Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 3, 895–911

Persistent URL: `http://dml.cz/dmlcz/145878`

Czechoslovak Mathematical Journal, 66 (141) (2016), 895–911

RATIONAL REALIZATION OF THE MINIMUM RANKS OF NONNEGATIVE SIGN PATTERN MATRICES

Wei Fang, Taiyuan, Wei Gao, Atlanta, Yubin Gao, Taiyuan,
Fei Gong, Guangming Jing, Zhongshan Li, Atlanta,
Yanling Shao, Taiyuan, Lihua Zhang, Atlanta

(Received December 31, 2015)

Dedicated to the memory of Miroslav Fiedler

Abstract. A sign pattern matrix (or nonnegative sign pattern matrix) is a matrix whose entries are from the set $\{+,-,0\}$ ($\{+,0\}$, respectively). The minimum rank (or rational minimum rank) of a sign pattern matrix $\mathcal{A}$ is the minimum of the ranks of the matrices (rational matrices, respectively) whose entries have signs equal to the corresponding entries of $\mathcal{A}$. Using a correspondence between sign patterns with minimum rank $r \geqslant 2$ and point-hyperplane configurations in $\mathbb{R}^{r-1}$ and Steinitz's theorem on the rational realizability of 3-polytopes, it is shown that for every nonnegative sign pattern of minimum rank at most 4, the minimum rank and the rational minimum rank are equal. But there are nonnegative sign patterns with minimum rank 5 whose rational minimum rank is greater than 5. It is established that every d-polytope determines a nonnegative sign pattern with minimum rank $d+1$ that has a $(d+1)\times(d+1)$ triangular submatrix with all diagonal entries positive. It is also shown that there are at most $\min\{3m, 3n\}$ zero entries in any condensed nonnegative $m\times n$ sign pattern of minimum rank 3. Some bounds on the entries of some integer matrices achieving the minimum ranks of nonnegative sign patterns with minimum rank 3 or 4 are established.

Keywords: sign pattern (matrix); nonnegative sign pattern; minimum rank; convex polytope; rational minimum rank; rational realization; integer matrix; condensed sign pattern; point-hyperplane configuration

MSC 2010: 15B35, 15B36, 52C35, 15A23

1. Introduction

An important part of combinatorial matrix theory is the study of sign pattern matrices (see [8], [17], and the references therein). A *sign pattern* (*matrix*) (or *non-*

negative sign pattern (*matrix*)) is a matrix whose entries are from the set $\{+,-,0\}$ ($\{+,0\}$ respectively). For a real matrix B, $\operatorname{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (or negative or zero) entry of B by $+$ ($-, 0$, respectively). For a sign pattern matrix $\mathcal{A}$, the *qualitative class* of $\mathcal{A}$, denoted $Q(\mathcal{A})$, is defined as

$$Q(\mathcal{A}) = \{A\colon\ A \text{ is a real matrix with } \operatorname{sgn}(A) = \mathcal{A}\}.$$

A *signature sign pattern* is a diagonal sign pattern matrix whose diagonal entries are from the set $\{+,-\}$. Two $m\times n$ sign patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *signature equivalent* or *diagonally equivalent* if there exist signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{A}_2 = \mathcal{D}_1\mathcal{A}_1\mathcal{D}_2$.

A square $n\times n$ sign pattern is called a *permutation sign pattern* if each row and column contains exactly one $+$ entry and $n-1$ zero entries. Two $m\times n$ sign pattern matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *permutationally equivalent* if there exist permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{A}_2 = \mathcal{P}_1\mathcal{A}_1\mathcal{P}_2$.

The product $\mathcal{PD}$ of a permutation sign pattern $\mathcal{P}$ and a signature sign pattern $\mathcal{D}$ is called a *signed permutation* (*sign pattern*). Two $m\times n$ sign pattern matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *signed permutationally equivalent* if there exist signed permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{A}_2 = \mathcal{P}_1\mathcal{A}_1\mathcal{P}_2$.

The minimum rank of a sign pattern matrix $\mathcal{A}$, denoted $\operatorname{mr}(\mathcal{A})$, is the minimum of the ranks of the real matrices in $Q(\mathcal{A})$. Similarly, the rational minimum rank of a sign pattern $\mathcal{A}$, denoted $\operatorname{mr}_{\mathbb{Q}}(\mathcal{A})$, is defined to be the minimum of the ranks of the rational matrices in $Q(\mathcal{A})$. The minimum ranks of sign pattern matrices have been the focus of a large number of papers (see e.g. [1], [2], [3], [4], [5], [6], [7], [9], [11], [12], [13], [18], [19], [20], [21], [22], [27]), and they have important applications in areas such as communication complexity [1], [24], [25], machine learning [14], neural networks [10], combinatorics [11], [16], [28], and discrete geometry [23].

It is clear that $\operatorname{mr}(\mathcal{A}) \leqslant \operatorname{mr}_{\mathbb{Q}}(\mathcal{A})$ for every sign pattern $\mathcal{A}$. When $\operatorname{mr}(\mathcal{A}) = \operatorname{mr}_{\mathbb{Q}}(\mathcal{A})$, we say that the minimum rank of $\mathcal{A}$ can be *realized rationally.* It is known (see [2], [5], [22], [27]) that for every $m\times n$ sign pattern $\mathcal{A}$ with $\operatorname{mr}(\mathcal{A}) \leqslant 2$ or $\operatorname{mr}(\mathcal{A}) \geqslant n-2$, its minimum rank can be realized rationally. However, it is shown in [21] and [11] that there exist sign patterns with minimum rank 3 whose rational minimum rank is greater than 3. In contrast, using a correspondence established in [11] between sign patterns with minimum rank $r \geqslant 2$ and point-hyperplane configurations in $\mathbb{R}^{r-1}$ and Steinitz's theorem (see [28]) on the rational realizability of 3-polytopes, we show in Section 2 that for every nonnegative sign pattern of minimum rank at most 4, the minimum rank and the rational minimum rank are equal, but there are nonnegative sign patterns with minimum rank 5 whose rational minimum rank is greater than 5. We also establish several other interesting properties of nonnegative sign patterns with minimum rank 3. In Section 3, we find upper bounds on the entries of some integer matrices achieving the minimum ranks of nonnegative sign patterns with minimum rank 3 or 4.

