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ON WELL-POSEDNESS FOR PARAMETRIC VECTOR QUASIEQUILIBRIUM PROBLEMS WITH MOVING CONES

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Abstract. In this paper we consider weak and strong quasiequilibrium problems with moving cones in Hausdorff topological vector spaces. Sufficient conditions for well-posedness of these problems are established under relaxed continuity assumptions. All kinds of well-posedness are studied: (generalized) Hadamard well-posedness, (unique) well-posedness under perturbations. Many examples are provided to illustrate the essentialness of the imposed assumptions. As applications of the main results, sufficient conditions for lower and upper bounded equilibrium problems and elastic traffic network problems to be well-posed are derived.

Keywords: quasiequilibrium problem; lower bounded equilibrium problem; upper bounded equilibrium problem; network traffic problem; well-posedness; C-upper semicontinuity; C-lower semicontinuity

MSC 2010: 49K40, 90C31, 91B50

1. Introduction

The equilibrium problem was introduced by Blum and Oettli [16] in 1994. The mathematical formulation of the problem unifies various important problems related to optimization, namely, constrained minimization, variational inequality, complementarity problem, Nash equilibria, minimax problem, fixed-point and coincidence-point problems, traffic network problem, etc. Due to the important role of the problem, it has been intensively studied for all of the main topics, such as existence theory [10], [14], [18], [22], [25], [28], [31], [35], [38], [40], stability and sensitivity analysis theory [1], [3], [4], [5], [6], [13], [15], and solution methods [17], [32], [37], [42], etc.

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Well-posedness may be understood in two ways. The first concept is the well-posedness in the sense of Hadamard [27]. This property includes the existence, uniqueness and continuous dependence of the optimal solution and optimal value on the perturbed data of the problem. The second concept was introduced by Tikhonov in 1966 in [43], where the author proposed it as the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness under perturbations (termed also parametric well-posedness) is a generalized notion of those of Hadamard and Tikhonov. Recently, the study of generalized well-posed concepts was extended to mathematical programming [30], optimization problem (see e.g., [39], [47]–[49]), variational inequality [24], Nash equilibria [34]. Well-posedness for equilibrium problems is a theme of great importance and has received increasing attention of many researchers recently. The sufficient and necessary conditions and metric characterizations of some types of generalized well-posedness for the class of equilibrium problems were considered. For more details, we refer the reader to [7], [8], [9], [12], [23], [29], [33], and the references therein.

Optimization-related problems with moving cones play an important role in many practical situations. For example, when we want to choose a product from the market, of course, the characteristics of the chosen product have to satisfy our requirements, the so-called requirement domain. Naturally, the higher quality of the product is the better choice, and hence the requirement domain is a cone which is called the requirement cone. Since our requirement cones may be different from those of the other ones, the requirement cone will depend on some parameters.

Motivated and inspired by the above observations, in this paper we consider the weak and strong quasiequilibrium problems with moving cones. We introduce the concepts of moving cone semicontinuity and investigate their properties. Under the moving cone semicontinuity assumptions, we establish sufficient and/or necessary conditions for such problems to be (uniquely) well-posed under perturbations and (generalized) Hadamard well-posed. Moreover, since we consider the problems with moving cones, the main results in [7], [8], [9], [12], [23], [29], [33] cannot be applied to such problems. Therefore, our results are new and different from the existing ones in the literature.

The rest of the paper is organized as follows. Section 2 is devoted to problem statements and preliminary facts. The concepts of moving cone semicontinuity and their properties are also introduced and discussed in this section. We study, in Section 3, the necessary and/or sufficient conditions for the weak and strong vector quasiequilibrium problems with moving cones to be (uniquely) well-posed under perturbations and (generalized) Hadamard well-posed. The applications of the main results to some special cases of quasiequilibrium problems are presented in Section 4.

