Czechoslovak Mathematical Journal

Harm Bart; Torsten Ehrhardt; Bernd Silbermann

Rank decomposition in zero pattern matrix algebras

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 3, 987–1005

Persistent URL: http://dml.cz/dmlcz/145884

Czechoslovak Mathematical Journal, 66 (141) (2016), 987–1005

RANK DECOMPOSITION IN ZERO PATTERN MATRIX ALGEBRAS

Harm Bart, Rotterdam, Torsten Ehrhardt, Santa Cruz,
Bernd Silbermann, Chemnitz

(Received February 29, 2016)

In thankful memory of Miroslav Fiedler

Abstract. For a block upper triangular matrix, a necessary and sufficient condition has been given to let it be the sum of block upper rectangular matrices satisfying certain rank constraints; see H. Bart, A. P. M. Wagelmans (2000). The proof involves elements from integer programming and employs Farkas' lemma. The algebra of block upper triangular matrices can be viewed as a matrix algebra determined by a pattern of zeros. The present note is concerned with the question whether the decomposition result referred to above can be extended to other zero pattern matrix algebras. It is shown that such a generalization does indeed hold for certain digraphs determining the pattern of zeros. The digraphs in question can be characterized in terms of forests, i.e., disjoint unions of rooted trees.

Keywords: block upper triangularity; additive decomposition; rank constraints; zero pattern matrix algebra; preorder; partial order; Hasse diagram; rooted tree; out-tree; in-tree

MSC 2010: 15A21, 05C05, 05C50

1. Introduction

Let m and n be positive integers. The linear space of all $m \times n$ complex matrices will be denoted by $\mathbb{C}^{m\times n}$. In the case $m = n$, we have here the unital algebra of the $n \times n$ complex matrices. Its unit element is I_n, the $n \times n$ identity matrix.

Let $n_1, \ldots, n_p$ be positive integers, and suppose

$$n = n_1 + \ldots + n_p. \tag{1.1}$$

Given a matrix $A \in \mathbb{C}^{n\times n}$, we can partition A, writing it as a block matrix

$$A = [A_{k,l}]_{k,l=1}^{p}, \quad A_{k,l} \in \mathbb{C}^{n_k\times n_l},\ k, l = 1, \ldots, p. \tag{1.2}$$

We say that A is *block upper triangular* with respect to the additive decomposition (1.1) if $A_{k,l} = 0$ whenever $k > l$. Of course this block form depends on the additive decomposition in question. In the following, explicit references to (1.1) in connection with partitioning into blocks will sometimes be suppressed, the situation being without ambiguity given the context.

The set of all $n \times n$ matrices that are block upper triangular with respect to (1.1) will be denoted by $\mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$. Note that $\mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$ is a Banach subalgebra of $\mathbb{C}^{n\times n}$. For $p = 1$ and (hence) $n_1 = n$, this Banach subalgebra of $\mathbb{C}^{n\times n}$ coincides with $\mathbb{C}^{n\times n}$; for $p = n$ and (hence) $n_1 = \ldots = n_p = 1$, it is equal to the Banach subalgebra of $\mathbb{C}^{n\times n}$ consisting of all upper triangular matrices.

Returning to the general situation, let A be as in (1.2). For $r = 1, \dots, p$, we call A *block upper rectangular of type* r if $A_{k,l} = 0 \in \mathbb{C}^{n_k \times n_l}$ whenever $k > r$ or $l < r$. Clearly this implies that $A \in \mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$, so block upper rectangularity implies block upper triangularity.

Let A, represented as in expression (1.2), be a block upper triangular matrix. By a *simple block submatrix* of A we mean a submatrix of A of the form

$$A[s,t] = [A_{k,l}]_{k,l=s}^{t} = \begin{bmatrix} A_{s,s} & \dots & A_{s,t-1} & A_{s,t} \\ 0 & & & A_{s+1,t} \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & A_{t,t} \end{bmatrix}.$$

Here s and t are from the set $\{1, \dots, p\}$.

The following result is taken from [5].

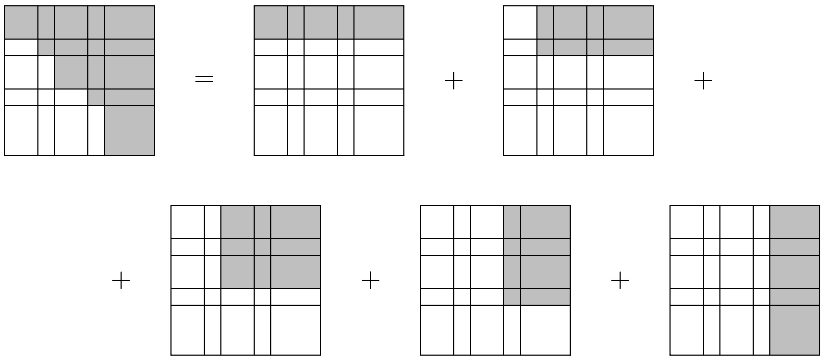
Theorem 1.1 (Rank decomposition theorem). *Let $A \in \mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$, and let $r_1, \dots, r_p$ be nonnegative integers. Write*

$$A = [A_{k,l}]_{k,l=1}^{p}, \quad A_{k,l} \in \mathbb{C}^{n_k \times n_l}, \ k, l = 1, \dots, p.$$

Then the following two statements are equivalent:

(1) *Matrix A admits a decomposition $A = A_1 + \ldots + A_p$ where for all integers $l \in \{1, \dots, p\}$, matrix A_l is block upper rectangular of type l and has rank not exceeding r_l.*
(2) *For each pair of integers s, t satisfying $1 \leqslant s \leqslant t \leqslant p$, the rank of $A[s,t]$ does not exceed $r_s + \ldots + r_t$.*

The figure below illustrates the decomposition in (2) for the case $p = 5$. The non-shaded blocks consist of zeros only. For a concrete numerical illustration, see Example 3.5 in [5].



Without the rank constraints imposed in (2), the decomposition of a block upper triangular matrix into a sum of block upper rectangular matrices is a triviality. In contrast, the proof of the rank decomposition theorem, as given in [5], involves elements from integer programming and uses Farkas' lemma (cf., [8], [11], [12]).

The algebra $\mathbb{C}_{\text{upper}}^{n_1,\ldots,n_p}$ of block upper triangular matrices can be viewed as a matrix algebra determined by a pattern of zeros. Thus one may ask: Can the rank decomposition theorem be extended to other zero pattern matrix algebras? More specifically, is a generalization possible to matrix algebras determined by a pattern of zeros corresponding to a preorder, i.e., a transitive and reflexive relation (or, if one prefers, a digraph) on $\{1, \ldots, n\}$? A negative answer would of course mean that there are situations where it is impossible to choose the decomposition in such a way that the summands are in the special zero pattern algebra in question. Such a situation has not been discovered yet. What we do have, are positive answers under certain conditions on the given preorder that allow for a large class of examples and that also came up in [1]. They are obtained in Section 3 and contained in Theorem 3.2. The proof of the theorem not only uses the rank decomposition theorem stated above but also employs specifics of its proof as given in [5]. This means that it again relies on elements from integer programming and Farkas' lemma. The preorders featuring in Theorem 3.2 can be characterized in terms of rooted trees. This is done in Section 4. The necessary preliminaries are presented in Section 2. Further Section 5 contains some concluding remarks.