Consider a nonnegative sign pattern $\mathcal{A}$. Observe that if $\mathcal{A}$ contains a zero row or zero column, then deletion of the zero row or zero column preserves the minimum rank. Similarly, if two nonzero rows (or columns) of $\mathcal{A}$ are identical, then deleting such a row (or column) also preserves the minimum rank. Clearly, deletion of a zero or duplicate row or column does not affect rational realizability of the minimum rank. Indeed, the deletion process can be reversed easily to create a matrix in the nonnegative sign pattern class of the original nonnegative sign pattern that achieves the minimum rank. Following [22], we say that a nonnegative sign pattern is *condensed* if it does not contain a zero row or a zero column and no two rows or two columns of it are identical. Clearly, given any nonzero nonnegative sign pattern $\mathcal{A}$, we can delete the zero rows and columns, and delete all except the first row or column from each maximal collection of identical nonzero rows or columns of $\mathcal{A}$ to get the condensed nonnegative sign pattern matrix $\mathcal{A}_{\rm c}$ of $\mathcal{A}$ with the same minimum rank.

For example, for the sign pattern

$$\mathcal{A} = \begin{bmatrix} 0 & + & + & + & + \\ + & 0 & 0 & + & + \\ + & 0 & 0 & + & + \\ + & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$\mathcal{A}_{\rm c} = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}.$$

Since every nonnegative sign pattern and its condensed sign pattern have the same minimum rank and the same rational minimum rank, without loss of generality, in most of the subsequent discussions, we may assume that the nonnegative sign patterns involved are condensed.

2. Rational realizability of the minimum ranks of nonnegative sign patterns

As shown in [11], every sign pattern with minimum rank $r \geqslant 2$ corresponds to a point-hyperplane configuration in $\mathbb{R}^{r-1}$. The following lemma shows a very special feature of the point-hyperplane configurations corresponding to nonnegative sign patterns: all the points in the configuration are in the same closed half space bounded by any of the hyperplanes in the configuration.

Clearly, for an $m \times n$ nonnegative sign pattern $\mathcal{A}$ and a signature sign pattern $\mathcal{D}$ with order n, $\operatorname{mr}(\mathcal{A}) = \operatorname{mr}(\mathcal{AD})$.

Lemma 2.1. *Let $\mathcal{A}$ be an $m\times n$ condensed nonnegative sign pattern with $\operatorname{mr}(\mathcal{A}) = r \geqslant 2$. Then there exist a suitable signature sign pattern $\mathcal{D}$ of order n and a real matrix $B \in Q(\mathcal{AD})$ such that B has a factorization $B = UV$ of the form*

$$U = \begin{bmatrix} 1 & u_{11} & \dots & u_{1r} \\ 1 & u_{21} & \dots & u_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_{m1} & \dots & u_{mr} \end{bmatrix}, \quad \text{and} \quad V = [v_1, \dots, v_n] = \begin{bmatrix} v_{r1} & v_{r2} & \dots & v_{rn} \\ \vdots & \vdots & \ddots & \vdots \\ v_{11} & v_{12} & \dots & v_{1n} \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proof. Let $B_1 \in Q(\mathcal{A})$ be a real matrix with $\operatorname{rank}(B_1) = r$ and let U_1 be a matrix whose columns form a maximal linearly independent list of columns of B_1. Then U_1 has size $m\times r$ and there exists an $r\times n$ matrix V_1 such that $B_1 = U_1V_1$. Since $\mathcal{A}$ is condensed, B_1 does not have any zero row or column. Note that each row of U_1 is nonzero and nonnegative. Since the total number of row vectors of U_1 and the column vectors of V_1 is the finite number $m+n$, there exist suitable Givens rotation matrices (through some small positive angles) $R(\theta_2, 1, 2), R(\theta_3, 1, 3), \dots, R(\theta_r, 1, r)$ of order r such that the real orthogonal matrix $G = R(\theta_2, 1, 2)R(\theta_3, 1, 3)\dots R(\theta_r, 1, r)$ satisfies the following two conditions.

(i) The first column of U_1G^T has only positive entries.

(ii) The last row of GV_1 has no zero entries.

Hence, there is a diagonal matrix D_1 with all diagonal entries positive and a diagonal matrix D_2 with all diagonal entries nonzero such that all the entries in the first column of $D_1U_1G^T$ are equal to 1, and all the entries in the last row of GV_1D_2 are equal to 1. Now, let $U = D_1U_1G^T$, $V = GV_1D_2$ and $B = UV$. Since multiplication on the left by D_1 preserves the sign pattern, we see that $B \in Q(\mathcal{AD})$, where $\mathcal{D} = \operatorname{sgn}(D_2)$. This completes the proof. □

It is shown in [11] that every sign pattern with minimum rank $r \geqslant 2$ corresponds to a point-hyperplane configuration in $\mathbb{R}^{r-1}$. In particular, for the sign pattern $\mathcal{AD}$ in the preceding lemma, through the special factorization $B = UV$, by identifying the ith row of U with the point $p_i = (u_{i2}, \dots, u_{ir}) \in \mathbb{R}^{r-1}$ $(i = 1, \dots, m)$ and identifying the jth column v_j of V with the hyperplane h_j in $\mathbb{R}^{r-1}$ satisfying the equation $[1\ x_1\ \dots\ x_{r-1}]v_j = 0$ $(j = 1, \dots, n)$, we get an m point-n hyperplane configuration in $\mathbb{R}^{r-1}$, denoted $C_{\mathcal{A}}$. Furthermore, p_i is above or below or in h_j if and only if the (i, j) entry of $\mathcal{A}D$ is $+$ or $-$ or 0, respectively.