2. Preliminaries

Let X, Λ be two metric spaces, and Y a Hausdorff topological vector space. Let $K\colon X\times\Lambda \rightrightarrows X$ and $C\colon \Lambda \rightrightarrows Y$ be two set-valued mappings such that, for each $\lambda \in \Lambda$, $C(\lambda)$ is a closed, convex, pointed and solid cone. Let $e\colon \Lambda \to Y$ be a continuous mapping such that, for each $\lambda \in \Lambda$, $e(\lambda) \in \operatorname{int} C(\lambda)$. Let $f\colon X \times X \times \Lambda \to Y$ be a vector valued mapping. For each $\lambda \in \Lambda$, we consider the following parametric vector quasiequilibrium problems:

(WQEP$_\lambda$) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for each $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda)).$$

(SQEP$_\lambda$) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for each $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \lambda) \in C(\lambda).$$

Instead of writing $\{(\mathrm{WQEP}_\lambda)\colon\ \lambda \in \Lambda\}$ and $\{(\mathrm{SQEP}_\lambda)\colon\ \lambda \in \Lambda\}$ for the sets of such problems, we will simply write (WQEP) and (SQEP), respectively, in the sequel. For each $\lambda \in \Lambda$, the solution sets of (WQEP$_\lambda$) and (SQEP$_\lambda$) are denoted by $S^w(\lambda)$ and $S^s(\lambda)$, respectively, i.e.,

$$S^w(\lambda) = \{x \in K(x, \lambda)\colon\ f(x, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda)) \quad \forall\, y \in K(x, \lambda)\},$$
$$S^s(\lambda) = \{x \in K(x, \lambda)\colon\ f(x, y, \lambda) \in C(\lambda) \quad \forall\, y \in K(x, \lambda)\}.$$

Definition 2.1. Let Q be a set-valued mapping from X into Y.

(i) Q is said to be upper semicontinuous (usc, shortly) at x_0 if for any open superset U of $Q(x_0)$, there is a neighborhood N of x_0 such that $Q(N) \subset U$.

(ii) Q is said to be lower semicontinuous (lsc, shortly) at x_0 if for any open subset U of Y with $Q(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that for all $x \in N$, $Q(x) \cap U \neq \emptyset$.

The mapping Q is said to be continuous at x_0 if it is both usc and lsc at x_0. We say that Q satisfies a certain property on a subset $A \subset X$ if Q satisfies it at every points of A. If $A = X$ we omit "on X" in the statement.

Lemma 2.1 ([11]).

(i) *Q is usc at x_0, if for each superset U of $Q(x_0)$ and for every sequence $\{x_n\}$ in X converging to x_0, there is n_0 such that for all $n \geqslant n_0$, $Q(x_n) \subset U$.*

(ii) *Q is lsc at x_0 if, for each sequence $x_n \to x_0$ and $y_0 \in Q(x_0)$, there exists $y_n \in Q(x_n)$ such that $y_n \to y_0$.*

Definition 2.2. Let Q be a set-valued mapping from X into Y.

(i) Q is said to be Hausdorff upper semicontinuous (H-usc, shortly) at x_0 if, for each neighborhood B of the origin θ_Y in Y, there exists a neighborhood N of x_0 such that, $Q(x) \subset Q(x_0) + B$ for every $x \in N$.

(ii) Q is said to be Hausdorff lower semicontinuous (H-lsc, shortly) at x_0 if, for each neighborhood B of the origin θ_Y in Y, there exists a neighborhood N of x_0 such that, $Q(x_0) \subset Q(x) + B$ for every $x \in N$.

Q is said to be Hausdorff continuous at x_0 if it is both H-usc and H-lsc at x_0.

Picking up the main ideas from [18], we introduce the concepts of moving cone upper and lower semicontinuity as follows.

Definition 2.3. Let X, Λ be two topological spaces, Y a topological vector space, and $C\colon \Lambda \rightrightarrows Y$ a set-valued mapping with pointed solid convex cone values. Let $g\colon X \times \Lambda \to Y$ be a vector valued mapping.