Among those are comments on the connection of the material presented here and other work by the authors on sums of idempotents and logarithmic residues. Roughly speaking, a logarithmic residue is a contour integral of an analytic vector-valued function. The paper [3] deals with logarithmic residues in matrix algebras. In the block upper triangular case, it characterizes those as sums of idempotents. Instrumental here is a characterization both of logarithmic residues and sums of idempotents in terms of certain rank/trace conditions. It is at this point that the rank decomposition theorem serves as an essential tool. In [2], similar characterizations have been obtained for several other matrix algebras determined by a pattern of zeros. The complete picture is, however, not yet clear. The present paper fits in an attempt to further clarify the subject.

2. Preliminaries

2.1. Digraphs. The term digraph is used as an abbreviation for directed graph. So, formally, a *digraph* is a pair $(G, \mathcal{G})$, where G is a set (here always finite) and $\mathcal{G}$ is a relation on G, i.e., a subset of $G \times G$. This state of affairs will also be articulated by saying that $\mathcal{G}$ is a digraph with *ground set* G. The elements of the ground set are sometimes called the *nodes* of the digraph. If $\mathcal{G}$ is a digraph and k, l are nodes of $\mathcal{G}$, the notation $k \to_{\mathcal{G}} l$ is an alternative for $(k, l) \in \mathcal{G}$. So it means that there is an *edge* (directed) from k to l. In the same vein, $k \not\to_{\mathcal{G}} l$ signals that $(k, l) \notin \mathcal{G}$, i.e., there is no edge from k to l.

2.2. Zero pattern algebras. Let $\mathcal{G}$ be a digraph on N, where N stands for the set $\{1, \ldots, n\}$. By $\mathbb{C}^{n\times n}[\mathcal{G}]$ we denote the set of all $n \times n$ complex matrices $A = [a_{k,l}]_{k,l=1}^{n}$ for which $a_{k,l} = 0$ whenever $k \not\to_{\mathcal{G}} l$. Obviously $\mathbb{C}^{n\times n}[\mathcal{G}]$ is a linear subspace of $\mathbb{C}^{n\times n}$. We are interested in the situation where $\mathbb{C}^{n\times n}[\mathcal{G}]$ is a subalgebra of $\mathbb{C}^{n\times n}$. The following theorem (the proof of which is straightforward) is a special case of a result in [7].

Theorem 2.1. *Let $\mathcal{G}$ be a digraph with ground set $N = \{1, \ldots, n\}$. Then $\mathbb{C}^{n\times n}[\mathcal{G}]$ is a subalgebra of $\mathbb{C}^{n\times n}$ if and only if $\mathcal{G}$ is transitive.*

For our purposes, we want $\mathbb{C}^{n\times n}[\mathcal{G}]$ to contain as unit element the $n \times n$ identity matrix I_n. This is the case if and only if $\mathcal{G}$ is reflexive. A digraph that is both transitive and reflexive is called a preorder. A subalgebra $\mathcal{A}$ of $\mathbb{C}^{n\times n}$ is said to be a *zero pattern matrix algebra* if there exists a (uniquely determined) preorder $\mathcal{G}$ on N (short for: with ground set N) such that $\mathcal{A} = \mathbb{C}^{n\times n}[\mathcal{G}]$. For background material see [10].

2.3. Total preorders. As before, let $\mathcal{G}$ be a preorder with ground set $N = \{1, \ldots, n\}$. Suppose $\mathcal{G}$ is *total*, that is $k \to_{\mathcal{G}} l$ or $l \to_{\mathcal{G}} k$ for any pair k, l of nodes in N. If $\mathcal{G}$ is antisymmetric as well, i.e., if $\mathcal{G}$ is a linear order, then $\mathcal{G}$ is permutation similar to the standard linear order on N. This means that $k \to_{\mathcal{G}} l$ if and only if $k \leqslant l$. The zero pattern algebra $\mathbb{C}^{n\times n}[\mathcal{G}]$ is then simply the algebra of $n \times n$ upper triangular matrices. Permutation similarity comes down, of course, to a renumbering of the nodes.

In the general, possibly non-antisymmetric, situation we have to settle for something less, but to a certain extent the upper triangular structure in retained. This can be seen by looking at the condensation of the total preorder (obtained by identifying nodes k and l whenever there is an edge $k \to l$ from k to l as well as vice versa). This condensation is then a linear order to which what is said in the previous paragraph applies. Hence, modulo a permutation similarity, $\mathcal{G}$ has *block upper triangular form*. By this we mean that there exist positive integers $n_1, \ldots, n_p$, called the *block sizes*, such that $n_1 + \ldots + n_p = n$ and

$$\mathcal{G} = \bigcup_{s,t=1,\ldots,p;\, s \leqslant t} N_s \times N_t, \tag{2.1}$$

where

$$N_r = \{n_1 + \ldots + n_{r-1} + 1, \ldots, n_1 + \ldots + n_r\}, \quad r = 1, \ldots, p. \tag{2.2}$$

In this situation, $\mathbb{C}^{n\times n}[\mathcal{G}]$ is the algebra of block upper triangular $n \times n$ matrices with block dimensions $n_1, \ldots, n_p$, i.e., the algebra $\mathbb{C}^{n_1,\ldots,n_p}_{\text{upper}}$. The paper [5], which provided the inspiration for the present note (see Section 1), is concerned with this algebra. The block sizes mentioned above are uniquely determined by $\mathcal{G}$.

2.4. Preorders of block upper triangular type. Now let us drop the totality requirement which was imposed in Subsection 2.3, and return to the situation where $\mathcal{G}$ is just a preorder. Then its condensation is a partial order. Such a partial order always has a linear extension. (This is easy to see in the situation considered here where the underlying set is finite. For completeness we mention that, assuming the Axiom of Choice, the result is even true when this set is infinite; cf. [13]). It follows that $\mathcal{G}$ is permutation similar to a preorder of what we will call *block upper triangular type*. Thus, modulo a permutation, the situation is as in the previous subsection, with the understanding that, instead of being equal to the right hand side of (2.1), the preorder $\mathcal{G}$ is now contained in it while, in addition,

$$\begin{aligned}
&N_s \times N_s \subset \mathcal{G}, \quad s = 1, \ldots, p;\\
&(N_s \times N_t) \cap \mathcal{G} = \emptyset \quad \text{or} \quad (N_s \times N_t) \subset \mathcal{G}, \quad s, t = 1, \ldots, p;\\
&(N_r \times N_s) \subset \mathcal{G} \quad \text{and} \quad (N_s \times N_t) \subset \mathcal{G} \Rightarrow (N_r \times N_t) \subset \mathcal{G}, \quad r, s, t = 1, \ldots, p.
\end{aligned}$$

Clearly, if $\mathcal{G}$ has block upper triangular form with block sizes $n_1, \ldots, n_p$, then $\mathcal{G}$ is of block upper triangular type with (the same) block sizes $n_1, \ldots, n_p$.

