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NEW RESULTS ABOUT SEMI-POSITIVE MATRICES

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We are pleased to dedicate this work to the very fond memory of Miroslav Fiedler, who helped to shape this subject and our views on it

Abstract. Our purpose is to present a number of new facts about the structure of semipositive matrices, involving patterns, spectra and Jordon form, sums and products, and matrix equivalence, etc. Techniques used to obtain the results may be of independent interest. Examples include: any matrix with at least two columns is a sum, and any matrix with at least two rows, a product, of semipositive matrices. Any spectrum of a real matrix with at least 2 elements is the spectrum of a square semipositive matrix, and any real matrix, except for a negative scalar matrix, is similar to a semipositive matrix. M-matrices are generalized to the non-square case and sign patterns that require semipositivity are characterized.

Keywords: sign semipositivity; semipositive matrix; M-matrix; spectrum; equivalence

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1. Introduction and background

Several aspects of semipositive matrices are explored: the sums and products that result from semipositive matrices, when generalized Z-matrices are semipositive, sign patterns that require semipositivity, the spectra and Jordan cannonical form of semipositive matrices, and equivalence on semipositive matrices.

Definition 1.1. An $m \times n$ matrix A is called *semipositive* (SP) if there exists $v \in \mathbb{R}^n$, $v \geqslant 0$, such that $Av > 0$. We may equivalently require $v > 0$ rather than $v \geqslant 0$ by a simple perturbation argument. In order for an $m \times n$ matrix A to be

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seminonnegative (SNN), it is only required that $Av \geqslant 0$ where $v \geqslant 0$ and $v \neq 0$. A *seminegative* (SN) matrix is the negative of a semipositive matrix. *Seminonpositive* matrices are defined similarly.

There are many mathematical references that develop some properties of semipositivity, including [1], [5], [4], [7], [13], [15], [16], [17], etc.

Definition 1.2. An $m \times n$ matrix A is called *left semipositive* (left SP) if there exists $v \in \mathbb{R}^n$, $v \geqslant 0$ such that $v^\top A > 0$. Again, we may equivalently require $v > 0$ rather than $v \geqslant 0$

Left seminonpositivity (SNP) and other 'left' variants are defined in the obviously analogous ways.

Definition 1.3. A *sign pattern* is a matrix whose entries are the symbols 0, +, and −. A real matrix A is said to *have sign pattern* B if A and B are the same size and $a_{ij} = 0$ when $b_{ij} = 0$, $a_{ij} > 0$ when $b_{ij} = +$, and $a_{ij} < 0$ when $b_{ij} = -$.

Example 1.4. The matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

is both semipositive and seminegative because $v = [2,\ 3,\ 1]^\top > 0$ and $Av = [1,\ 1,\ 1]^\top > 0$, and $w = [2,\ 1,\ 3]^\top > 0$ and $Aw = [-1,\ -1,\ -1]^\top < 0$.

Lemma 1.5 (Theorem of the alternative). *For a given matrix $A \in M_{m,n}(\mathbb{R})$ exactly one of the following is true:*

1. *A is* SP.
2. *A is left* SNP.

There are many variations of the theorem of the alternative, which can be seen in [14], and this is a restatement of one of them.

Since a permutation maps the positive vectors to themselves, the following is clear.

Lemma 1.6. *A matrix, $A \in M_{m,n}(\mathbb{R})$, is semipositive if and only if , whenever P is a permutation matrix, AP is semipositive. The same is true for left multiplication by a permutation matrix.*

Lemma 1.7. *Any matrix in $M_{m,n}(\mathbb{R})$ with a positive column is* SP.

Proof. Suppose that column j is all positive. Then make the jth entry large enough compared to the other entries of the vector, v, and then $Av > 0$. □

Remark 1.8. If a general matrix A is transformed to $B = DAE$, with D and E positive diagonal matrices, we say that B is a scaling of A. We note that SP matrices are scale invariant and that another view of SP matrices is that they are those matrices scaleable to matrices with row sums 1.