(i) g is said to be C-upper semicontinuous (C-usc, for short) at (x_0, λ_0) if, for any neighborhood V of θ_Y in Y, there exists a neighborhood U of (x_0, λ_0) such that, for all $(x, \lambda) \in U$,

$$g(x, \lambda) \in g(x_0, \lambda_0) + V - C(\lambda_0).$$

(ii) g is said to be C-lower semicontinuous (C-lsc, for short) if $-g$ is C-upper semicontinuous.

The mapping g is said to be C-continuous at (x_0, λ_0) if it is both C-usc and C-lsc at (x_0, λ_0). We say that g satisfies a certain property on a subset $A \subset X \times \Lambda$ if g satisfies it at every point of A. If $A = X \times \Lambda$ we omit "on $X \times \Lambda$" in the statement.

Proposition 2.1. *Let X, Λ, C, g be as in Definition* 2.3. *The following conditions are equivalent to each other.*

(i) *g is C-upper semicontinuous.*

(ii) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $d \in \operatorname{int} C(\lambda_0)$, there is a neighborhood U of (x_0, λ_0) such that $g(x, \lambda) \in g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0)$, for all $(x, \lambda) \in U$.*

(iii) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $a \in Y$, $g^{-1}(a - \operatorname{int} C(\lambda_0))$ is open.*

Proof. First, we show that (i) and (ii) are equivalent to each other. Let $(x_0, \lambda_0) \in X \times \Lambda$ and $d \in \operatorname{int} C(\lambda_0)$. Let $V = d - \operatorname{int} C(\lambda_0)$. Then V is a neighborhood of θ_Y in Y. If (i) is satisfied, then there is a neighborhood U of (x_0, λ_0) such that

$$g(x, \lambda) \in g(x_0, \lambda_0) + V - C(\lambda_0) \quad \forall\, (x, \lambda) \in U.$$

Since $C(\lambda_0)$ is a convex cone, $-\operatorname{int} C(\lambda_0) - C(\lambda_0) = -\operatorname{int} C(\lambda_0)$, we conclude that

$$g(x, \lambda) \in g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U,$$

i.e., (ii) holds. Conversely, let $(x_0, \lambda_0) \in X \times \Lambda$ and let V be a neighborhood of θ_Y. Since $C(\lambda_0)$ is a pointed solid cone, there is $d \in \operatorname{int} C(\lambda_0)$ such that $d \in V$. If (ii) is satisfied, there exists a neighborhood U of (x_0, λ_0) such that

$$g(x, \lambda) \in g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U,$$

and hence, $g(x, \lambda) \in g(x_0, \lambda_0) + V - C(\lambda_0)$ for all $(x, \lambda) \in U$, i.e., (i) holds.

Next, we prove that (ii) and (iii) are equivalent to each other. Assume that, for each $(x_0, \lambda_0) \in X \times \Lambda$ and $a \in Y$, $g^{-1}(a - \operatorname{int} C(\lambda_0))$ is open. Let $d \in \operatorname{int} C(\lambda_0)$, we have $(x_0, \lambda_0) \in g^{-1}(g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0))$, which is open. Thus, there is a neighborhood U of (x_0, λ_0) such that $U \subset g^{-1}(g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0))$, i.e.,

$$g(x, \lambda) \in g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U.$$

Conversely, for each $a \in Y$ and $(x_0, \lambda_0) \in g^{-1}(a - \operatorname{int} C(\lambda_0))$, let $d = a - g(x_0, \lambda_0)$. Since $d \in \operatorname{int} C(\lambda_0)$, there is a neighborhood U of (x_0, λ_0) such that

$$g(x, \lambda) \in g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U,$$

i.e., $U \subset g^{-1}(g(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0))$. Hence, $g^{-1}(a - \operatorname{int} C(\lambda_0))$ is open. □

Next, we propose other properties of the C-upper semicontinuity.