An illustrative example of a preorder of block upper triangular type (not having block upper triangular form) is the preorder with ground set $\{1, \ldots, 12\}$ given by

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} &
\begin{pmatrix}
* & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & * \\
* & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & * \\
0 & 0 & * & * & * & * & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{pmatrix}
\end{array}, \tag{2.3}
$$

having as its condensation the partial order

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} &
\begin{pmatrix}
\star & \mathbf{0} & \star & \mathbf{0} & \star \\
\mathbf{0} & \star & \mathbf{0} & \star & \star \\
\mathbf{0} & \mathbf{0} & \star & \mathbf{0} & \star \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \star & \star \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \star
\end{pmatrix}
\end{array}.
$$

on $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, which is compatible with the standard linear order there.

Returning to the general situation, we formulate a question which comes up naturally: Does an extension of the rank decomposition theorem (Theorem 1.1) hold for zero pattern algebras $\mathbb{C}^{n \times n}[\mathcal{G}]$ with $\mathcal{G}$ a preorder of generalized block upper triangular type? We do not (yet) have a full answer concerning this issue. What has been obtained are positive answers under certain conditions on the given preorder that allow for a large class of examples and that also came up in [1]. The positive results in question will be presented a little later (Section 3). The additional restrictions are of a type also encountered in [1]. We will recall them in the next section.

3. Rank decomposition

We begin with some preparations.

Let n be a positive integer, and let M be an $n \times n$ matrix. Replacing certain columns of M by zero columns and leaving the other columns of M unchanged, we can transform the matrix M, without changing its rank, into a matrix whose nonzero columns are linearly independent. A specific procedure for doing this is as follows. Write $M = [c_1 \dots c_n]$ with $c_1, \dots, c_n \in \mathbb{C}^n$, where $\mathbb{C}^n$ stands for the linear space of all complex vectors of length n. So c_j is the j-th column of M. For $l = 1, \dots, n$, we leave the l-th column of M unchanged whenever c_l is not a linear combination of the columns $c_1, \dots, c_{l-1}$ preceding c_l; otherwise we replace c_l by a zero column. Here, of course, a vector is a linear combination of an empty collection of columns if and only if it is the zero vector. In this way, the case $l = 1$ is covered without ambiguity.

The procedure described above will be referred to as the *left to right column reduction* of M; the unambiguously defined resulting matrix—written M_{red}—will be called the *left to right column reduced form* of M. Note that the first column of M and that of M_{red} are always identical. Observe also that in the transformation process from M to M_{red}, zero columns and zero rows of M are unchanged. Anticipating on what we shall encounter in the proof of Proposition 3.1 below, we mention here already that the left to right column reduction M_{red} of M can be obtained from M by multiplying M from the right by an invertible upper triangular $n \times n$ matrix having ones on its diagonal. Henceforth matrices of that type will be called *monic.*

In the reminder of this section, n is a positive integer and $\mathcal{G}$ will be a preorder on $N = \{1, \dots, n\}$ of block upper triangular type with block sizes $n_1, \dots, n_p$ (adding up to n). Thus $\mathbb{C}^{n\times n}[\mathcal{G}]$ is a subalgebra of the algebra $\mathbb{C}^{n_1,\dots,n_p}_{\mathrm{upper}}$ of block upper triangular $n \times n$ matrices with block dimensions $n_1, \dots, n_p$.

We call $\mathcal{G}$ *out-ultra transitive* when

$$k \to_{\mathcal{G}} l \text{ and } k \to_{\mathcal{G}} m \;\Rightarrow\; l \to_{\mathcal{G}} m \text{ or } m \to_{\mathcal{G}} l.$$

Likewise, $\mathcal{G}$ is said to be *in-ultra transitive* if

$$k \to_{\mathcal{G}} m \text{ and } l \to_{\mathcal{G}} m \;\Rightarrow\; k \to_{\mathcal{G}} l \text{ or } l \to_{\mathcal{G}} k.$$

The preorder (2.3) featuring in Subsection 2.4 is out-ultra transitive but not in-ultra transitive.

The following result is of an auxiliary nature and crucial for what comes later on.

Proposition 3.1. *Let $\mathcal{G}$ be a preorder with ground set $N = \{1, \dots, n\}$, where n is a positive integer, and suppose $\mathcal{G}$ is of block upper triangular type with block sizes*

$n_1, \dots, n_p$. Let $A \in \mathbb{C}^{n\times n}[\mathcal{G}]$, and write A in partitioned form

$$A = \left[A_{k,l}\right]_{k,l=1}^{p}, \quad A_{k,l} \in \mathbb{C}^{n_k\times n_l}, \quad k,l = 1,\dots,p.$$

Then the following statements are true:

(1) *If $\mathcal{G}$ is out-ultra transitive, then there exists a monic $n\times n$ matrix $R \in \mathbb{C}^{n\times n}[\mathcal{G}]$ such that for each pair of integers s,t satisfying $1 \leqslant s \leqslant t \leqslant p$, the nonzero columns in the simple block submatrix $(AR)[s,t]$ of AR are linearly independent.*

(2) *If $\mathcal{G}$ is in-ultra transitive, then there exists a monic $n\times n$ matrix $L \in \mathbb{C}^{n\times n}[\mathcal{G}]$ such that for each pair of integers s,t satisfying $1 \leqslant s \leqslant t \leqslant p$, the nonzero rows in the simple block submatrix $(LA)[s,t]$ of LA are linearly independent.*

In view of certain results obtained in [1], the question arises whether or not the matrices AR and LA in (1) and (2), respectively, are canonical forms in the sense that they are uniquely determined by A. This is not the case. Counterexamples are easy to find.

Proof. We restrict ourselves to proving (1). If we would be merely interested in finding a monic matrix R in the algebra $\mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$, we could simply refer to Proposition 2.1 in [5]. In the present context, however, the thrust of the statement lies in the claim that R can be taken from the algebra $R \in \mathbb{C}^{n\times n}[\mathcal{G}]$ which is generally strictly contained in $\mathbb{C}_{\text{upper}}^{n_1,\dots,n_p}$. This makes it necessary to adapt the argument given in [5].

Suppose (1) is satisfied. The proof goes by induction with the number of blocks p as the parameter. First consider the situation where $p = 1$. Then $n = n_1$ and $\mathcal{G} = N \times N$. Thus $\mathbb{C}^{n\times n}[\mathcal{G}]$ is simply the full matrix algebra $\mathbb{C}^{n\times n}$, and things are covered by the preparatory material of the beginning of the present section (take $M = A = A[1,1]$).

Next take $p \geqslant 2$, put $n_0 = n - n_1$, and consider the matrix $A_0 = A[2,p]$. Let $\mathcal{G}_0$ be the digraph on $\{1,\dots,n_0\}$ given by

$$k \to_{\mathcal{G}_0} l \;\Leftrightarrow\; k + n_1 \to_{\mathcal{G}} l + n_1.$$

Then $\mathcal{G}_0$ is preorder on $\{1,\dots,n_0\}$ of block upper triangular type with block sizes $n_2,\dots,n_p$. Also, the matrix A_0 is (more precisely, can be identified with) a member of the algebra $\mathbb{C}^{n_0\times n_0}[\mathcal{G}_0]$. Finally, along with $\mathcal{G}$, the preorder $\mathcal{G}_0$ is out-ultra transitive.