Conversely, given any point-hyperplane configuration C in $\mathbb{R}^{r-1}$ consisting of m points $p_1, \ldots, p_m$ and n nonvertical (i.e., the normal vector is not perpendicular to the x_{r-1}-axis) hyperplanes $h_1, \ldots, h_n$, we may write $p_i = (u_{i2}, \ldots, u_{ir})$, and suppose that h_j is given by the equation $[\,1 \;\; x_1 \;\; \ldots \;\; x_{r-1}\,]v_j = 0$, where the last component of v_j is 1. Let U, V be defined as in Lemma 2.1. Then $B = UV$ is a matrix of rank at most r. Furthermore, $\mathcal{A} = \operatorname{sgn}(B) = [a_{ij}]$ is an $m \times n$ sign pattern with $\operatorname{mr}(\mathcal{A}) \leqslant r$ such that $a_{ij} = +$ or $-$ or 0 if and only if p_i is above or below or in h_j, respectively. Alternatively, each hyperplane in the configuration can be oriented by prescribing one of its two sides as the positive side. Then the configuration gives rise to a sign pattern $\mathcal{A} = [a_{ij}]$ such that $a_{ij} = +$ or $-$ or 0 if and only if p_i is on the positive side of or on the negative side of or in h_j, respectively. As observed in [11], re-orienting the hyperplanes amounts to negating certain columns of the corresponding sign pattern. Two point-hyperplane configurations are said to be *equivalent* if their corresponding sign patterns are signed permutationally equivalent.

Lemma 2.2. *Let $\mathcal{A}$ be a condensed nonnegative sign pattern with minimum rank 3, and let its corresponding point-hyperplane configuration C consist of m points $p_1, \ldots, p_m$ and n lines $l_1, \ldots, l_n$ in $\mathbb{R}^2$. Let $P = \{p_1, p_2, \ldots, p_m\}$ and $L = \{l_1, l_2, \ldots, l_n\}$. Then*

(1) *all points of P are in the same closed halfplane bounded by l_j for each $j = 1, \ldots, n$ (namely, no two points of P are on opposite sides of any line l_j in L);*
(2) *if three distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in P$ are collinear with p_{i_2} between p_{i_1} and p_{i_3}, then the only possible line in L passing through p_{i_2} is the line containing p_{i_1}, p_{i_2}, and p_{i_3};*
(3) *there are at most three points of P on the same line in L and there are at most 3 lines in L passing through the same point in P.*

Proof. (1) Note that as in the preceding discussion, every column in $\mathcal{AD}$ is either nonnegative or nonpositive. Assume that there exist two points p_i and p_k on opposite sides of l_j. Then the jth column of $\mathcal{AD}$ must have at least one $+$ entry and one $-$ entry, which is a contradiction.

(2) Assume that three distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in P$ are collinear with p_{i_2} between p_{i_1} and p_{i_3}, and there is a line $l_j \in L$ passing through p_{i_2} but not containing p_{i_1}. Then p_{i_1} and p_{i_3} are on opposite sides of l_j, contradicting (1).

(3) Assume that there are four distinct points, p_1, p_2, p_3, p_4 of P appearing in this order on the same straight line $l_j \in L$. By (2), there is no other line in L that passes through p_2 or p_3. As a result, p_2 and p_3 are on the same side of all other lines in L. It follows that the rows of $\mathcal{A}$ corresponding to p_2 and p_3 are identical, contradicting the fact that $\mathcal{A}$ is row condensed. To show that there are at most 3

lines in L passing through the same point in P, we consider the point-hyperplane configuration C' corresponding to A^T. Note that C' may be obtained from C by transforming the n lines in L into n points $p'_1, \dots, p'_n$ by the dual transform (which transforms the line not passing through the origin given by $\{x \in \mathbb{R}^2 \colon \langle a, x\rangle = 0\}$ to the point a and vice versa, see [23]) and also by transforming the m points in P into the m lines by the dual transform. Since concurrent lines are transformed to collinear points under the dual transform, the last part of (3) follows. $\square$

As an immediate consequence, we obtain the following result on the number of zero entries in a condensed nonnegative sign pattern with minimum rank 3.

Theorem 2.3. *Let $\mathcal{A}$ be an $m \times n$ condensed nonnegative sign pattern with* $\operatorname{mr}(\mathcal{A}) = 3$. *Then each row and column of $\mathcal{A}$ has at most* 3 *zero entries. Hence there are at most* $\min\{3m, 3n\}$ *zero entries in $\mathcal{A}$.*

Proof. By Lemma 2.2, each point in $C_{\mathcal{A}}$ is on at most three lines and each line passes through at most 3 points. Thus each row and column of $\mathcal{A}$ has at most 3 zero entries since the rows and columns correspond to points and hyperplanes in $C_{\mathcal{A}}$, respectively. $\square$

We remark that the upper bound on the number of zero entries given in the preceding theorem is the best possible. For example, let C be the point-hyperplane configuration in $\mathbb{R}^2$ whose hyperplanes are (the extensions of) the edges of a fixed n-gon G (with the side including the interior of G being the positive side) and whose points are the vertices of G and the midpoints of the edges of G. Then the resulting condensed nonnegative sign pattern $\mathcal{A}$ corresponding to C has size $2n \times n$ and minimum rank 3 (see Theorem 2.9 below) and has exactly $3n$ zero entries (with exactly 3 zero entries in each column).

It is also worth noting that every condensed nonnegative sign pattern $\mathcal{A}$ with minimum rank 3 must have at least 3 zero entries. Indeed, if a condensed nonnegative sign pattern $\mathcal{A}$ with minimum rank 3 has at most 2 zero entries, then $\mathcal{A}$ must be 3×3 (otherwise, $\mathcal{A}$ would have two positive rows or columns) and there must be two zero entries on distinct rows and columns. It follows that up to permutational equivalence, $\mathcal{A} = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix}$, so that $\operatorname{mr}(\mathcal{A}) = 2$, a contradiction. The minimum number of zero entries, 3, is achieved by nonnegative sign patterns such as $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ and $\begin{bmatrix} + & + & + \\ 0 & + & + \\ 0 & 0 & + \end{bmatrix}$.

We now arrive at one of the key facts of this paper.

Theorem 2.4. *Let $\mathcal{A}$ be an $m \times n$ nonnegative sign pattern with* $\operatorname{mr}(\mathcal{A}) = 3$. *Then* $\operatorname{mr}_{\mathbb{Q}}(\mathcal{A}) = 3$.