Proposition 2.2. *Let X, Λ, C be as in Definition 2.3 and let $f, g\colon X \times \Lambda \to Y$ be vector valued mappings. If f and g are C-upper semicontinuous, then*
(i) *$f + g$ is C-upper semicontinuous;*
(ii) *kf is C-upper semicontinuous, for each $k \in (0, \infty)$.*

Proof. (i) For each $(x_0, \lambda_0) \in X \times \Lambda$ and $d \in \operatorname{int} C(\lambda_0)$, since f and g are C-usc, there are two neighborhoods U_1 and U_2 of (x_0, λ_0) such that

$$f(x, \lambda) \in f(x_0, \lambda_0) + \frac{1}{2}d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U_1,$$
$$g(x, \lambda) \in g(x_0, \lambda_0) + \frac{1}{2}d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U_2.$$

Therefore,

$$(f + g)(x, \lambda) \in (f + g)(x_0, \lambda_0) + d - \operatorname{int} C(\lambda_0) \quad \forall\, (x, \lambda) \in U = U_1 \cap U_2.$$

Applying Proposition 2.2, we conclude that $f + g$ is C-upper semicontinuous.

(ii) The C-upper semicontinuity of kf can be proved similarly. □

Passing to the C-lower semicontinuity, we also establish conclusions similar to those in Propositions 2.1 and 2.2.

Proposition 2.3. *Let X, Λ, C, g be as in Definition* 2.3. *The following conditions are equivalent to each other.*

(i) *g is C-lower semicontinuous.*

(ii) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $d \in \operatorname{int} C(\lambda_0)$, there is a neighborhood U of (x_0, λ_0) such that $g(x, \lambda) \in g(x_0, \lambda_0) - d + \operatorname{int} C(\lambda_0)$ for all $(x, \lambda) \in U$.*

(iii) *For each $(x_0, \lambda_0) \in X \times \Lambda$ and $a \in Y$, $g^{-1}(a + \operatorname{int} C(\lambda_0))$ is open.*

Proposition 2.4. *Let X, Λ, C, f, g be as in Proposition* 2.2. *If f and g are C-lower semicontinuous, then*

(i) *$f + g$ is C-lower semicontinuous;*

(ii) *kf is C-lower semicontinuous for each $k \in (0, \infty)$.*

If $X = \Lambda = Y = \mathbb{R}$, $C(\lambda) \equiv \mathbb{R}_+$, and $g(x, \lambda) \equiv g(x)$, then Propositions 2.1 and 2.3 imply that the C-upper (lower) semicontinuity of g reduces to the ordinary upper (lower, respectively) semicontinuity of such function.

Definition 2.4. Let $\lambda \in \Lambda$ and let $\{\lambda_n\} \subset \Lambda$ be a sequence converging to λ. A sequence $\{x_n\}, x_n \in K(x_n, \lambda_n)$, is said to be an approximating sequence for (WQEP$_\lambda$) or (SQEP$_\lambda$) corresponding to $\{\lambda_n\}$, if there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that, for each $n \in \mathbb{N}$ and $y \in K(x_n, \lambda_n)$,

$$\begin{aligned} & f(x_n, y, \lambda_n) + \varepsilon_n e(\lambda_n) \in (Y \setminus -\operatorname{int} C(\lambda_n)), \\ \text{or} \quad & f(x_n, y, \lambda_n) + \varepsilon_n e(\lambda_n) \in C(\lambda_n), \ \text{respectively.} \end{aligned}$$

Definition 2.5. The problem (WQEP) or (SQEP) is said to be well-posed under perturbations (well-posed, shortly), if for each $\lambda \in \Lambda$,

(i) (WQEP$_\lambda$) or (SQEP$_\lambda$) has solutions;

(ii) for any sequence $\{\lambda_n\} \subset \Lambda$ converging to λ, every approximating sequence $\{x_n\}$ for (WQEP$_\lambda$) or (SQEP$_\lambda$) corresponding to $\{\lambda_n\}$ has a subsequence converging to an element in $S^w(\lambda)$ or $S^s(\lambda)$, respectively.