2. Sums and products of SP matrices

We begin with a section on sums and products of SP matrices. We are mainly concerned with which matrices occur as sums or products of SP matrices.

Theorem 2.1. *Suppose $A \in M_{m,n}(\mathbb{R})$, with $n \geqslant 2$. Then there exist* SP *matrices $B, C \in M_{m,n}(\mathbb{R})$ such that $A = B + C$.*

Proof. First, pick B so that the entries of the first column are all 1s, the second column is the second column of A with one subtracted from each entry, and all of the other columns match the columns of A. Then set the first column of C to be the first column of A with 1 subtracted from each entry, let the second column have every entry equal to 1, and make all of the other columns have entries equal to 0. Then $A = B + C$ and B and C are both SP as each has a positive column. $\square$

Remark 2.2. The requirement that $n \geqslant 2$ in Theorem 2.1 is necessary. If $n = 1$, A has a negative entry, and $A = B + C$, then either B or C must also have a negative entry and thus is not SP. So, when $n = 1$, A is the sum of two SP matrices if and only if A, itself, is SP.

We now turn to products.

Lemma 2.3. *Let $m \geqslant 2$, $n \geqslant 1$, and suppose that $0 \neq C \in M_{m,n}(\mathbb{R})$. If $\{v_1, w\} \in \mathbb{R}^m$ is a linearly independent set and $v_2 \in \mathbb{R}^n$ is such that $\{Cv_2, w\}$ is a linearly independent set, then there exist $A \in M_m(\mathbb{R})$ and $B \in M_{m,n}(\mathbb{R})$ such that $Av_1 = w$, $Bv_2 = w$, and $C = AB$.*

Proof. Choose $A \in M_m(\mathbb{R})$ to be an invertible matrix such that $Av_1 = w$ and $Aw = Cv_2 \neq 0$. Set $B = A^{-1}C$, so that $C = AB$. Then $Bv_2 = A^{-1}Cv_2 = w$, and the stated requirements are fulfilled. $\square$

Theorem 2.4. *If $m \geqslant 2$, $n \geqslant 1$, and $C \in M_{m,n}(\mathbb{R})$, then there exits matrices $A \in M_m(\mathbb{R})$ and $B \in M_{m,n}(\mathbb{R})$, both* SP, *such that $C = AB$.*

Proof. If rank $C \geqslant 1$, positive vectors v_1, v_2, and w may be chosen so as to fulfill the hypothesis of Lemma 2.3. The positivity of these vectors means that A

and B are SP, completing the proof of this theorem. If $C = 0$ and $m \geqslant 2$, we may choose

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

of the required sizes, so that $C = AB$. Since A and B both have positive columns, they are SP. □

Remark 2.5. The requirement that $m \geqslant 2$ in Theorem 2.2 is necessary. Suppose that $C \in M_{1,n}(\mathbb{R})$ and all of the entries of C are negative. If $C = AB$ then either A has a negative entry in which case it is not SP, or B has all negative entries, and is therefore not SP.

Despite it being simple enough to write any matrix as a sum or product of SP matrices, it cannot be done for both left and right SP matrices. For sums, an easy counterexample is the zero matrix. Suppose that $A+B = 0$ where A and B are both left and right SP. Then $A = -B$, so B is left and right SN as well. But then B is SP and left SN, and therefore left SNP, which violates Lemma 1.5.

3. Generalized M-matrices

In order for $A \in M_{m,n}(\mathbb{R})$ to be SP, A must have at least one positive entry in every row. When is a matrix with just one positive entry per row SP? It can be easily seen that a positive entry in every column is not necessary for A to be SP, as one may use a vector that weights the columns, with no positive entries, to be arbitrarily small compared to the columns that contain positive entries. Therefore one can ignore the columns with no positive entries.

Definition 3.1. An $n \times n$ matrix is a Z-*matrix* if every off diagonal entry is nonpositive. If in addition, every diagonal entry is positive, then a Z-matrix is a Z+ *matrix*.

Definition 3.2. An M-*matrix* is a Z-matrix whose leading principal minors are positive, see [3], [2], [6], [8].

Note that an M-matrix is both left and right semipositive.

Definition 3.3. An $m \times n$ matrix is a *generalized* Z+ *matrix* when there is exactly one positive entry in every row.

Definition 3.4. An $m \times n$ generalized Z+ matrix is a *proper* Z+ *matrix* if it has at least one positive entry in every column.

Definition 3.5. An $m \times n$ proper Z+ matrix is a *permuted proper* Z+ *matrix* if whenever the i, j entry is positive, all entries k, l are nonpositive for $k > i$ and $l < j$.

Definition 3.6. An $m \times n$ permuted proper Z+ matrix is a *generalized* M-*matrix* if every $n \times n$ submatrix that is a Z+ matrix is an M-matrix.

Example 3.7. The following is an example of the sign pattern of a permuted proper Z+ matrix:

$$\begin{bmatrix} + & - & - & - \\ + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & + & - \\ - & - & - & + \\ - & - & - & + \\ - & - & - & + \end{bmatrix}.$$

Now, it is known that a "tall" matrix is SP if and only if all of its maximal square submatrices are SP. This will be helpful in verifying our main result. Now, our main result is a straightforward consequence.

Lemma 3.8 ([10]). *The matrix $A \in M_{m,n}(\mathbb{R})$, with $m \geqslant n$, is* SP *if and only if every $n \times n$ submatrix is* SP.

Lemma 3.9. *A permuted proper* Z+ *matrix is* SP *if and only if every $n \times n$ submatrix that is a* Z+ *matrix is also an M-matrix.*

P r o o f. The reverse implication is the content of Lemma 3.8.

Now suppose that A is an $m \times n$ matrix with the stated properties. We proceed by induction on n. The claim is trivially true when $n = 1$. Now suppose the claim holds for all matrices with less than n columns. We want to show that A is SP by checking that every $n \times n$ submatrix is SP. We already know this is true for all the submatrices which are Z-matrices. Let B be an $n \times n$ submatrix that is not a Z-matrix. Then B must have at least one nonpositive column. Let B' denote B with all of its nonpositive columns deleted. B is SP if and only if B' is SP. B' is a matrix with more rows than columns, exactly one positive entry per row, no nonpositive columns, and with the rows organized in the desired way. All of its maximal Z-submatrices are principal submatrices of the maximal M-submatrices of A, so they are M-matrices as well. Thus, by the induction hypothesis, B is SP. The matrix B was arbitrary, so all of the $n \times n$ submatrices of A are SP, and A is SP. $\square$

Theorem 3.10. *A matrix, $A \in M_{m,n}(\mathbb{R})$, with exactly one positive entry per row is* SP *if and only if deletion of entry-wise nonpositive columns leaves a matrix that, when permuted to a permuted proper* Z+ *matrix, is a generalized* M*-matrix.*

This result settles the question of which matrices with exactly as many positive entries as rows are SP. If a matrix A has exactly as many positive entries as rows, then we may assume A has exactly one positive entry per row, or else A has a nonpositive row and is not SP. If A has more columns than rows, then A has nonpositive columns which we can delete. If A is square, then we may permute the rows of A so that it is a Z-matrix, and the question of which Z-matrices are M-matrices is already settled by determinental criteria. So we may assume A has fewer columns than rows and no nonpositive columns. The rows can be organized in any way one pleases by the permutation invariance of semipositivity. The organization above was chosen for convenience.

4. Sign semipositivity

As noted previously, an SP matrix must have at least one positive entry in each row, and if a sign pattern has at least one $+$ in each row, then it allows SP [12]. See also [11]. So which sign patterns require SP (sign semipositivity)?

Definition 4.1. An $m \times n$ matrix with a given sign pattern is *sign semipositive* if any $A \in M_{m,n}(\mathbb{R})$ with the same sign pattern is semipositive.

Definition 4.2. The matrix $A \in M_{m,n}(\mathbb{R})$ has a *positive front* if there is a permutation of rows and columns such that in each row, the first nonzero entry is positive. A *negative front* is defined similarly.

Example 4.3. The matrix A from Example 1.4 has a positive front since the columns may be permuted to the form

$$A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Note that any matrix with the same sign pattern as A' will be semipositive, since by choosing vector v with a "large" first coordinate, a second coordinate "small" relative to the first (enough so that the first row in the product $A'v$ is positive), and a third coordinate "small" relative to the second (so that both the second and third rows are of the product $A'v$ are positive).

In fact, any matrix with a positive front is semipositive. To show this, we will need to utilize a lemma.

Lemma 4.4. *If A has a positive front, then A is semipositive.*

P r o o f. If A can be permuted to a matrix A' that has a positive front, by Lemma 1.6 it is enough to show that A' is semipositive, since semipositivity is not changed under permutation. If A' has z zero columns as its first z columns, set the first z entries of v to be zero. Set the remaining d entries of v as $x^d, x^{d-1}, \ldots, x$, where x is a fixed number to be specified later. Each row of the product $A'v$ will be a polynomial of degree less than or equal to d, whose first entry is positive (since the first nonzero entry in each row is positive, and there are no zero rows since A has a positive front). Since the coefficient of the monomial with the largest degree in each of these polynomials is positive, the limit as x approaches infinity for all of these polynomials is infinity. Therefore there is some $x > 0$ where each of these polynomials are greater than zero, which implies that A' is semipositive, and thus A is semipositive. □

However, when presented with a matrix A with a large number of columns (say, n), it may be difficult to check all $n!$ permutations of the columns of the matrix. However, there is an algorithm that can determine whether A has a positive front that works in at most n steps.

A l g o r i t h m. To determine if an $m \times n$ matrix A has a positive front, first check that the matrix has no zero row. If it does, this matrix does not have a positive front. If it does not, construct a sequence of matrices with $A_1 := A$. Construct A_2 by removing the 0 columns in A_1. Afterwards for $k > 1$, construct A_{k+1} recursively in the following fashion:

1. Determine if A_k contains a column without a negative entry. If all p columns of A_k contain a negative entry, this matrix does not have a positive front. Otherwise, choose a jth column of A_k, which we will denote by a_{kj}, which has no negative entries. Set A_{k+1} by deleting the jth column of A_k, as well as any row in A_k in which a_{kj} contains a positive entry (leaving the rest of the matrix) and repeat step 1, unless this deletion will result in the loss of the entire matrix. If this is the case, set A_k as the "final matrix" and proceed to step 2.
2. Define a function $C(a_{kj})$ which takes the vector a_{kj} and returns the original column of A associated with a_{kj}.
3. If A has r columns of entirely 0, set the first r columns of A' to be 0. Then, set the next column to be $C(a_{3j})$, the next column to be $C(a_{4,j}), \ldots$, and the next column to be $C(a_{kj})$ where A_k is the final matrix. If there are columns of

A that have not been mapped to by this C function and are nonzero, set them in an arbitrary order after the $C(a_{kj})$ column.

Note that this process must terminate in at most n steps since after n repetitions of step 1, the entire matrix will be "deleted".

Theorem 4.5. *A' exists (this algorithm terminates) if and only if A has a positive front.*

Proof. Assume A' exists. If A' exists, then A has a positive front, namely A'. To show this, we will show that A' is a permutation of the columns of A, and then the first nonzero entry of A' is positive. To show that A' is a permutation of A, note that if $C(a_{ij}) = a_k$ (where a_{ij} is defined to be the jth column of A_i and a_k is the kth column of A), then a_k is not mapped to again by C, since every entry of a_k is deleted to obtain A_{i+1}. Therefore, as A' contains every column of A and has each column only once, A' is a permutation of the columns of A. The 1st nonzero entry of each row must also be positive, since for A' to exist every entry of A must at some point have been "deleted", and the only way for a row to be deleted is if its first nonzero entry is positive. Now assume A' does not exist. This means that for some A_k, every column contains a negative entry or A has a zero row. If A has a zero row it does not have a positive front, so assume that there is some A_k in which every column contains a negative entry. Matrix A is $m \times n$; assume A_k is $p \times q$, with $A_k = (a_{k1}|\ldots|a_{kq})$, in which a_{kj} is the jth column of A_k (the vertical lines separating columns). Let B denote a permutation of A. In this permutation, let a denote the leftmost column indicated by $\{C(a_{k1}), \ldots, C(a_{kq})\}$. In some row, in which a has a negative entry, that entry will be the first nonzero entry in the row. This is because every member of $a_{k1}, \ldots, a_{kq}$ contains a negative entry, which implies that none of the other columns of A had a positive entry in one of the rows in which a had a negative entry and since a is leftmost. This implies that one of the negative entries of a is the leftmost entry of some row. Therefore a positive front does not exist for A. $\square$

Using this process, it is possible to show that a positive front is necessary for a sign pattern to be semipositive.

Lemma 4.6. *If a sign pattern is sign semipositive, then it has a positive front.*

Proof. To show this, assume that A does not have a positive front. The first possibility is that A has a zero row, in which case $A^\top$ has a zero column and thus is seminonpositive (since if $A^\top$ has an ith 0 column the vector with a 1 in the ith entry and 0s elsewhere will yield a zero vector), so A is not semipositive. The other possibility is that at some point in the algorithm above, at the matrix A_k (a $p \times q$

matrix) there is a negative entry in each column. Let $C(a_{kj})$ be defined as it was before in step 2 of the algorithm above. Construct a row vector $v^\top$ which is 0 in the ith row if the ith row of A was previously deleted in the algorithm, and 1 otherwise. Choose a matrix B to have the same sign pattern as A, but so that the column sums of any column with a negative entry is zero. Therefore $v^\top B = 0$, but $v^\top$ is nonzero since if every entry were zero, every row would have been deleted and this algorithm would have terminated. Therefore B is left seminonpositive and thus not semipositive; therefore the sign pattern of A does not require semipositivity. $\square$

Finally, combining Lemma 4.4 and Lemma 4.6, we give a characterization of sign semipositivity.

Theorem 4.7. *An $m \times n$ sign pattern is sign semipositive if and only if it has a positive front.*

5. Spectral theory of SP matrices

Of course, SP matrices represent a generalization of entry-wise nonnegative matrices. Here we will only consider square matrices. The inverse eigenvalue problem for nonnegative matrices (NIEP) is known to be notoriously difficult. We give a complete solution to the SP inverse eigenvalue problem (SPIEP) and, further, the similarity class problem for SP matrices. Interestingly, except for nonpositive scalar matrices, there is an SP matrix in the similarity class of every real matrix.

The key observation is:

Lemma 5.1. *Every $n \times n$ non-scalar real matrix is similar to a real matrix with a positive first column.*

Proof. Unless one is scalar and the other is not, two matrices in $M_2(\mathbb{R})$ are similar if and only if they have the same trace and determinant. Furthermore, unless the diagonal entries are the eigenvalues, both off-diagonal entries will be nonzero. Thus, a simple calculation shows that the claim of the lemma is valid in the 2×2 case. Now, let $A \in M_n(\mathbb{R})$ be non-scalar. It is clearly similar to a non-diagonal matrix $B \in M_n(\mathbb{R})$, which, in turn, has a 2×2 non-scalar principal submatrix that we may assume, without loss of generality, is in the first two rows and columns and has a positive first column.

Now, if the remaining entries in the first column of B are all nonzero, the proof is completed by performing similarity on B by a diagonal matrix of $\pm$ 1s, so as to adjust the signs of the entries in the first column to be all positive and complete the proof. If not all entries in the first column are nonzero, we may make them so, by a sequence of similarities of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & 0 \\ 0 & 0 & I \end{bmatrix},$$

in which the second I may not be present. Then, we may proceed as in the case of a totally nonzero first column to complete the proof. □

Now we may show

Theorem 5.2. *Except for nonpositive scalar matrices, every $n \times n$ real matrix is similar to an* SP *matrix.*

Proof. If $A \in M_n(\mathbb{R})$ is not a scalar matrix, then according to Lemma 5.1, there is a matrix with positive column in its similarity class. By Lemma 1.7, this matrix is SP. Since any positive diagonal matrix is SP, positive scalar matrices are, as well, which completes the proof. □

As a corollary, we have

Corollary 5.3. *Let n be given and suppose that $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ is a multi-set of complex numbers that is the spectrum of a real matrix. Then Λ is the spectrum of an* SP *matrix in $M_n(\mathbb{R})$, unless $n = 1$ and $\lambda_1 \leqslant 0$.*

6. Equivalence theory of SP matrices

Recall that two matrices $A, B \in M_{m,n}(\mathbb{R})$ are said to be *equivalent* if there exist invertible matrices $S \in M_m(\mathbb{R})$ and $T \in M_n(\mathbb{R})$ such that

$$B = SAT.$$

Of course this "equivalence" is an equivalence relation on $M_{m,n}(\mathbb{R})$. It is known that two matrices in $M_{m,n}(\mathbb{R})$ are equivalent if and only if they have the same rank, see [9]. Now, if S (T) may be taken to be I, equivalence is called right (left) equivalence. For each equivalence, left and right equivalence, we may characterize the equivalence classes that include an SP matrix. Of course, the 0 matrix is the only matrix in its (right or left) equivalence class (the rank of 0 class), so that no SP matrix is equivalent or right or left equivalent to 0.

Now we have

Theorem 6.1. *Any matrix of rank at least* 1 *is equivalent to an* SP *matrix.*

P r o o f. Begin with $A \in M_{m,n}(\mathbb{R})$. Now suppose that matrices S, T reduce A to its reduced echelon form. Therefore the equivalence relation SAT will result in a matrix with a positive front for the 1 through rth row where $r =$ rank of A. Then left multiply SAT by the matrix that will add the rth row to the $r+1$th rows. The resulting matrix will be SP and is clearly equivalent to the original matrix A. $\square$

Corollary 6.2. *Any matrix of rank at least* 1 *is left equivalent to an* SP *matrix.*

The operations performed by matrices S, T can be done with exclusively left multiplication using elementary row operations. Therefore, in addition to any matrix being equivalent to an SP matrix, any matrix is left equivalent to an SP matrix.

Right equivalence is more subtle.

Lemma 6.3. *If a matrix is right equivalent to an* SP *matrix, then it has a positive vector in its range.*

P r o o f. Begin with matrices $A \in M_{m,n}(\mathbb{R})$, $B \in M_{m,p}(\mathbb{R})$, and $C \in M_{p,n}(\mathbb{R})$ and a nonnegative vector, $x \in \mathbb{R}^n$, such that $Ax = v$ where $v > 0$ so that A is SP. By assumption B is right equivalent to A by some matrix C and therefore $A = BC$. This implies that $Ax = BCx$ and therefore $Ax = Bu$ where $u = Cx$. And finally $v = Bu$ and this implies that for some vector u, Bu will result in a positive vector, v, and therefore B has a positive vector in its range. $\square$

On the other hand, we have

Lemma 6.4. *If a matrix has a positive vector in its range then it is right equivalent to an* SP *matrix.*

P r o o f. Begin with a matrix $A \in M_{m,n}(\mathbb{R})$ and $x \in \mathbb{R}^n$ where $Ax = v$ and $v > 0$. Choose a matrix $B \in M_{n,p}(\mathbb{R})$, such that $x = Bu$ where $u \geqslant 0$. Then $ABu = v$ and therefore AB is SP and A is right equivalent to an SP matrix. $\square$

It follows from the two lemmas that

Theorem 6.5. *A matrix is right equivalent to an* SP *matrix if and only if it has a positive vector in its range.*

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