P r o o f. Without loss of generality, we may assume that $\mathcal{A}$ is a condensed nonnegative sign pattern.

By Lemma 2.1, there exist a suitable signature pattern $\mathcal{D}$ and a matrix $B \in Q(\mathcal{A}\mathcal{D})$ such that $B = UV$, where

$$U = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ \vdots & \vdots & \vdots \\ 1 & a_m & b_m \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

We associate $\mathcal{A}$ with its point-line configuration $C_{\mathcal{A}} = P \cup L$, where $P = \{p_1, \dots, p_m\}$ and $L = \{l_1, \dots, l_n\}$, as in Lemma 2.2. We complete the proof by finding a rational point-line configuration (namely, a configuration whose all points are rational points and each of whose lines contains two distinct rational points of $\mathbb{R}^2$) that is equivalent to $C_{\mathcal{A}}$.

Consider the convex polytope $K = \operatorname{conv}(P)$. The fact that $\operatorname{mr}(\mathcal{A}) = 3$ ensures that not all the points in P are collinear (see [11]). Hence, K is a polygonal region, whose boundary is a polygon with $k \geqslant 3$ vertices $p_{i_1}, \dots, p_{i_k}$ (in counterclockwise order). For convenience, we may assume that each line of L is oriented so that its positive side contains the center of mass of K. Observe that each point of P is either a vertex of $\operatorname{conv}(P)$, or a point in the relative interior of an edge of K, or an interior point of K. Let

$$\begin{aligned} P_0 &= \{p \in P \colon\ p \text{ is a vertex of } K\}, \\ P_1 &= \{p \in P \colon\ p \text{ is in the relative interior of an edge of } K\}, \text{ and} \\ P_2 &= \{p \in P \colon\ p \text{ is an interior point of } K\}. \end{aligned}$$

Clearly, $P = P_0 \cup P_1 \cup P_2$, and $P_0 = \{p_{i_1}, \dots, p_{i_k}\}$. Note that if a point $p_{i_0} \in P_2$, then it corresponds to a row of $\mathcal{A}$ that is positive (namely, all the entries in that row are positive). Thus there is at most one point in P_2, and replacing such a point (if any) with the center of mass of K results in an equivalent configuration. In the case a point $p_i \in P_1$, by Lemma 2.2 (1), the only possible line in L containing p_i is the line extending the edge containing p_i; such a point p_i can be replaced by the midpoint of the edge of K containing this point and the resulting configuration is equivalent to the original one. We now replace the 2-polytope K with consecutive vertices $p_{i_1}, \dots, p_{i_k}$ by a 2-polytope K' with k consecutive rational vertices $p'_{i_1}, \dots, p'_{i_k}$ (also in counterclockwise order) such that no edge is vertical. If $P_2 \neq \emptyset$ with $P_2 = \{p_{i_0}\}$, we let p'_{i_0} be the center of mass of K'. If a point $p_j \in P_1$, then there are vertices p_{i_t} and $p_{i_{t+1}}$ (with the understanding that $i_{k+1} = i_1$) such that p_j is an interior point of the edge $p_{i_t}p_{i_{t+1}}$, and we let p'_j be the midpoint of $p'_{i_t}p'_{i_{t+1}}$. Let $P' = \{p'\colon\ p \in P\}$, and for each $k \in \{0, 1, 2\}$, let $P'_k = \{p'\colon\ p \in P_k\}$.

Observe that every line in L contains exactly 0, or 1, or 2 vertices of $K = \operatorname{conv}(P)$. If a line $l_{j_0} \in L$ does not intersect K, then its corresponding column of $\mathcal{A}$ is positive (and hence, there is at most one such line in L), and this line can be replaced by any horizontal line below K; in this case, we define l'_{j_0} to be a rational horizontal line below K'. If a line l_j in L passes through two consecutive vertices p_{i_t} and $p_{i_{t+1}}$ of K, then we let l'_j be the line through p'_{i_t} and $p'_{i_{t+1}}$. If a line $l_i \in L$ passes through exactly one vertex of p_{i_t} of K, then we define l'_i to be any nonvertical rational line passing through p'_{i_t} that has only one intersection point with $K' = \operatorname{conv}(P')$. Let $L' = \{l'\colon\ l \in L\}$. We orient each line l'_j in L' so that the interior of K' is on the positive side l'_j.

Thus we obtain a rational point-line configuration $C' = P' \cup L'$, such that for all i, j, p_i is on the positive side of or on the negative side of or in l_j if and only if p'_i is on the positive side of or on the negative side of or in l'_j, respectively. Hence, the rational point-line configuration C' is equivalent to C. Using C', we can define rational matrices U' and V' of sizes $m \times 3$ and $3 \times n$, respectively, as in the discussion preceding Lemma 2.2, and the sign pattern of the resulting rational matrix $B' = U'V'$ is $\mathcal{A}\mathcal{D}'$, where $\mathcal{D}'$ is a signature sign pattern with negative diagonal entries corresponding to the lines in L' that are above the interior of $\operatorname{conv}(P')$. It follows that $B'\mathcal{D}'$ is a rational matrix in $Q(\mathcal{A})$ achieving the minimum rank 3. $\square$

There is an interesting relationship between the numbers of rows and columns of a condensed nonnegative sign pattern with minimum rank 3.

Theorem 2.5. *Let $\mathcal{A}$ be an $m \times n$ condensed nonnegative sign pattern with $\operatorname{mr}(\mathcal{A}) = 3$. Then $n \leqslant 2m + 1$ and $m \leqslant 2n + 1$, where each upper bound is tight.*

Proof. By considering $\mathcal{A}^T$ instead, it suffices to show that $n \leqslant 2m + 1$. By Lemma 2.2 and the proof of Theorem 2.4, it is easy to see that for a point-line configuration $C = P \cup L$ corresponding to the sign pattern $\mathcal{A}$, since there are m points in P, there is at most one line in L disjoint with $\operatorname{conv}(P)$, at most m lines in L passing through exactly one vertex of $\operatorname{conv}(P)$, and at most m lines in L that are extensions of edges of $\operatorname{conv}(P)$. Moreover, every line in L must be one of the above types. It follows that $|L| \leqslant 2m + 1$, namely, $n \leqslant 2m + 1$.

Note that for the point-line configuration obtained by taking the points to be the vertices of a fixed m-gon and the lines to be the edges of the m-gon along with a horizontal line below the m-gon and m additional lines each of which is a line intersecting the m-gon at only one point, then the corresponding condensed nonnegative sign pattern is $m \times (2m+1)$ and has minimum rank 3 (see Theorem 2.9). This completes the proof. $\square$

We now arrive at another key result of this paper, based on Steinitz's theorem (see [28]) on the rational realizability of 3-polytopes. Basic terms and facts about convex polytopes may be found in [28].

Theorem 2.6. *Let* $\mathcal{A}$ *be an* $m \times n$ *nonnegative sign pattern with* $\mathrm{mr}(\mathcal{A}) = 4$. *Then* $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) = 4$.

Proof. Without loss of generality, we assume that $\mathcal{A}$ is condensed. Similarly to the proof of Theorem 2.4, it suffices to find a rational point-hyperplane configuration in $\mathbb{R}^3$ equivalent to $C_{\mathcal{A}} = P \cup H$, where $P = \{p_1, \ldots, p_m\} \subset \mathbb{R}^3$ and $H = \{h_1, \ldots, h_n\}$ consists of hyperplanes. Let $K = \mathrm{conv}(P)$, which must be a 3-polytope, see [11]. Then each point of P is either a vertex of K, or an interior point of K, or in the relative interior of either an edge or a facet of K. In other words, each point of P is in the relative interior point of a face of K of dimension 0, 1, 2, or 3. Of course, a 0-dimensional face is a vertex, a 1-dimensional face is an edge, a 2-dimensional face is a facet, and the 3-dimensional face is K itself. Let the vertices of K be $p_{i_1}, \ldots, p_{i_k}$. By Steinitz's theorem (see [28]), there is a rational 3-polytope K' whose vertices are rational perturbations $p'_{i_1}, \ldots, p'_{i_k}$ of the corresponding vertices of K and whose face-lattice is isomorphic to that of K under the mapping $p_{i_t} \mapsto p'_{i_t}$. By using suitable rational Givens rotations if necessary, we may assume that no facet of K' is vertical (namely, the normal vector of any facet is not perpendicular to the z-axis).

The fact that $\mathcal{A}$ is condensed ensures that there is at most one point in P in the relative interior of each face of K. For each point $p \in P$ in the relative interior of a face F of K, we define p' to be the center of mass of the corresponding face F' of K'. Clearly, each p' is a rational point since all the vertices of K' are rational points and every face of K' is the convex hull of a subset of the vertices of K'.

Since $\mathcal{A}$ is nonnegative, by Lemma 2.1, in the configuration $C_{\mathcal{A}} = P \cup H$ no two vertices of K are on opposite sides of any hyperplane $h \in H$, and hence the intersection of each hyperplane $h \in H$ with K is either empty or is a face F of K, with $\dim(F) = 0, 1$, or 2.

Consider a hyperplane $h \in H$. If $h \cap K = \emptyset$, let h' be any rational horizontal hyperplane below K. If $\dim(h \cap K) = 0$ with $h \cap K = \{p_{i_t}\}$, let h' be a rational hyperplane (namely, a hyperplane passing through a rational point and having a rational normal vector) such that $h' \cap K' = p'_{i_t}$. More generally, if $h \cap K = F$ for some face F of K, let h' be a rational hyperplane such that $h' \cap K' = F'$. This is clearly possible since F' is rational and when $\dim(F') = \dim(F) \leqslant 1$, there is a cone of possible choices of normal vectors of h', see [28].

It is evident that the rational configuration $C' = P' \cup H'$ is equivalent to $C_{\mathcal{A}}$. We can use C' to reverse the process of Lemma 2.1 to create two rational factors U and V of sizes $m \times 4$ and $4 \times n$, respectively, to obtain a rational matrix $B = UV$ of rank at most 4 whose sign pattern is signature equivalent to $\mathcal{A}$. It follows that $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) \leqslant 4$. Therefore, $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) = 4$. $\square$

The argument in the preceding proof can be adapted to show the following result, whose formal proof is omitted as the main ideas are the same.

Theorem 2.7. *Let $\mathcal{A}$ be a condensed nonnegative sign pattern with minimum rank $r \geqslant 5$. Then $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) = r$ if and only if for a point-hyperplane representation $C_{\mathcal{A}} = P \cup H$, where P is the set of all points of $C_{\mathcal{A}}$, the face lattice of the convex polytope $K = \mathrm{conv}(P)$ is isomorphic to that of a rational convex polytope K'.*

The construction of the possible hyperplanes in H' (according to their intersections with K') in the proof of Theorem 2.6 reveals an upper bound on n for any $m \times n$ condensed nonnegative sign pattern with minimum rank 4. By considering the transpose, we get the dual upper bound.

Theorem 2.8. *Let $\mathcal{A}$ be an $m \times n$ condensed nonnegative sign pattern with $\mathrm{mr}(\mathcal{A}) = 4$. Then*

$$n \leqslant \sum_{k=0}^{3} \binom{m}{k} = \frac{1}{6}m^3 + \frac{5}{6}m + 1 \quad \text{and} \quad m \leqslant \sum_{k=0}^{3} \binom{n}{k} = \frac{1}{6}n^3 + \frac{5}{6}n + 1.$$

As indicated in the preceding theorems, the study of the minimum ranks of nonnegative sign patterns leads to investigation of convex polytopes. We now show that due to the two-way correspondence between sign pattern matrices and point-hyperplane configurations, a convex polytope also determines a nonnegative sign pattern naturally, and this approach has been exploited previously, see [15].