By induction hypothesis we may now assume that there exists a monic matrix R_0 in $\mathbb{C}^{n_0\times n_0}[\mathcal{G}_0]$ such that for each pair of integers s,t with $2 \leqslant s \leqslant t \leqslant p$, the nonzero columns in the simple block submatrix $(A_0R_0)[s,t]$ of A_0R_0 are linearly independent. So, writing $B = A_0R_0$ and

$$B = \left[B_{k,l}\right]_{k,l=2}^{p}, \quad B_{k,l} \in \mathbb{C}^{n_k\times n_l}, \quad k,l = 2,\dots,p,$$

we have that $B_{k,l} = 0$ whenever $2 \leqslant l < k \leqslant p$, and the nonzero columns in the matrices $\left[B_{k,l}\right]_{k,l=s}^{t}$, $2 \leqslant s \leqslant t \leqslant p$, are linearly independent.

Introduce $\hat{R}_0 = I_{n_1} \oplus R_0$ and $\hat{A} = A\hat{R}_0$, where $\oplus$ signals the operation of taking the matrix direct sum. Then $\hat{R}_0$ is a monic $n \times n$ matrix in $\mathbb{C}^{n\times n}[\mathcal{G}]$ and, consequently, $\hat{A}$ is in $\mathbb{C}^{n\times n}[\mathcal{G}]$ too. Note that $\hat{A}[2,p] = A_0R_0 = B$. Thus for each pair of integers s,t satisfying $2 \leqslant s \leqslant t \leqslant p$, the nonzero columns in the simple block submatrix $\hat{A}[s,t]$ of $\hat{A}$ are linearly independent.

Now apply left to right column reduction to $\hat{A}$, resulting in $\hat{A}_{\rm red}$. From the proof of Proposition 2.1 in [5] we know that $\hat{A}_{\rm red}$ has the desired properties, i.e., in each simple block submatrix of $\hat{A}_{\rm red}$, the nonzero columns are linearly independent. Recall that $\hat{A}_{\rm red}$ can be obtained from $\hat{A}$ by multiplying it from the right by a monic $n \times n$ matrix $\hat{R}$. But then $\hat{A}_{\rm red} = A\hat{R}$ with $R = \hat{R}_0\hat{R}$ monic. As $\hat{R}_0$ is already in the algebra $\mathbb{C}^{n\times n}[\mathcal{G}]$, it remains to show that the matrix $\hat{R}$ can be chosen in such a way that it belongs to $\mathbb{C}^{n\times n}[\mathcal{G}]$ as well. This is what we turn to now. In doing so, it is helpful to keep in mind that the left to right reduction $\hat{A}_{\rm red}$ of $\hat{A}$ is obtained from $\hat{A}$ by replacing certain columns by zero columns and leaving the other columns unchanged.

Let $\hat{c}_1, \ldots, \hat{c}_n \in \mathbb{C}^n$ be the columns of $\hat{A}$. Also, for $l = 1, \ldots, n$, let c_l be the vector $\hat{c}_l$ with the first n_1 coordinates removed. Then $c_1, \ldots, c_{n_1}$ vanish, and $c_{n_1+1}, \ldots, c_n$ are the columns of $\hat{A}[2,p] = B$. Denote by Γ the set of all l's in $\{n_1+1, \ldots, n\}$ such that the column c_l of B does not vanish. Recall that the nonzero columns of B are linearly independent. Using the block upper triangularity of $\hat{A}$, one verifies easily that the columns $\hat{c}_l$, $l \in \Gamma$, are among the nonzero columns of $\hat{A}_{\rm red}$. In other words, Γ is a subset of $\hat{\Gamma}$, where the latter is the collection of all $l \in \{1, \ldots, n\}$ for which $\hat{c}_l$ is a nonzero column of $\hat{A}_{\rm red}$. Evidently, for $l \in \hat{\Gamma} \setminus \Gamma$, the last $n - n_1$ coordinates of $\hat{c}_l$ are zero, and so there must be a nonzero entry among the first n_1 coordinates of $\hat{c}_l$.

Take $l \in \{1, \ldots, n\} \setminus \hat{\Gamma}$, and assume $\hat{c}_l$ is a nonzero column of $\hat{A}$. Then $\hat{c}_l$ is a linear combination of the columns $\hat{c}_j$ with $j \in \hat{\Gamma}$, $j < l$. So there exist scalars $\alpha_j^{(l)}$, $j \in \hat{\Gamma}$, $j < l$, such that

$$\hat{c}_l = \sum_{j\in\hat{\Gamma},\ j<l} \alpha_j^{(l)} \hat{c}_j.$$

The columns $\hat{c}_j$, $j \in \hat{\Gamma}$, $j < l$ are linearly independent. Hence the scalars $\alpha_j^{(l)}$, $j \in \hat{\Gamma}$, $j < l$, are uniquely determined. Using that c_t vanishes for $t \in \{1, \ldots, n\} \setminus \Gamma$, the above identity yields

$$0 = c_l = \sum_{j\in\hat{\Gamma},\ j<l} \alpha_j^{(l)} c_j = \sum_{j\in\Gamma,\ j<l} \alpha_j^{(l)} c_j.$$

As the vectors $c_j, j \in \Gamma$, are linearly independent, it follows that

$$\alpha_j^{(l)} = 0, \quad j \in \Gamma,\ j < l,$$

so that

$$\hat{c}_l = \sum_{j \in \hat{\Gamma} \setminus \Gamma,\ j < l} \alpha_j^{(l)} \hat{c}_j.$$

Let $u_1, \ldots, u_n$ stand for the standard unit vectors in $\mathbb{C}^{n \times n}$. Define the $n \times n$ matrix $\hat{R}$ by stipulating that its l-th column $\hat{r}_l$ is given by

$$\hat{r}_l = u_l - \sum_{j \in \hat{\Gamma} \setminus \Gamma,\ j < l} \alpha_j^{(l)} u_j \tag{3.1}$$

when $l \in \{1, \ldots, n\} \setminus \hat{\Gamma}$, $\hat{c}_l \neq 0$, and that $\hat{r}_l = u_l$ otherwise. Clearly $\hat{R}$ is monic. For $l \in \{1, \ldots, n\} \setminus \hat{\Gamma}$, $\hat{c}_l \neq 0$, the l-th column of $\hat{A}\hat{R}$ is

$$\hat{c}_l - \sum_{j \in \hat{\Gamma} \setminus \Gamma,\ j < l} \alpha_j^{(l)} \hat{c}_j,$$

hence it vanishes. For $l \in \hat{\Gamma}$, the l-th column of $\hat{A}\hat{R}$ is $\hat{c}_l$. For $l \in \{1, \ldots, n\}$, $\hat{c}_l = 0$, the l-th column of $\hat{A}\hat{R}$ is again $\hat{c}_l$, which now comes down to the l-th column of $\hat{A}\hat{R}$ being zero. The conclusion is that $\hat{A}\hat{R} = \hat{A}_{\mathrm{red}}$.

We finish the argument by making clear that the matrix $\hat{R}$ as introduced in the previous paragraph belongs to $\mathbb{C}^{n \times n}[\mathcal{G}]$. It is here that the out-ultra transitivity of $\mathcal{G}$ will play a crucial role.

Suppose $\hat{r}_{k,l}$ (with $k < l$) is a nonzero off diagonal entry of $\hat{R}$. We need to show that $k \to_{\mathcal{G}} l$. As $\hat{r}_{k,l} \neq 0$, necessarily $l \in \{1, \ldots, n\} \setminus \hat{\Gamma}$ and $\hat{c}_l \neq 0$, and we have the expression (3.1) for the l-th column $\hat{r}_l$ of $\hat{R}$. This gives, employing the Kronecker delta notation,

$$\hat{r}_{k,l} = u_k^{\top} \hat{r}_l = u_k^{\top} \left(u_l - \sum_{j \in \hat{\Gamma} \setminus \Gamma,\ j < l} \alpha_j^{(l)} u_j \right) = - \sum_{j \in \hat{\Gamma} \setminus \Gamma,\ j < l} \alpha_j^{(l)} \delta_{k,j},$$

and it follows that $k \in \hat{\Gamma} \setminus \Gamma$. For $s = 1, \ldots, p$, put

$$N_s = \{n_1 + \ldots + n_{s-1} + 1, \ldots, n_1 + \ldots + n_s\},$$

and choose r such that $l \in N_r$. If $k \in N_r$ too, then $(k, l) \in N_r \times N_r$. The latter set is contained in $\mathcal{G}$, so in this case the desired result $k \to_{\mathcal{G}} l$ is trivial. Assume therefore that $k \notin N_r$. Then $k \in N_s$ for some $s < r$. As $k \in \hat{\Gamma} \setminus \Gamma$, we know that among the first n_1 coordinates of $\hat{c}_k$ there is at least one nonzero entry. Thus $\mathcal{G} \cap (N_1 \times N_s)$ is nonempty, and we conclude that $N_1 \times N_s$ is contained in $\mathcal{G}$. Similarly $N_1 \times N_r$ is contained in $\mathcal{G}$. Here is the argument. The column $\hat{c}_l$ does not vanish and l is not in $\hat{\Gamma}$. The latter implies that $l \notin \Gamma$. So $c_l = 0$. But then among the first n_1 coordinates of $\hat{c}_l$ there is at least one nonzero entry.

We now have $N_1 \times N_s \subset \mathcal{G}$ and $N_1 \times N_r \subset \mathcal{G}$. By out-ultra transitivity, it ensues that $N_s \times N_r \subset \mathcal{G}$ or $N_r \times N_s \subset \mathcal{G}$. The latter possibility is ruled out because of block triangularity and the inequality $s < r$. Hence $N_s \times N_r \subset \mathcal{G}$, and the desired result is immediate from $(k, l) \in N_s \times N_r$. $\square$

We now come to our main goal in this paper, a generalization of the rank decomposition theorem (Theorem 1.1).

Theorem 3.2. *Let $\mathcal{G}$ be a preorder with ground set $N = \{1, \ldots, n\}$, where n is a positive integer, and suppose $\mathcal{G}$ is of block upper triangular type with block sizes $n_1, \ldots, n_p$. Further assume that either $\mathcal{G}$ is out-ultra transitive or $\mathcal{G}$ is in-ultra transitive. Let $A \in \mathbb{C}^{n \times n}[\mathcal{G}]$, write A in partitioned form*

$$A = [A_{k,l}]_{k,l=1}^{p}, \quad A_{k,l} \in \mathbb{C}^{n_k \times n_l}, \ k, l = 1, \ldots, p,$$

and let $r_1, \ldots, r_p$ be nonnegative integers. Then the following two statements are equivalent:

(1) *Matrix A admits a decomposition $A = A_1 + \ldots + A_p$ where for all integers $l \in \{1, \ldots, p\}$, matrix A_l belongs to $\mathbb{C}^{n \times n}[\mathcal{G}]$, is block upper rectangular of type l, and has rank not exceeding r_l.*
(2) *For each pair of integers s, t satisfying $1 \leqslant s \leqslant t \leqslant p$, the rank of $A[s, t]$ does not exceed $r_s + \ldots + r_t$.*

Two remarks are in order.

First, from the proof as given below, we see that we have something extra. Namely, that the summands $A_1, \ldots, A_p$ can be chosen in such a way that their images are contained in the image of A, and their null spaces contain the null space of A. This fact, not noted in [5], was known to us when we wrote [2], and used there in Subsection 4.3.

Second, compared to Theorem 1.1, the rank decomposition theorem taken from [5], the essential new feature lies in the requirement that the terms $A_1, \ldots, A_p$ belong to an algebra which is generally strictly contained in $\mathbb{C}_{\text{upper}}^{n_1, \ldots, n_p}$ (cf., the first paragraph in the proof of Proposition 3.1). So (here too) it is necessary to modify certain parts of the argument given in [5].

P r o o f. From Theorem 1.1 we know that (1) implies (2). So it needs to be shown here that (1) can be derived from (2). We only give the argument under the assumption that $\mathcal{G}$ is out-ultra transitive.

Suppose (2) is satisfied, and let $R \in \mathbb{C}^{n\times n}[\mathcal{G}] = \mathbb{C}^{n\times n}[\mathcal{G}]$ be as in (1) of Proposition 3.1. Thus, writing $B = AR$, for each pair of integers s, t satisfying $1 \leqslant s \leqslant t \leqslant p$, the nonzero columns in the simple block submatrix $B[s,t]$ of B are linearly independent. Note that for each pair of integers s, t satisfying $1 \leqslant s \leqslant t \leqslant p$, the rank of $B[s,t]$ does not exceed $r_s + \ldots + r_t$. In the present situation, the latter means that the number of nonzero columns in the simple block submatrix $B[s,t]$ of B does not exceed $r_s + \ldots + r_t$.

The rank decomposition theorem now guarantees the existence of a decomposition $B = B_1 + \ldots + B_p$ where for all integers $l \in \{1, \ldots, p\}$, matrix B_l belongs to $\mathbb{C}_{\text{upper}}^{n_1,\ldots,n_p}$, is block upper rectangular of type l, and has rank not exceeding r_l. This, however, is not enough for what we want. Indeed, it has to be made clear that matrices $B_1, \ldots, B_p$ can be chosen in such a way that they belong to the algebra $\mathbb{C}^{n\times n}[\mathcal{G}]$. As was already announced in the introduction, this makes it necessary to go into the specifics of the proof of the rank decomposition theorem as given in [5].

The proof in question involves a distribution of the columns of B over the summands $B_1, \ldots, B_p$ keeping their position (column number) intact. [This involves getting a nonnegative integer solution of a totally unimodular system of linear equations. For such a solution to exist it is necessary and sufficient that there is a nonnegative real solution. The circumstances under which this is the case are described by Farkas' lemma (cf., [8]). Here they come down to the (rank) condition on the simple block submatrices of B.] Thus the summands $B_1, \ldots, B_p$ are obtained from B by replacing certain (nonzero) columns from B by zero columns. As $B = AR$ belongs to the algebra $\mathbb{C}^{n\times n}[\mathcal{G}]$, it is immediate that so do the matrices $B_1, \ldots, B_p$. It is also clear that the images of $B_1, \ldots, B_p$ are contained in that of B. Further, using the linear independence of the columns of $B = B[1,p]$, one checks without difficulty that the null space of B is contained in the null spaces of $B_1, \ldots, B_p$.