The problem (WQEP) or (SQEP) is said to be uniquely well-posed under perturbations (uniquely well-posed, shortly) if for each $\lambda \in \Lambda$ the solution set $S^w(\lambda)$ or $S^s(\lambda)$ is a singleton, and every approximating sequence for (WQEP$_\lambda$) or (SQEP$_\lambda$) tends to the unique solution, respectively.

Definition 2.6. The problem (WQEP) or (SQEP) is said to be generalized Hadamard well-posed if, for each $\lambda \in \Lambda$,

(i) (WQEP$_\lambda$) or (SQEP$_\lambda$) has solutions;

(ii) for any sequence $\{\lambda_n\} \subset \Lambda$ converging to λ and $x_n \in S^w(\lambda_n)$ or $x_n \in S^s(\lambda_n)$, $\{x_n\}$ has a subsequence converging to some point of $S^w(\lambda)$ or $S^s(\lambda)$, respectively.

The problem (WQEP) or (SQEP) is said to be Hadamard well-posed if for each $\lambda \in \Lambda$ the solution set $S^w(\lambda)$ or $S^s(\lambda)$ is a singleton, and every $x_n \in S^w(\lambda_n)$ or $x_n \in S^s(\lambda_n)$, $\{x_n\}$ converges to the unique solution, respectively.

For each $\lambda \in \Lambda$, $\varepsilon \geqslant 0$, we denote the ε-solution sets of (WQEP$_\lambda$) and (SQEP$_\lambda$) by $\widetilde{S^w}(\lambda, \varepsilon)$ and $\widetilde{S^s}(\lambda, \varepsilon)$, respectively, defined as follows:

$$\widetilde{S^w}(\lambda, \varepsilon) = \{x \in K(x, \lambda) \colon\ f(x, y, \lambda) + \varepsilon e(\lambda) \in (Y \setminus -\operatorname{int} C(\lambda)) \quad \forall\, y \in K(x, \lambda)\},$$
$$\text{and } \widetilde{S^s}(\lambda, \varepsilon) = \{x \in K(x, \lambda) \colon\ f(x, y, \lambda) + \varepsilon e(\lambda) \in C(\lambda)) \quad \forall\, y \in K(x, \lambda)\}.$$

Lemma 2.2 ([9]). *Let $S\colon\ X \rightrightarrows Y$ be a set-valued mapping between two topological spaces. Then the following assertions hold.*

(i) *If $S(\bar{x})$ is compact, then S is usc at $\bar{x}$ if and only if for any sequence $\{x_n\}$ convergent to $\bar{x}$ and $y_n \in S(x_n)$, there is a subsequence $\{y_{n_k}\}$ convergent to some $y \in S(\bar{x})$.*

(ii) *If, in addition, $S(\bar{x}) = \{\overline{y}\}$ is a singleton, then the above limit point y must be $\overline{y}$ and the whole $\{y_n\}$ converges to $\overline{y}$.*

3. Main results

Since the existence theory of the equilibrium problems has been intensively studied, in this paper we only focus on the well-posedness of such problems, and hence we always assume that the solutions of the problems considered exist in a neighborhood of the reference point.

Theorem 3.1. *Assume that, for given $\lambda \in \Lambda$, the following assumptions hold:*

(i) *$f(\cdot, \cdot, \lambda)$ is C-upper semicontinuous;*

(ii) *$K(\cdot, \lambda)$ is continuous with compact values.*

Then the solution sets $S^w(\lambda)$ and $S^s(\lambda)$ are closed in X.