Theorem 2.9. *Let K be a convex polytope of dimension $d \geqslant 1$ that has m vertices and n facets, embedded in $\mathbb{R}^d$. Let P be the set of vertices of K and let H be the set of hyperplanes each of which contains precisely one facet of K, with each hyperplane oriented so that the interior of K is on its positive side. Then the $m \times n$ nonnegative sign pattern $\mathcal{A}$ corresponding to the point-hyperplane configuration $P \cup H$ has minimum rank $d+1$ and up to permutational equivalence, $\mathcal{A}$ contains an upper triangular submatrix of order $d+1$ all of whose diagonal entries are positive.*

P r o o f. As indicated in the discussion preceding Lemma 2.2, using the configuration $P \cup H$, we get an $m \times (d+1)$ matrix U using the coordinates of the points in P and a $(d+1) \times n$ matrix V using the equations of the hyperplanes in H. Hence, the matrix $B = UV$ has rank at most $d+1$. Since the nonnegative sign pattern $\mathcal{A}$ corresponding to the configuration $P \cup H$ is signed permutationally equivalent to $\mathrm{sgn}(B)$, we have $\mathrm{mr}(\mathcal{A}) \leqslant d+1$.

We prove the opposite inequality $\mathrm{mr}(\mathcal{A}) \geqslant d+1$ and the result about triangular submatrices by induction on d. For $d=1$, a 1-polytope K is just a line segment in $\mathbb{R}^1$, and K has two vertices $p_1 < p_2$, which are also the facets of K. Let $h_1 = p_2$ and $h_2 = p_1$, with the positive side containing the midpoint of K. Then the nonnegative sign pattern corresponding to the configuration $\{p_1, p_2\} \cup \{h_1, h_2\}$ is $\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$, which has minimum rank 2 and is upper triangular with positive diagonal entries. Thus the result holds for $d=1$.

Suppose that $d \geqslant 2$ and the result holds for every $(d-1)$-polytope. Let K be any d-polytope, with m vertices and n facets. By rotating K suitably if necessary, we may assume that K has a unique highest vertex (namely, the vertex with the largest x_d coordinate), denoted p_0. Let K_0 be a vertex figure (see [28]) of K at p_0, namely, K_0 is the intersection of a hyperplane h_0 slightly below the point p_0 such that h_0 strictly separates p_0 from the remaining vertices of K. Clearly, h_0 intersects every edge of K with p_0 as an endpoint at an interior point of the edge. It is well known that the vertex figure K_0 of K has dimension $\dim(K) - 1 = d-1$ (see [28]). Note that every vertex of K_0 is on an edge of K that has p_0 as an endpoint. By the induction hypothesis, K_0 has d vertices $v_1, \ldots, v_d$ and d facets $F_1, \ldots, F_d$ (viewed as hyperplanes) which form a configuration that gives rise to an upper triangular nonnegative sign pattern of order d with all diagonal entries positive. For each $i = 1, \ldots, d$, let p_i be the vertex of K below h_0 on the extension of $p_0 v_i$, and let $h_i = \mathrm{conv}(\{p_0\} \cup F_i)$. Then h_i is contained in a facet of K, $i = 1, \ldots, d$. Let h_{d+1} be any facet of K not containing p_0. We identify each set h_i with the unique hyperplane containing it. Then the subconfiguration consisting of the points $p_1, \ldots, p_d, p_0$ and the hyperplanes $h_1, \ldots, h_d, h_{d+1}$ yields an upper triangular nonnegative sign pattern of order $d+1$ with all diagonal entries positive. Since such an upper triangular sign pattern is a submatrix of $\mathcal{A}$ up to permutation equivalence, it follows that $\mathrm{mr}(\mathcal{A}) \geqslant d+1$. Since we also have $\mathrm{mr}(\mathcal{A}) \leqslant d+1$, we conclude that $\mathrm{mr}(\mathcal{A}) = d+1$. This completes the proof. □

We point out that the preceding result resolves an open problem posed in [15], as this result implies the existence of a fooling-set submatrix (which includes a nonsingular triangular matrix as a special case, see [15]) of order $d+1$ of the nonnegative sign pattern (or the $(0,1)$-matrix) determined by the convex polytope K of dimension d, while the previously known lower bound for the order of a largest fooling-set submatrix is $\sqrt{d}$.

Note that in general, up to permutation equivalence, a nonnegative sign pattern $\mathcal{A}$ with minimum rank r may not have a triangular submatrix of order r with all diagonal entries positive. For instance, the sign pattern $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ has minimum rank 3, but it is not permutationally equivalent to an upper triangular matrix.

In view of Theorems 2.7 and 2.9 and the fact that for each $d \geqslant 4$ there are d-polytopes that are not rationally realizable (see [28]), the following result is immediate.

Theorem 2.10. *For each integer $r \geqslant 5$, there exists a nonnegative sign pattern $\mathcal{A}$ with $\operatorname{mr}(\mathcal{A}) = r$ and $\operatorname{mr}_{\mathbb{Q}}(\mathcal{A}) > r$.*

3. Integer realization of the minimum rank

As shown in Section 2, for each nonnegative sign pattern $\mathcal{A}$ with minimum rank at most 4, its minimum rank can be realized rationally, and hence, there exist integer matrices in $Q(\mathcal{A})$ realizing the minimum rank. We now give upper bounds on the entries of some integer matrices in $Q(\mathcal{A})$ realizing the minimum rank.

General (not necessarily nonnegative) sign patterns with minimum rank 2 are characterized in [22]. The condensed nonnegative sign patterns $\mathcal{A}$ with $\operatorname{mr}(\mathcal{A}) = 2$ are quite simple. By considering the corresponding point-hyperplane configuration in $\mathbb{R}^1$ (in which a hyperplane is also a point), we see that each row and each column contains at most one zero entry. Suppose that $\mathcal{A}$ has three or more zero entries, then up to permutation equivalence, $\mathcal{A}$ contains the sign pattern $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ as a submatrix, which has minimum rank 3, a contradiction. Thus $\mathcal{A}$ contains one or two zero entries and no two zero entries can occur in the same row or the same column. It follows that up to permutation equivalence, $\mathcal{A}$ contains $\begin{bmatrix} 0 & + \\ + & + \end{bmatrix}$ or $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ as a submatrix, and all other entries are $+$. Consequently, $\mathcal{A}$ has two or three rows and two or three columns, and its minimum rank is achieved by an integer matrix in $Q(\mathcal{A})$ with entries from the set $\{0, 1, 2\}$. Thus we arrive at the following result.

Theorem 3.1. *Let $\mathcal{A}$ be any nonnegative sign pattern such that $\operatorname{mr}(\mathcal{A}) = 2$. Then its condensed sign pattern $\mathcal{A}_{\mathrm{c}}$ has at most three rows (columns) and contains at least one and at most two zero entries, with no two zero entries on the same row or column, and there is an integer matrix in $Q(\mathcal{A})$ with entries from the set $\{0, 1, 2\}$ that achieves the minimum rank of $\mathcal{A}$.*

In order to achieve the minimum rank of a condensed nonnegative sign pattern of size $m \times n$ with minimum rank 3 by an integer matrix, by following the steps of the proof of Theorem 2.4, it suffices to construct a 2-polytope (a convex polygonal region) in $\mathbb{R}^2$ with m integral vertices whose coordinates are even integers and the slopes of whose non-horizontal edges are odd integers, so that the midpoint of each edge is also an integral point and there is an integral interior point of the 2-polytope.

Lemma 3.2. *Let m be a positive integer with $m \geqslant 3$ and let $t = \lceil m/4 \rceil$. Then there is a 2-polytope K in $\mathbb{R}^2$ with $4t \geqslant m$ integral vertices with even coordinates such that the absolute values of the x-coordinates of the vertices of K are at most $2t$, the absolute values of the y-coordinates of the vertices of K are at most $2t^2$, and the slopes of the edges of K are odd integers with absolute value at most $2t-1$. Further, the y-intercepts of the extensions of the edges of K are at most $4t^2$ in absolute value, and at each vertex of K, there is a line with even slope that intersects K at exactly one point and has the y-intercept with absolute value at most $4t^2$.*

P r o o f. Consider the graphs of $y = f_1(x) = \frac{1}{2}x^2 - 2t^2$ and $y = f_2(x) = 2t^2 - \frac{1}{2}x^2$ over the interval $[-2t, 2t]$. These curves meet at the points $(\pm 2t, 0)$ and form the boundary of a convex set in $\mathbb{R}^2$ symmetric about the x-axis and the y-axis. Let K be the 2-polytope whose vertices are the following $4t$ points on the two curves above: $(2k, \pm(2t^2 - 2k^2))$, $k = -t, -t+1, \ldots, t-1, t$. Obviously, the coordinates of all the vertices of K are even integers; the absolute values of the x-coordinates of the vertices of K are at most $2t$; and the absolute values of the y-coordinates of the vertices of K are at most $2t^2$. Note that the slopes of the edges of K are odd integers with absolute values at most $2t-1$. Observe that the absolute values of the y-intercepts of the extensions of the edges of K are integers that increase as the edges are further away from the y-axis. Thus the largest y-intercept absolute value of the edge extensions of K is $2t(2t-1)$. Further, at the vertices with x-coordinate $\pm 2t$, the line with slope $2t$ is a supporting line of K with y-intercept $\pm 4t^2$ whose intersection with K is a vertex. For every vertex whose x-coordinate has absolute value less than $2t$, its incident edges of K have odd integer slopes that differ by 2, so there is a supporting line of K through such a vertex such that its intersection with K is just a point, its slope is an even integer, and its y-intercept has absolute value less than $4t^2$; in fact, this supporting line is the tangent line to one of the two curves given above. $\square$

Theorem 3.3. *Let $\mathcal{A}$ be a condensed nonnegative sign pattern of size $m \times n$ with $\operatorname{mr}(\mathcal{A}) = 3$. Let $t = \lceil m/4 \rceil$. Then there is an integer matrix $B = [b_{ij}] \in Q(\mathcal{A})$ with entries bounded above by $10t^2$ that achieves the minimum rank of $\mathcal{A}$.*

P r o o f. Let $C = \{p_1, \ldots, p_m\} \cup \{l_1, \ldots, l_n\}$ be a point-hyperplane configuration corresponding to $\mathcal{A}$. Suppose that the 2-polytope $\widehat{K} = \operatorname{conv}(\{p_1, \ldots, p_m\})$ has s vertices. If necessary, we may delete an odd number of vertices from the top (deleting the vertices with the largest y-coordinates first) of K and delete the bottom vertex of K, to obtain a 2-polytope K' that has the remaining s vertices of K as its vertices. Clearly, either the top (bottom) vertex of K is retained in K' or K' has two top (bottom) vertices with the same y-coordinate. Note that K' is symmetric about the y-axis. As in the proof of Theorem 2.4, we may replace $\widehat{K}$ by the integral 2-polytope K' and apply the preceding lemma to construct an equivalent integral point-line configuration $C' = \{p'_1, \ldots, p'_m\} \cup \{l'_1, \ldots, l'_n\}$ (all of whose points have integral coordinates and all of its lines have integer slopes and integer y-intercepts). As far as the points of C' are concerned, this construction is possible since each vertex of K' as well as the midpoint of each edge of K' is an integral point (as the coordinates of the vertices of K' are even integers) and the midpoint of the intersection of the y-axis with K' is an integral point in the interior of K'. The line $y = -2t^2 - 1$ is an integral line below K', and Lemma 3.2 ensures that the needed integral lines that support K' are available, and all such lines have integer slopes with absolute values at most $2t$ and have y-intercepts with absolute values at most $4t^2$. As in the proof of Theorem 2.4, the integral point-line configuration C' gives rise to integer matrices

$$U = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ \vdots & \vdots & \vdots \\ 1 & a_m & b_m \end{bmatrix}, \quad \text{and } V = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \\ 1 & 1 & \ldots & 1 \end{bmatrix},$$

where (a_i, b_i) are the coordinates of p'_i, and c_j and d_j are the negatives of the y-intercept and the slope of l_j, respectively. By Lemma 3.2, we have $|a_i| \leqslant 2t$, $|b_i| \leqslant 2t^2$, $|c_j| \leqslant 4t^2$, and $|d_j| \leqslant 2t$ for all i and j. It follows that the (i,j)-entry of the matrix $B' = [b'_{ij}] = UV$ satisfies $|b'_{ij}| = |c_j + a_i d_j + b_i| \leqslant |c_j| + |a_i||d_j| + |b_i| \leqslant 4t^2 + 2t2t + 2t^2 = 10t^2$. Since multiplying certain columns of B_1 by -1 yields a nonnegative integer matrix $B \in Q(\mathcal{A})$, the desired conclusion follows. $\square$

It is known [26] that the combinatorial type of every 3-polytope with m vertices can be realized in an integer grid of width $O(2^{7.55m})$. We use this result to derive an upper bound for the entries of some integer matrix that achieves the minimum rank of a nonnegative sign pattern with minimum rank 4.

Theorem 3.4. *Let $\mathcal{A}$ be a condensed nonnegative sign pattern of size $m \times n$ with $\operatorname{mr}(\mathcal{A}) = 4$. Then there is an integer matrix $B \in Q(\mathcal{A})$ with entries at most $O(m2^{22.65m})$ that achieves the minimum rank of $\mathcal{A}$.*

P r o o f. Let $C = P \cup H = \{p_1, \dots, p_m\} \cup \{h_1, \dots, h_n\}$ be a point-hyperplane configuration corresponding to $\mathcal{A}$ in $\mathbb{R}^3$. Then the 3-polytope $K = \operatorname{conv}(\{p_1, \dots, p_m\})$ has at most m vertices. By [26], the combinatorial type of K can be achieved by an integral 3-polytope K' (with all the vertices being integral points) in an integer grid of width $O(2^{7.55m})$ in the first orthant. Expanding K' 12 times if necessary, we may assume that the coordinates of each vertex of K' are multiples of 12. We follow the procedure in the proof of Theorem 2.6 to construct an integral point-hyperplane configuration $C' = P' \cup H'$ equivalent to C (where all points in P' have integer coordinates and all the hyperplanes in H' are given by linear equations with integer coefficients). Each point in P' is either a vertex of K' or a point in the relative interior of a face of K'. Since the coordinates of the vertices are multiples of 12, the midpoint of each edge of K' is an integer point, the center of mass of any triangle whose vertices are some vertices of a facet of K' is an integral point, and the interior of K' contains the integral center of mass of a tetrahedron whose vertices are four noncoplanar vertices of K'. Thus all the points in P' can be constructed in the same integer grid.

We now construct integral hyperplanes in H'. Of course, a hyperplane is an integral hyperplane provided it passes through an integral point and it has an integral normal vector. The intersection of each hyperplane in H' with K' is either empty or is a face of K' of dimension at most 2. An integral hyperplane not intersecting K' is given by $z = 0$. A hyperplane containing a facet of K' contains two consecutive edges of the facet and a normal vector of the hyperplane is given by the cross product of the two integer vectors obtained by treating the two consecutive edges as vectors (we may orient the edges of the facet in counterclockwise fashion when viewed from the outside so that the resulting normal vector is pointing outward). It follows that each coordinate of an integral normal vector of each facet is at most $O(2^{2\cdot 7.55m}) = O(2^{15.10m})$. Since an equation of a hyperplane in $\mathbb{R}^3$ passing through the point (x_0, y_0, z_0) and having a normal vector (b, c, d) is given by $-(bx_0 + cy_0 + dz_0) + bx + cy + dz = 0$, the hyperplane has an equation of the form $a + bx + cy + dz = 0$, where the coefficients are integers and $|a| \leqslant O(2^{3\cdot 7.55m}) = O(2^{22.65m})$, and $|b|, |c|, |d| \leqslant O(2^{15.10m})$.

A hyperplane whose intersection with K' is an edge of K' may be constructed with its normal vector being the sum of the two outward integral normal vectors of the two facets containing this edge, and hence, an equation with integer coefficients of such a hyperplane satisfies the same conditions as above.

Finally, a hyperplane whose intersection with K' is a vertex of K' may be constructed with its normal vector being the sum of the outward integral normal vectors of the at most $m-1$ facets containing this vertex. Hence, the hyperplane has an equation of the form $a + bx + cy + dz = 0$, where the coefficients are integers and $|a| \leqslant O(m2^{3\cdot 7.55m}) = O(m2^{22.65m})$, and $|b|, |c|, |d| \leqslant O(m2^{15.10m})$.

The configuration C' gives rise to two integer matrices

$$U = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_m & z_m \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ c_1 & \dots & c_n \\ d_1 & \dots & d_n \end{bmatrix},$$

where (x_i, y_i, z_i) are the integer coordinates of the point p'_i in P', and $a_j + b_j x + c_j y + d_j z = 0$ is an equation with integer coefficients of the hyperplane h'_j in H'. As shown above, $|x_i|, |y_i|, |z_i| \leqslant O(2^{7.55m})$, $|a_j| \leqslant O(m2^{3\cdot 7.55m}) = O(m2^{22.65m})$, and $|b_j|, |c_j|, |d_j| \leqslant O(m2^{15.10m})$. It follows that the entries of the integer matrix $B_1 = [b'_{ij}] = UV$ of rank 4 satisfy

$$|b'_{ij}| = |a_j + b_j x_i + c_j y_i + d_j z_i| \leqslant 4\,O(m2^{22.65m}) = O(m2^{22.65m}).$$

Since multiplying certain columns of B_1 by -1 yields a nonnegative integer matrix $B \in Q(\mathcal{A})$, the desired conclusion follows. $\square$

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Authors' addresses: Wei Fang, School of Instrument and Electronics, North University of China, Taiyuan, Shanxi 030051, China, e-mail: `441125541@qq.com`; Wei Gao, Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia 30302, USA, e-mail: `wgao2@gsu.edu`; Yubin Gao, Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China, e-mail: `ybgao.nuc@gmail.com`; Fei Gong, Guangming Jing (corresponding author), Zhongshan Li, Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia 30302, USA, e-mail: `gfmichelle@hotmail.com`, `gjing1@gsu.edu`, `zli@gsu.edu`; Yanling Shao, Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China, e-mail: `ylshao@nuc.edu.cn`; Lihua Zhang, Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia 30302, USA, e-mail: `jsahuatian@hotmail.com`.