For $l = 1, \ldots, p$, put $A_l = B_l R^{-1}$. As is well known, matrix subalgebras of $\mathbb{C}^{n\times n}$ containing I_n are inverse closed. Hence R^{-1} belongs to $\mathbb{C}^{n\times n}[\mathcal{G}]$, and the same is true for the products $A_l = B_l R^{-1}$. Obviously

$$A = BR^{-1} = (B_1 + \ldots + B_p)R^{-1} = A_1 + \ldots + A_p,$$

and this is a decomposition of A with summands from $\mathbb{C}^{n\times n}[\mathcal{G}]$. Now take $l \in \{1, \ldots, p\}$. As B_l is block upper rectangular of type l and R^{-1} is monic (along with R), we have that A_l is block upper rectangular of type l. Also the ranks of A_l and B_l coincide. So, along with that of B_l, the rank of A_l does not exceed r_l. It is also evident that the image of A_l, being the same as that of B_l, is contained in the image of B which in turn is the same as that of A. Finally, for the null spaces we have the following: $\operatorname{Ker} A = R[\operatorname{Ker} B] \subset R[\operatorname{Ker} B_l] = \operatorname{Ker} A_l$, as desired. □

4. Ultra-transitivity: characterizations in terms of rooted trees

In this section we will characterize out-ultra transitivity and in-ultra transitivity by making a connection with the notion of a rooted tree. It is convenient to consider digraphs on arbitrary ground sets, not necessarily equal to $\{1, \ldots, n\}$ for some positive integer n. The definitions of out-ultra transitivity and in-ultra transitivity carry over to this formally more general situation in a straightforward fashion.

First we present some well-known facts and results from graph theory (cf., [9]). We adapt terminology and notation a bit to suit our purposes. Let $\mathcal{G}$ be a digraph with ground set G, that is, $\mathcal{G}$ is a subset of $G \times G$. Let $g \in G$ be a node of $\mathcal{G}$. The *in-degree* of a node g is the number of edges $k \to g$, $k \in G$. Similarly, the *out-degree* of g is the number of edges $g \to l$, $l \in G$.

Let $\mathcal{T}$ be a directed tree with a set of nodes T. Recall that a tree is a connected, cycle-free, loopless graph. We call $\mathcal{T}$ an *out-tree* with *root* r if all edges are directed away from the (unique) node $r \in T$. It is a well-known fact (part of folklore) that this is equivalent with $\mathcal{T}$ being a connected, loopless digraph such that r has in-degree 0 and all other nodes have in-degree 1. Note that an out-tree is just a rooted tree in the usual sense: the tree 'grows' *out* of the unique root. An *in-tree* is obtained from an out-tree by reversing the direction of all edges. Now all edges are directed towards the root r, being the (unique) node of $\mathcal{T}$ with out-degree 0, whereas all other nodes now have out-degree 1. We call the disjoint union of out-trees an *out-forest*. Similarly, the disjoint union of in-trees is named *in-forest*.

A partial order $\mathcal{G}$ with (finite) ground set G can be considered as a digraph. Let k and m be two distinct nodes of $\mathcal{G}$. We say that $l \in G$ is *strictly between* k and m if l is distinct from both k and m while $k \to_{\mathcal{G}} l$ and $l \to_{\mathcal{G}} m$ are edges in $\mathcal{G}$. The node m is said to *cover* k if $k \to_{\mathcal{G}} m$ is an edge in $\mathcal{G}$ and there is no node strictly between k and m. The *Hasse diagram* $\mathcal{G}_{\downarrow}$ of $\mathcal{G}$ is the digraph with ground set G such that $k \to_{\mathcal{G}_{\downarrow}} m$ is an edge in $\mathcal{G}_{\downarrow}$ if and only if m covers k in $\mathcal{G}$. Note that $\mathcal{G}_{\downarrow}$ can be obtained from $\mathcal{G}$ by deleting all reflexivity loops $g \to_{\mathcal{G}} g$ and all edges $k \to_{\mathcal{G}} m$ for which there exists $l \in G$, $l \neq k, m$ with $k \to_{\mathcal{G}} l \to_{\mathcal{G}} m$. Conversely, let $\mathcal{H}$ be the Hasse diagram of a partial order with ground set G. The *reflexive transitive closure* $\mathcal{H}^{\uparrow}$ of $\mathcal{H}$ is obtained from $\mathcal{H}$ by adding all loops (g, g) and all edges $(k, m) \in G \times G$ such that there is a directed path from k to l in $\mathcal{H}$.

The following two facts are well known in the theory of partially ordered sets, and belong to folklore (cf., [6]). Let $\mathcal{G}$ be a partial order on G. Then we have $(\mathcal{G}_{\downarrow})^{\uparrow} = \mathcal{G}$. Let $\mathcal{H}$ be a Hasse diagram of a partial order. Then we have $(\mathcal{H}^{\uparrow})_{\downarrow} = \mathcal{H}$. Loosely speaking, this means that there is a one-to-one correspondence between the partial orders on a set G and the Hasse diagrams on G.

We now come to the second main result of this paper.

Theorem 4.1. *Let $\mathcal{G}$ be a partial order on the finite set G. Then the following statements are equivalent:*

(1) *the partial order $\mathcal{G}$ is in-ultra transitive;*
(2) *the Hasse diagram $\mathcal{G}_\downarrow$ of $\mathcal{G}$ is an out-tree forest on G;*
(3) *there exists an out-tree forest $\mathcal{F}$ on G such that the reflexive transitivity closure $\mathcal{F}^\uparrow$ of $\mathcal{F}$ is identical to $\mathcal{G}$.*

Also, if $\mathcal{G}$ is in-ultra transitive, then the Hasse diagram $\mathcal{G}_\downarrow$ of $\mathcal{G}$ is the unique out-tree forest $\mathcal{F}$ on G such that $\mathcal{F}^\uparrow = \mathcal{G}$.

Thus there is a one-to-one correspondence between in-ultra transitive partial orders and out-tree forests. There is also a one-to-one correspondence between out-ultra transitive partial orders and in-tree forests. Indeed, Theorem 4.1 remains true if in-ultra transitivity is replaced by out-ultra transitivity and out-tree forests by in-tree forests.

P r o o f. Suppose (1) is satisfied, so the given partial order $\mathcal{G}$ is in-ultra transitive. Take any component $\mathcal{T}$ of the Hasse diagram $\mathcal{G}_\downarrow$ of $\mathcal{G}$. We have to prove that $\mathcal{T}$ is an out-tree. Assume that there exists a node m of $\mathcal{T}$ with in-degree at least two, say (k, m) and (l, m) are two distinct edges. Then these are also edges in $\mathcal{G}$. By in-ultra transitivity of $\mathcal{G}$, there exists an edge between k and l in $\mathcal{G}$, say the edge (k, l). But now l is between k and m in $\mathcal{G}$, which means that (k, m) is not an edge in $\mathcal{T}$. This yields a contradiction. So all nodes in $\mathcal{T}$ have in-degree at most 1.

Let T be the set of nodes of $\mathcal{T}$. Then the partial order $\mathcal{G}$ induces a partial order on T that has $\mathcal{T}$ as Hasse diagram. Take a minimal node r in this partial order on T. If r covered a node s in $\mathcal{G}$, then s would necessarily be a node of the component $\mathcal{T}$, contradicting the minimality of r in $\mathcal{T}$. Hence r does not cover any node in $\mathcal{G}$, so it has in-degree 0. Since $\mathcal{T}$ is connected, it has at least $|T| - 1$ edges. Here $|T|$ stands for the cardinality of T. It follows that the sum of the in-degrees in $\mathcal{T}$ is at least $|T| - 1$. On the other hand, the in-degree of each node of $\mathcal{T}$ is at most 1, and r has in-degree 0. So the sum of the in-degrees is at most $|T| - 1$. We conclude that all nodes except r have in-degree exactly 1. Now folklore tells us that $\mathcal{T}$ is a rooted tree.

This proves that (1) implies (2). If (2) holds, then $\mathcal{F} = \mathcal{G}_\downarrow$ is an out-tree forest such that $\mathcal{F}^\uparrow = \mathcal{G}$, and we have (3).

Next assume (3) is satisfied. We want to establish (1). Suppose $k \to_{\mathcal{G}} m$ and $l \to m$ are two distinct edges in $\mathcal{G}$ pointing towards m. Then, in $\mathcal{F}$, there exists a directed path P_k from k to m as well as a directed path P_l from l to m. Assume that k is not on P_l and l is not on P_k. Going from k to m on P_k, let s be the first common node of P_k and P_l. Then, in $\mathcal{F}$, node s has an incoming edge on P_k as well as on P_l. Necessarily these two edges must be distinct. But this would mean that s has in-degree at least 2 in $\mathcal{F}$. Since this is impossible, we infer that k is on P_l or l is on P_k. In the first case there is an edge $l \to_{\mathcal{G}} k$, in the second there is $k \to_{\mathcal{F}} l$. Thus we have shown that $\mathcal{G}$ is in-ultra transitive.

We have now proved the first part of the theorem, i.e., the equivalence of (1), (2) and (3). It remains to deal with the second part. Suppose $\mathcal{G}$ is in-ultra transitive. Then the Hasse diagram $\mathcal{G}_\downarrow$ of $\mathcal{G}$ is an out-tree forest on G and $(\mathcal{G}_\downarrow)^\uparrow = \mathcal{G}$. Let $\mathcal{F}$ be another out-tree forest with reflexive transitivity closure identical to $\mathcal{G}$. So $\mathcal{G} = \mathcal{F}^\uparrow$ and, consequently, $\mathcal{G}_\downarrow = (\mathcal{F}^\uparrow)_\downarrow$. The latter, however, coincides with $\mathcal{F}$, and we arrive at $\mathcal{F} = \mathcal{G}_\downarrow$ so that uniqueness is guaranteed. $\square$

In Theorem 4.1, the given digraph $\mathcal{G}$ is a partial order. We now relax on this and consider the situation where $\mathcal{G}$ is a preorder (so we drop the antisymmetry requirement). Recall that the condensation of $\mathcal{G}$ is a partial order, so there is a Hasse diagram assigned to it.

Theorem 4.2. *Let $\mathcal{G}$ be a preorder on a finite ground set. Then $\mathcal{G}$ is in-ultra transitive if and only if the Hasse diagram of the condensation of $\mathcal{G}$ is an out-tree forest.*

Again this theorem remains true if in-ultra transitivity is replaced by out-ultra transitivity and out-tree forests by in-tree forests.

P r o o f. Without difficulty one verifies that the preorder $\mathcal{G}$ is in-ultra transitive if and only if so is its condensation. The desired result is now immediate from Theorem 4.1. $\square$

We close this section with an example illustrating Theorem 4.1. Let $\mathcal{G}$ be the partial order on the set $\mathcal{G} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ given by the scheme

$$\mathcal{G} = \begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{matrix} &
\begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & 0 & * & * & 0 & * & * & * & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 & * & 0 \\
* & * & 0 & * & 0 & 0 & * & 0 & 0 & 0 & * & 0 & *
\end{pmatrix}
\end{array}.$$

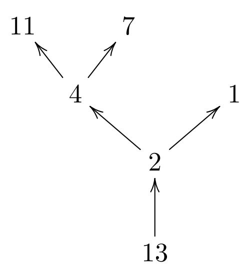
Then $\mathcal{G}$ is in-ultra transitive, and the Hasse diagram $\mathcal{G}_\downarrow$ of $\mathcal{G}$ is

$$\mathcal{G}_\downarrow = \begin{array}{cc} & \begin{array}{ccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ *&0&0&*&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&*&0&0&0 \\ 0&0&0&0&0&0&*&0&0&0&*&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&*&0&0&*&0&0&0&0&0&*&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&*&0&0&0&*&0&0&0&0 \\ 0&*&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \end{array}.$$

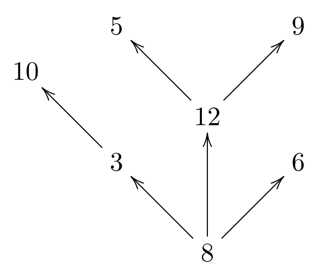
Write $G_1 = \{1, 2, 4, 7, 11, 13\}$ and $G_2 = \{3, 5, 6, 8, 9, 10, 12\}$. Then G_1, G_2 is a partition of G. For $j = 1, 2$, introduce $G_\downarrow(j) = G_\downarrow \cap (G_j \times G_j)$, i.e.,

$$G_\downarrow(1) = \begin{array}{cc} & \begin{array}{cccccc} 1&2&4&7&11&13 \end{array} \\ \begin{array}{c} 1\\2\\4\\7\\11\\13 \end{array} & \begin{pmatrix} 0&0&0&0&0&0 \\ *&0&*&0&0&0 \\ 0&0&0&*&*&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&*&0&0&0&0 \end{pmatrix} \end{array}, \quad G_\downarrow(2) = \begin{array}{cc} & \begin{array}{ccccccc} 3&5&6&8&9&10&12 \end{array} \\ \begin{array}{c} 3\\5\\6\\8\\9\\10\\12 \end{array} & \begin{pmatrix} 0&0&0&0&0&*&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ *&0&*&0&0&0&* \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&*&0&0&*&0&0 \end{pmatrix} \end{array}.$$

One easily verifies that $\mathcal{G}$ is the disjoint union of $G_\downarrow(1)$ and $G_\downarrow(2)$. Also these digraphs are out-trees with roots 2 and 13, respectively, a fact which immediately comes to attention from the following alternative way to depict $G_\downarrow(1)$ and $G_\downarrow(2)$:

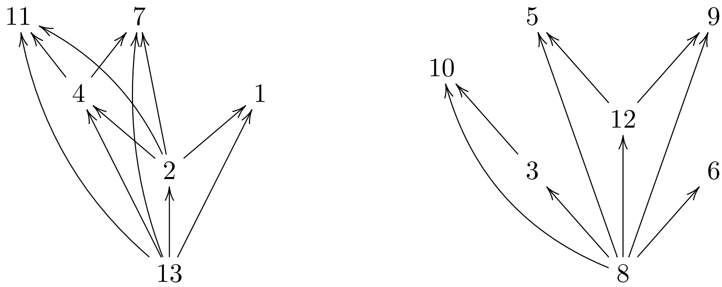


$G_\downarrow(1)$



$G_\downarrow(2)$

Thus $\mathcal{G}_{\downarrow}$ is an out-tree forest, in line with Theorem 4.1. Omitting the reflexivity loops, the digraph $\mathcal{G}$ we started with can be represented as:



These pictures are in line with Theorem 4.1.

5. Concluding remarks

5.1. Ultra closures. Let $\mathcal{G}$ be a preorder with ground set $N = \{1, \ldots, n\}$, where n is a positive integer, and suppose $\mathcal{G}$ is of block upper triangular type with block sizes $n_1, \ldots, n_p$. By $\mathcal{G}_{\mathrm{out}}$ we mean the smallest preorder containing $\mathcal{G}$ which is of block upper triangular type with (the given) block sizes $n_1, \ldots, n_p$, and having the additional property of being out-ultra transitive. The existence of this *out-ultra transitive closure* of $\mathcal{G}$ is easily established (see [1] for details). Its natural counterpart is the *in-ultra transitive closure* $\mathcal{G}_{\mathrm{in}}$ of $\mathcal{G}$. Theorem 3.2 now allows for a modification in which the ultra transitivity condition on the given preorder is dropped at the expense of accepting rank decompositions with summands in $\mathbb{C}^{n\times n}[\mathcal{G}_{\mathrm{out}}]$ or $\mathbb{C}^{n\times n}[\mathcal{G}_{\mathrm{in}}]$.

A caveat is in order here: the closures depend on the given block-structure. Thus, if the preorders $\mathcal{G}$ and $\mathcal{G}'$ are both of block upper triangular type, and $\mathcal{G}$ and $\mathcal{G}'$ are permutation similar, the closures of $\mathcal{G}$ and $\mathcal{G}'$ introduced above need not be permutation similar.

5.2. Logarithmic residues and sums of idempotents. As has been indicated at the end of the introduction, the ideas for the present paper came up in our study of logarithmic residues and sums of idempotents (cf., [4], [3] and [2], and the references given there). In line with this, we now formulate a concrete conjecture on in-ultra transitive preorders. Of course the conjecture has a counterpart where in-ultra transitivity is replaced by out-ultra transitivity.

Conjecture A. *Let $\mathcal{G}$ be a preorder on $\{1, \ldots, n\}$, where n is a positive integer. Suppose $\mathcal{G}$ is in-ultra transitive. Then, for $A \in \mathbb{C}^{n \times n}[\mathcal{G}]$ the following statements are equivalent:*

(i) *A is a sum of rank one idempotents in $\mathbb{C}^{n \times n}[\mathcal{G}]$;*
(ii) *A is a sum of idempotents in $\mathbb{C}^{n \times n}[\mathcal{G}]$;*
(iii) *A is a logarithmic residue in $\mathbb{C}^{n \times n}[\mathcal{G}]$;*
(iv) *A satisfies the $\mathcal{G}$-rank/trace conditions.*

For the meaning of (iv), we refer to [2], Section 2. In the case considered in Theorem 1.1, the statement in question comes down to a rank/trace condition on the simple block submatrices of the given matrix A. In the more general setting, these are replaced by submatrices of A corresponding to $\mathcal{G}$-convex subsets of $\{1, \ldots, n\}$. These are defined as follows: $J \subset \{1, \ldots, n\}$ is *$\mathcal{G}$-convex* if $k, m \in J$, $k \to_{\mathcal{G}} l \to_{\mathcal{G}} m$ implies $l \in J$.

In terms of [2], the conclusion in the above conjecture can be reformulated by saying that in-ultra transitive preorders are *rank/trace complete*. Disjoint unions of rank/trace complete preorders are rank/trace complete again (cf., Theorem 6.3 in [2]). This observation, combined with Theorem 4.1, gives that for the case when the underlying preorder actually is a partial order, the above conjecture comes down to this:

Conjecture B. *Let $\mathcal{G}$ be a partial order on $\{1, \ldots, n\}$, where n is a positive integer. Assume that the Hasse diagram $\mathcal{G}_{\downarrow}$ of $\mathcal{G}$ is an out-tree. Then $\mathcal{G}$ is rank/trace complete. Formulated otherwise, if $\mathcal{T}$ is an out-tree on $\{1, \ldots, n\}$, then the reflexive transitivity closure $\mathcal{T}^{\uparrow}$ of $\mathcal{T}$ is rank/trace complete.*

Acknowledgement. The authors thank Henry Martyn Mulder for suggestions that strongly influenced the presentation of the material in Section 4. They also thank the anonymous referee for constructive comments on the first version of the paper.

References

[1] *H. Bart, T. Ehrhardt, B. Silbermann*: Echelon type canonical forms in upper triangular matrix algebras. To appear in Oper. Theory, Adv. Appl.

[2] *H. Bart, T. Ehrhardt, B. Silbermann*: Sums of idempotents and logarithmic residues in zero pattern matrix algebras. Linear Algebra Appl. *498* (2016), 262–316.

[3] *H. Bart, T. Ehrhardt, B. Silbermann*: Sums of idempotents and logarithmic residues in matrix algebras. Operator Theory and Analysis. The M. A. Kaashoek Anniversary Volume (Bart, H. et al. eds.), Oper. Theory, Adv. Appl. *122* (2001), 139–168.

[4] *H. Bart, T. Ehrhardt, B. Silbermann*: Logarithmic residues in Banach algebras. Integral Equations Oper. Theory *19* (1994), 135–152.

[5] *H. Bart, A. P. M. Wagelmans*: An integer programming problem and rank decomposition of block upper triangular matrices. Linear Algebra Appl. *305* (2000), 107–129.
[6] *G. Birkhoff*: Lattice Theory. Colloquium Publications Vol. 25, American Mathematical Society, Providence, 1967.
[7] *R. L. Davis*: Algebras defined by patterns of zeros. J. Comb. Theory *9* (1970), 257–260.
[8] *S.-C. Fang, S. Puthenpura*: Linear Optimization and Extensions: Theory and Algorithms. Prentice-Hall, Englewood Cliffs, 1993.
[9] *F. Harary*: Graph Theory. Addison-Wesley Series in Mathematics, Reading, Mass., Addison-Wesley Publishing Company, 1969.
[10] *T. J. Laffey*: A structure theorem for some matrix algebras. Linear Algebra Appl. *162–164* (1992), 205–215.
[11] *C. H. Papadimitriou, K. Steiglitz*: Combinatorial Optimization: Algorithms and Complexity. Prentice-Hall, Englewood Cliffs, 1982.
[12] *A. Schrijver*: Theory of Linear and Integer Programming. Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Chichester, 1986.
[13] *E. Szpilrajn*: Sur l'extension de l'ordre partiel. Fundamenta Mathematicae *16* (1930), 386–389 (In French.); available at
`http://matwbn.icm.edu.pl/ksiazki/fm/fm16/fm16125.pdf`.

Authors' addresses: Harm Bart, Econometric Institute, Erasmus University Rotterdam, Burgemeester Oudlaan 50, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, e-mail: `bart@ese.eur.nl`; Torsten Ehrhardt, Mathematics Department, University of California, 1156 High St., Santa Cruz, CA-95064, California, USA, e-mail: `tehrhard@ucsc.edu`; Bernd Silbermann, Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Str. 39, 09107 Chemnitz, Germany, e-mail: `silbermn.toeplitz@googlemail.com`.