P r o o f. Since the techniques of the proofs are similar, we discuss only the closedness of $S^w(\lambda)$. Let an arbitrary sequence $\{x_n\} \subset S^w(\lambda)$ such that $x_n \to \bar{x} \in X$. We need to show that $\bar{x}$ belongs to $S^w(\lambda)$. Since x_n is a solution of (WQEP$_\lambda$) for each $n \in \mathbb{N}$, $x_n \in K(x_n, \lambda)$, and

$$f(x_n, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda)) \quad \forall\, y \in K(x_n, \lambda). \tag{3.1}$$

Since $K(\cdot, \lambda)$ is usc with compact values, $\bar{x} \in K(\bar{x}, \lambda)$. If $\bar{x} \notin S^w(\lambda)$, then there exists $y_0 \in K(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y_0, \lambda) \in -\operatorname{int} C(\lambda).$$

Due to the openness of $-\operatorname{int} C(\lambda)$, there is a neighborhood V of θ_Y such that $(f(\bar{x}, y_0, \lambda) + V) \subset -\operatorname{int} C(\lambda)$. Since $f(\cdot,\cdot,\lambda)$ is C-usc at $(\bar{x}, y_0)$, and $x_n \to \bar{x}$, there exist a neighborhood U of y_0 and $n_0 \in \mathbb{N}$ such that for all $y \in U$ and $n \geqslant n_0$ we have

$$f(x_n, y, \lambda) \in f(\bar{x}, y_0, \lambda) + V - C(\lambda) \subset -\operatorname{int} C(\lambda) - C(\lambda) = -\operatorname{int} C(\lambda). \tag{3.2}$$

Due to $y_0 \in K(\bar{x}, \lambda)$, $K(\bar{x}, \lambda) \cap U \neq \emptyset$. Since $K(\cdot, \lambda)$ is lsc at $\bar{x}$, there exists $n_1 \in \mathbb{N}$ such that

$$K(x_n, \lambda) \cap U \neq \emptyset \quad \forall\, n \geqslant n_1. \tag{3.3}$$

Choose up $n \geqslant \max\{n_0, n_1\}$. Combining (3.2) with (3.3), we infer the existence of $\overline{y} \in K(x_n, \lambda) \cap U$ such that

$$f(x_n, \overline{y}, \lambda) \in -\operatorname{int} C(\lambda),$$

which is in contradiction with (3.1). Therefore, $\bar{x} \in S^w(\lambda)$. □

Now we pass to the semicontinuity results of the approximating solution sets, which play an important role in the well-posedness for the corresponding problems.

Theorem 3.2. *Assume that X is compact, f is C-upper semicontinuous, and K is continuous with compact values. Then*

(a) *$\widetilde{S^w}$ is upper semicontinuous with compact values on $\Lambda \times \mathbb{R}_+$ if for given $\lambda \in \Lambda$ and an arbitrary sequence $\{\lambda_n\} \subset \Lambda$ converging to λ there exists λ_{n_0} such that $C(\lambda) \subset C(\lambda_{n_0})$.*

(b) *$\widetilde{S^s}$ is upper semicontinuous with compact values on $\Lambda \times \mathbb{R}_+$ if for given $\lambda \in \Lambda$ and an arbitrary sequence $\{\lambda_n\} \subset \Lambda$ converging to λ there exists λ_{n_0} such that $C(\lambda_{n_0}) \subset C(\lambda)$.*

Proof. As an example we present only the proof for (b). We first prove the upper semicontinuity with compact values of $S^s(\cdot) = \widetilde{S^s}(\cdot, 0)$ on Λ. For each $\lambda \in \Lambda$, Theorem 3.1 yields that $S^s(\lambda)$ is a closed subset of X, so $S^s(\lambda)$ is compact. We show that S^s is usc on Λ. Assume, to arrive at a contradiction, that there exists $\lambda \in \Lambda$ such that S^s is not usc at λ. Then there are an open superset V of $S^s(\lambda)$ and a sequence $\{\lambda_n\} \subset \Lambda$, $\lambda_n \to \lambda$ such that, for each $n \in \mathbb{N}$, there exists $x_n \in S^s(\lambda_n) \backslash V$. Since $x_n \in S^s(\lambda_n)$, $x_n \in K(x_n, \lambda_n)$ and

$$f(x_n, y, \lambda_n) \in C(\lambda_n) \quad \forall\, y \in K(x_n, \lambda_n).$$

By the compactness of X, we can assume that x_n converges to $\bar{x}$ for some $\bar{x} \in X$. Since K is usc with compact values, K is closed, and hence $\bar{x} \in K(\bar{x}, \lambda)$. Since $x_n \notin V$ for all n, $\bar{x}$ does not belong to $S^s(\lambda) \subset V$, i.e., there exists $y_0 \in K(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y_0, \lambda) \in (Y \setminus C(\lambda)),$$

and hence, there is a neighborhood B of θ_Y such that

$$(f(\bar{x}, y_0, \lambda) + B) \subset (Y \setminus C(\lambda)).$$

Since f is C-usc at $(\bar{x}, y_0, \lambda)$, there is a neighborhood U of y_0 and $n_0 \in \mathbb{N}$ such that for all $y \in U$ and $n \geqslant n_0$,

$$f(x_n, y, \lambda_n) \in f(\bar{x}, y_0, \lambda) + B - C(\lambda) \subset (Y \setminus C(\lambda)) - C(\lambda) \subset (Y \setminus C(\lambda)).$$

Since $y_0 \in K(\bar{x}, \lambda)$, we have $K(\bar{x}, \lambda) \cap U \neq \emptyset$. By the lower semicontinuity of K at $(\bar{x}, \lambda)$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$,

$$K(x_n, \lambda_n) \cap U \neq \emptyset.$$

Applying the assumption in (b), we obtain the existence of n_2, where $n_2 \geqslant \max\{n_0, n_1\}$, $C(\lambda_{n_2}) \subset C(\lambda)$, and hence

$$(Y \setminus C(\lambda)) \subset (Y \setminus C(\lambda_{n_2})).$$

Therefore, $f(x_{n_2}, y, \lambda_{n_2}) \notin C(\lambda_{n_2})$ for some $y \in K(x_{n_2}, \lambda_{n_2}) \cap U$, which is impossible as $x_{n_2} \in S^s(\lambda_{n_2})$. Hence, S^s is usc at λ.

Now we show that $\widetilde{S^s}$ is usc with compact values on $\Lambda \times \mathbb{R}_+$. Let $g\colon X \times X \times \mathbb{R}_+ \times \Lambda \to Y$ be defined by

$$g(x, y, \varepsilon, \lambda) = f(x, y, \lambda) + \varepsilon e(\lambda).$$

Since f is C-usc and e is continuous, Proposition 2.2 implies that g is C-usc. Using the techniques as in the proof for S^s, where f is replaced by g, we conclude that $\widetilde{S^s}$ is upper semicontinuous with compact values on $\Lambda \times \mathbb{R}_+$. $\square$

The essentialness of the assumptions imposed in (a) and (b) of Theorem 3.2 is shown by the following examples.

Example 3.1. Let $X = [0, \frac{1}{2}\pi]$; $\Lambda = [0, 2]$; $Y = \mathbb{R}^2$, $K(x, \lambda) = [\frac{1}{2}\lambda x, x]$, $e(\lambda) \equiv (1; 1)$, $f(x, y, \lambda) = (x + \cos x; \lambda - y)$,

$$C(\lambda) = \begin{cases} \mathbb{R}^2_+, & \text{if } \lambda \neq 1, \\ \mathbb{R} \times \mathbb{R}_+, & \text{if } \lambda = 1. \end{cases}$$

Then X is compact, f and K are continuous with compact values. Direct calculations give

$$S^w(\lambda) = \begin{cases} [0, \frac{1}{2}\pi], & \text{if } \lambda \neq 1, \\ [0, 1], & \text{if } \lambda = 1, \end{cases}$$

and

$$\widetilde{S^w}(\lambda, \varepsilon) = \begin{cases} [0, \frac{1}{2}\pi], & \text{if } \lambda \neq 1, \\ [0, \varepsilon + 1] \cap [0, \frac{1}{2}\pi], & \text{if } \lambda = 1. \end{cases}$$

It is obvious that S^w and $\widetilde{S^w}$ are not usc at 1 and $(1, 0)$, respectively. The reason is that the imposed assumption in (a) is violated.

Example 3.2. Let $X = [0, 2]$; $\Lambda = [0, 1]$; $Y = \mathbb{R}^2$, $K(x, \lambda) = [\lambda x, x]$, $e(\lambda) \equiv (1; 1)$, $f(x, y, \lambda) = (\mathrm{e}^{x+y}; \lambda - y)$,

$$C(\lambda) = \begin{cases} \mathbb{R}^2_+, & \text{if } \lambda = \frac{1}{2}, \\ \mathbb{R}_+ \times \mathbb{R}, & \text{if } \lambda \neq \frac{1}{2}. \end{cases}$$

Then X is compact, f and K are continuous with compact values. Easy calculations yield

$$S^s(\lambda) = \begin{cases} [0, \frac{1}{2}], & \text{if } \lambda = \frac{1}{2}, \\ [0, 2], & \text{if } \lambda \neq \frac{1}{2}, \end{cases}$$

and

$$\widetilde{S^s}(\lambda, \varepsilon) = \begin{cases} [0, \frac{1}{2} + \varepsilon] \cap [0, 2], & \text{if } \lambda = \frac{1}{2}, \\ [0, 2], & \text{if } \lambda \neq \frac{1}{2}. \end{cases}$$

It is clear that S^s and $\widetilde{S^s}$ are not usc at $\frac{1}{2}$ and $(\frac{1}{2}, 0)$, respectively. The reason is that the imposed assumption in (b) is violated.

Although the imposed assumptions in (a) and (b) of Theorem 3.2 cannot be dispensed with in the statement, we can replace such assumptions together with the imposed condition on f as follows.

Theorem 3.3. *Assume that X is compact and K is continuous with compact values. Then*

(a) $\widetilde{S^w}$ *is upper semicontinuous with compact values on $\Lambda \times \mathbb{R}_+$ if one of the following two conditions holds:*

(i) *f is continuous and $Y \setminus -\operatorname{int} C(\cdot)$ is closed.*

(ii) *f is C-upper semicontinuous and $Y \setminus -\operatorname{int} C(\cdot)$ is H-upper semicontinuous.*

(b) $\widetilde{S^s}$ *is upper semicontinuous with compact values on $\Lambda \times \mathbb{R}_+$ if one of the following two conditions holds:*

(i) *f is continuous and C is closed in Λ.*

(ii) *f is C-upper semicontinuous and C is H-upper semicontinuous.*

Proof. We verify only (a) as an example. The proof of part (b) is similar.

(i) For given $\lambda \in \Lambda$, we first prove that S^w is usc at λ. Suppose to the contrary that S^w is not usc at λ. Then we can pick a superset U of $S^w(\lambda)$ and a sequence $\{\lambda_n\}$ in Λ converging to λ such that there is $x_n \in S^w(\lambda_n) \setminus U$, for all n. Since X is compact, we can assume that x_n tends to $\bar{x}$, for some $\bar{x} \in X$. By the upper semicontinuity with compact values of K at $(\bar{x}, \lambda)$, we have $\bar{x} \in K(\bar{x}, \lambda)$. For each $y \in K(\bar{x}, \lambda)$, since K is lsc at $(\bar{x}, \lambda)$, there is $y_n \in K(x_n, \lambda_n)$, $y_n \to y$. As $x_n \in S^w(\lambda_n)$, we have

$$f(x_n, y_n, \lambda_n) \in (Y \setminus -\operatorname{int} C(\lambda_n)).$$

Combining the closedness of $Y \setminus -\operatorname{int} C(\cdot)$ with the continuity of f, we conclude that

$$f(\bar{x}, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda)),$$

i.e., $\bar{x} \in S^w(\lambda)$, which is impossible as $x_n \notin U$ for all n. