further, since $r \leqslant 2$, Hölder inequality yields

$$\begin{aligned} &\int_Q\int_0^1 (1+|Dv+s(Du-Dv)|^2)^{(r-2)/2}|Du-Dv|\,\mathrm{d}s\,|D\varphi| \\ &\leqslant \Big(\int_Q\int_0^1 (1+|Dv+s(Du-Dv)|^2)^{(r-2)/2}\,\mathrm{d}s\,|Du-Dv|^2\Big)^{1/2}\Big(\int_Q |\nabla\varphi|^2\Big)^{1/2} \end{aligned}$$

and the required estimate follows. □

Lemma 3 ([9], Theorem 6.10). *Let $\varphi \in W_0^{1,\varphi}(Q)$. There exists a constant c independent of Q and φ such that*

$$\int_Q \Phi(|\nabla\varphi|) \leqslant c \int_Q \Phi(|D\varphi|).$$

Lemma 4 ([1], Lemma 3.3). *Let $s \in (1,\infty)$. There is a constant c such that for a cube Q and for all $f \in L^{s'}(Q)$ there exists a function $\varphi \in W^{1,s}(Q)$ such that*

$$\begin{aligned} &\operatorname{div}\varphi = |f|^{s'-2}f - \fint_Q |f|^{s'-2}f \quad \text{in } Q,\\ &\varphi|_{\partial Q} = 0 \end{aligned}$$

and

$$\|\varphi\|_{1,s,Q} \leqslant c\|f\|_{s',Q}^{1/(s-1)}.$$

For the case $s=2$ we write c_{div} instead of c.

3. Proof of the main theorem

The proof of the main theorem relies on the following lemma.

Lemma 5 ([19], Lemma 2.7). *Let $\mathcal{O} \subset \mathbb{R}^n$, $1 \leqslant p < q < s < \infty$, $f \in L^{q/p}(\mathcal{O})$, $g \in L^{q/p}(\mathcal{O})$ and $w \in L^p(\mathcal{O})^n$. Further, let $Q \subset \mathcal{O}$ be a cube and Q_k be dyadic cubes obtained from Q. Then there exists $\varepsilon > 0$ independent of Q and $\mathcal{O}$ such that the following implication holds:*

If for every dyadic sub-cube $Q_k \subset Q$ there exists $w_a \in L^p(4\widetilde{Q}_k \cap \mathcal{O})^n$ with the following properties:

$$\left(\fint_{2\widetilde{Q}_k\cap\mathcal{O}} |w_a|^s\right)^{1/s} \leqslant \frac{C}{2}\left(\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |w_a|^p\right)^{1/p}, \tag{3.1}$$

$$\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |w_a|^p \leqslant C\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |w|^p + C\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |g|, \tag{3.2}$$

$$\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |w-w_a|^p \leqslant \varepsilon\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |w|^p + C\fint_{4\widetilde{Q}_k\cap\mathcal{O}} |f|, \tag{3.3}$$

then $w \in L^q(Q)^n$. Positive constants C and ε are independent of Q_k, w_a and w. Furthermore, there exists a positive constant c independent of f, g and w such that

$$\fint_Q |w|^q \leqslant c\left(\fint_{4Q\cap\mathcal{O}} |f|^{q/p} + \fint_{4Q\cap\mathcal{O}} |g|^{q/p} + \left(\fint_{4Q\cap\mathcal{O}} |w|^p\right)^{q/p}\right). \tag{3.4}$$

Let $Q'/4$ be an arbitrary dyadic cube obtained from Q. We consider a reference system

$$\text{(R)}\qquad \begin{aligned} -\operatorname{div} T(Dv,\pi) + \nabla\pi &= 0 \quad \text{in } Q',\\ \operatorname{div} v &= 0 \quad \text{in } Q',\\ v|_{\partial Q'} &= u, \end{aligned}$$

where we assume that $⨍_{Q'} \pi = ⨍_{Q'} p$. The existence of such solution is due to monotone operator method, see [23], Chapter 2.

Lemma 6. *Let v, π be a solution to* (R) *on a domain Q'. Let $Q \subset \overline{Q} \subset Q'$ be a cube. Let $\alpha \in (0,1)$ and $\xi \in C^\infty(Q)$ be such that $\xi|_{\alpha Q} = 1$, $\operatorname{supp}\xi \subset Q$ and $\xi(x) \in [0,1]$ for every $x \in Q$. Then there exists $\delta < 1$ such that for all $i \in \{1,\ldots,d\}$*

$$\int_Q \partial_i T(Dv,\pi)(\xi^2\partial_i\nabla v) \geqslant \lim_{h\to 0}\frac{1}{h^2} I_{(v(x+he_i),v(x),\alpha Q)} - \delta \lim_{h\to 0}\frac{1}{h^2} I_{(v(x+he_i),v(x),Q)}.$$

Proof. We start with the term

$$J_h := \int_Q (T(Dv(x+he_i),p(x+he_i)) - T(Dv(x),p(x)))(Dv(x+he_i)-Dv(x))\xi^2.$$

Note that T is symmetric and thus $\int_Q \partial_i T(Dv,\pi)(\xi^2\partial_i\nabla v) = \lim\limits_{h\to 0} J_h/h^2$. We integrate (2.1) with $u = v(x+he_i)$ over Q and we get

$$I_{(v(x+he_i),v(x),\alpha Q)} \leqslant \frac{2}{\gamma_1} J_h + \frac{\gamma_3^2}{\gamma_1^2}\int_Q |\pi(x+he_i)-\pi(x)|^2. \tag{3.5}$$

For a general function f we define the operator $\delta_{-h,i}$ as $\delta_{-h,i} f = f(x-he_i) - f(x)$. Furthermore, we define a test function φ as

$$\begin{aligned} \operatorname{div}\varphi &= \pi(x+he_i) - \pi(x) - ⨍_Q (\pi(x+he_i)-\pi(x)) \quad \text{on } Q\\ \varphi &= 0 \quad \text{on } \partial Q. \end{aligned}$$

We test (R) by $\delta_{-h,i}\varphi$ and, as far as $\delta_{-h,i}K = 0$ for every constant K, we get according to Lemma 2

$$\begin{aligned} \int_Q |\pi(x+he_i)-\pi(x)|^2 &\leqslant \gamma_2 c_{\mathrm{div}} \sqrt{I_{(v(x+he_i),v(x),Q)}}\, \|\pi(x+he_i)-\pi(x)\|_{2,Q}\\ &\quad + \gamma_3 c_{\mathrm{div}} \int_Q |\pi(x+he_i)-\pi(x)|^2, \end{aligned}$$

consequently,

$$\frac{\gamma_3^2}{\gamma_1^2}\int_Q |\pi(x+he_i)-\pi(x)|^2 \leqslant \Big(\frac{\gamma_2\gamma_3 c_{\mathrm{div}}}{\gamma_1(1-\gamma_3 c_{\mathrm{div}})}\Big)^2 I_{(v(x+he_i),v(x),Q)}.$$

We would like to emphasize that due to (A3) it holds that

$$\Big(\frac{\gamma_2\gamma_3 c_{\mathrm{div}}}{\gamma_1(1-\gamma_3 c_{\mathrm{div}})}\Big)^2 < 1.$$

In order to get the claim it suffices to put this estimate into (3.5), divide by h^2 and let $h \to 0$. $\square$

Estimates (3.1) and (3.2) are verified in the following lemma.

Lemma 7. *There exists a constant c independent of v, π, u, p and Q' such that*

$$\Big(\fint_{Q'/2} |V(Dv)|^q\Big)^{1/q} \leqslant c\Big(\fint_{Q'} |V(Dv)|^2\Big)^{1/2}, \tag{3.6}$$

where $q \in (1, 2d/(d-2)]$ (or $q \in (1,\infty)$ in the case $d = 2$).

Furthermore,

$$\fint_{Q'} |V(Dv)|^2 \leqslant c \fint_{Q'} |V(Du)|^2. \tag{3.7}$$

Proof. For simplicity we assume that $d = 3$. This assumption will be commented later. Using the standard bootstrap argument one may also easily show that $v \in W^{2,p}_{\mathrm{loc}}(Q')$ and $\nabla V(Dv) \in L^2_{\mathrm{loc}}(Q')$. See [21].

Let $q\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear function with $\nabla q = \langle \nabla v\rangle_{Q'}$. Let α and α' be such that $3/8 \leqslant \alpha < \alpha' \leqslant 3/4$ and let $\xi \in C^\infty(Q)$ be a smooth test function such that $\xi|_{\alpha Q'} = 1$, $\operatorname{supp}\xi \subset \alpha' Q'$, $\xi(x) \in [0,1]$ for $x \in \alpha' Q'$ and $|\nabla^j \xi| \leqslant C/((\alpha'-\alpha)^j |\operatorname{diam} Q'|)$. We test (R) by $\varphi = \operatorname{curl}(\xi^2 \operatorname{curl}(v-q))$.

We would like to point out that $\operatorname{curl} v = (\partial_3 v_2 - \partial_2 v_3, \partial_1 v_3 - \partial_3 v_1, \partial_2 v_1 - \partial_1 v_2)$. It follows that φ is not a suitable test function. Instead, one has to take $\psi = \operatorname{curl}(\xi^2 \operatorname{curl}_h(v-q))$, where curl_h is defined as $\operatorname{curl}_h(v) = (\delta_{h,3}v_2 - \delta_{h,2}v_3, \delta_{h,1}v_3 - \delta_{h,3}v_1, \delta_{h,2}v_1 - \delta_{h,1}v_2)$. Note that $\operatorname{div}\varphi = \operatorname{div}\psi = 0$. For the sake of clarity we work with φ instead of ψ. After some cumbersome manipulation we get

$$\begin{aligned}\int_{\alpha' Q'} \nabla T(Dv,\pi)\xi^2\nabla^2 v = \int_{\alpha' Q'} & T(Dv,\pi)(\nabla(\nabla(\xi^2)\times\operatorname{curl}(v-q)) \\ &+ \operatorname{div}(\nabla(\xi^2)\otimes\nabla(v-q)) - \nabla(\nabla(\xi^2)\nabla(v-q))).\end{aligned}$$

In the same way as in the proof of [8], Lemma 3.5, we derive

$$\int_{\alpha' Q'} \nabla T(Dv,\pi)\xi^2\nabla^2 v \leqslant \frac{c(\varepsilon)}{(\alpha'-\alpha)^2|\operatorname{diam} Q'|^2}\int_{\alpha' Q'}|V(Dv)|^2+\varepsilon\int_{\alpha' Q'}|\nabla V(Dv)|^2,$$

where ε comes from Young's inequality. Further, since

$$\sum_{i=1}^{d}\lim_{h\to 0}\frac{1}{h^2}I_{(v(x+he_i),v(x),\alpha' Q')}\sim\int_{\alpha' Q'}|\nabla V(Dv)|^2,$$

Lemma 6 gives

$$\begin{aligned}\sum_{i=1}^{d}\lim_{h\to 0}\frac{1}{h^2}I_{(v(x+he_i),v(x),\alpha Q')} &\leqslant (\delta+\varepsilon)\sum_{i=1}^{d}\lim_{h\to 0}\frac{1}{h^2}I_{(v(x+he_i),v(x),\alpha' Q')}\\ &\quad+\frac{c(\varepsilon)}{(\alpha'-\alpha)^2|\operatorname{diam} Q'|^2}\int_{\alpha Q'}|V(Dv)|^2.\end{aligned}$$

As far as ε could be chosen such that $\delta+\varepsilon<1$, we may use an algebraic lemma (for example [12], Lemma 6.1) in order to get

$$\int_{3Q'/8}|\nabla V(Dv)|^2\sim\sum_{i=1}^{d}\lim_{h\to 0}\frac{1}{h^2}I_{(v(x+he_i),v(x),1/2Q')}\leqslant\frac{c(\varepsilon)}{(\operatorname{diam} Q')^2}\int_{3Q'/4}|V(Dv)|^2.$$

The estimate (3.6) follows due to the Poincaré inequality and a standard covering argument.

The case of general dimension is just a matter of a suitable test function φ. See [19], Proof of Lemma 3.3, for more details.

To prove the second estimate we test (R) by $\varphi=u-v$. We get

$$\int_{Q'}T(Dv,\pi)(Du-Dv)=0.$$

Consequently, due to Lemma 2

$$\begin{aligned}\int_{Q'}|V(Dv)|^2\sim\int_{Q'}T(Dv,\pi)Dv&=\int_{Q'}T(Dv,\pi)Du\\ &\leqslant\delta\int_{Q'}\Phi^*(|T(Dv,\pi)|)+c_\delta\int_{Q'}\Phi(|Du|)\\ &\leqslant\delta\int_{Q'}|V(Dv)|^2+c_\delta\int_{Q'}|V(Du)|^2,\end{aligned}$$

where δ comes from Young's inequality. The estimate (3.7) follows immediately. $\square$

P r o o f of Theorem 1. In order to use Lemma 5 it remains to verify (3.3). We subtract (R) from (P). Thus, for all $\varphi \in W_0^{1,r}(Q')$, it holds that

$$\int_{Q'} (T(Du,p) - T(Dv,\pi))D\varphi - \int_{Q'} (p-\pi)\operatorname{div}\varphi = \int_{Q'} F\nabla\varphi. \tag{3.8}$$

Taking $\varphi = u - v$ in (3.8), we get

$$\begin{aligned} I_{(u,v,Q')} &\leqslant \frac{2}{\gamma_1}\int_{Q'} (T(Du,p) - T(Dv,\pi))(Du - Dv) + \frac{\gamma_3^2}{\gamma_1^2}\int_{Q'} |p-\pi|^2 \\ &= c\int_{Q'} T(F,0)(\nabla(u-v)) + \frac{\gamma_3^2}{\gamma_1^2}\int_{Q'} |p-\pi|^2, \end{aligned} \tag{3.9}$$

where we use Lemma 2. Let φ be a solution to

$$\begin{aligned} \operatorname{div}\varphi &= p - \pi \quad \text{in } Q', \\ \varphi|_{\partial Q'} &= 0. \end{aligned}$$

We use this φ as a test function in (3.8). Lemma 2 yields

$$\begin{aligned} \|p-\pi\|_{2,Q'}^2 &\leqslant \int_{Q'} (T(Du,p) - T(Dv,\pi))D\varphi - \int_{Q'} T(F,0)\nabla\varphi \\ &\leqslant \gamma_2\sqrt{I_{(u,v,Q')}}\,\|\nabla\varphi\|_{2,Q'} + \gamma_3\|p-\pi\|_{2,Q'}\|\nabla\varphi\|_{2,Q'} + \int_{Q'} T(F,0)\nabla\varphi \end{aligned}$$

and, due to Lemma 4 and Hölder inequality,

$$(1 - c_{\mathrm{div}}\gamma_3)\|p-\pi\|_{2,Q'} \leqslant c_{\mathrm{div}}\gamma_2\sqrt{I_{(u,v,Q')}} + \|T(F,0)\|_{2,Q'}.$$

Consequently

$$\|p-\pi\|_{2,Q'}^2 \leqslant \Big(\Big(\frac{c_{\mathrm{div}}\gamma_2}{1 - c_{\mathrm{div}}\gamma_3}\Big)^2 + \varepsilon\Big)I_{(u,v,Q')} + c(\varepsilon)\|T(F,0)\|_{2,Q'}^2.$$

We get, according to Lemma 3, Lemma 2, Young's inequality and (3.7), that

$$\begin{aligned} \int_{Q'} T(F,0)(\nabla(u-v)) &\leqslant c(\varepsilon_1)\int_{Q'} \Phi^*(\Phi'(|F|)) + \varepsilon_1\int_{Q'} \Phi(|Du - Dv|) \\ &\leqslant c(\varepsilon_1)\int_{Q'} \Phi(|F|) + \varepsilon_1\int_{Q'} \Phi(|Du|) + \varepsilon_1\int_{Q'} \Phi(|Dv|) \\ &\leqslant c(\varepsilon_1)\int_{Q'} \Phi(|F|) + \varepsilon\int_{Q'} |V(Du)|^2. \end{aligned}$$

We put all these estimates into (3.9) and we get

$$I_{(u,v,Q')} \leqslant \Big(\Big(\frac{\gamma_3 c_{\mathrm{div}}\gamma_2}{\gamma_1(1-c_{\mathrm{div}}\gamma_3)}\Big)^2+\varepsilon\Big) I_{(u,v,Q')} + c(\varepsilon)\int_{Q'}\Phi(|F|) + c\int_{Q'}|T(F,0)|^2 + \varepsilon_1\int_{Q'}|V(Du)|^2.$$

Due to (A3) there exists ε such that it holds that $((\gamma_3 c_{\mathrm{div}}\gamma_2)/(\gamma_1(1-c_{\mathrm{div}}\gamma_3)))^2 + \varepsilon < 1$ and thus

$$I_{(u,v,Q')} \leqslant c\int_{Q'}(\Phi(|F|)+|T(F,0)|^2) + \varepsilon_1\int_{Q'}|V(Du)|^2.$$

According to Lemma 2

$$\fint_{Q'}|V(Du)-V(Dv)|^2 \leqslant \varepsilon_1 \fint_{Q'}|V(Dv)|^2 + c\fint_{Q'}(\Phi(|F|)+|T(F,0)|^2).$$

Further, since $\Phi(|F|)\sim(1+|F|^2)^{(r-2)/2}|F|^2 \leqslant |F|^r$ and $|T(F,0)|^2\sim(1+|F|^2)^{r-2}\times |F|^2 \leqslant (1+|F|^2)^{(r-2)/2} \leqslant |F|^r$ we get

$$\fint_{Q'}|V(Du)-V(Dv)|^2 \leqslant \varepsilon \fint_{Q'}|V(Dv)|^2 + c\fint_{Q'}|F|^r.$$

All assumptions of Lemma 5 are met and the estimate (1.1) follows directly from (3.4).

Let $\overline{p} = p - \fint_Q p$. We consider a test function φ fulfilling

$$\operatorname{div}\varphi = |\overline{p}|^{s'-2}\overline{p} - \fint_Q |\overline{p}|^{s'-2}\overline{p} \quad \text{in } Q,$$
$$\varphi|_{\partial Q} = 0,$$

where s is such that $s' = qr'$. Lemma 4 yields

$$\|\nabla\varphi\|_{s,Q} \leqslant c\|\overline{p}\|_{s',Q}^{1/(s-1)}. \tag{3.10}$$

We multiply $(\mathrm{P})_1$ by this φ in order to get

$$\|\overline{p}\|_{s',Q}^{s'} \leqslant (\|T(Du,p)\|_{s',Q} + \|T(F,0)\|_{s',Q})\|\nabla\varphi\|_{s,Q},$$

and, together with (3.10), we obtain

$$\|\overline{p}\|_{s',Q} \leqslant c(\|T(Du,p)\|_{s',Q} + \|T(F,0)\|_{s',Q}). \tag{3.11}$$

Further, since $t^{r'} \leqslant c\Phi^*(t)$, we have

$$|T(Du,p)|^{qr'} \leqslant c(\Phi^*(\Phi'(|Du|)))^q \sim ((1+|Du|^2)^{(r-2)/2}|Du|^2)^q.$$

This together with (3.11) yields

$$\fint_Q |\overline{p}|^{qr'} \leqslant c\fint_Q ((1+|Du|^2)^{(r-2)/2}|Du|^2)^q + c\fint_Q ((1+|F|^2)^{(r-2)/2}|F|^2)^q.$$

Using (1.1) one may easily conclude (1.2). □

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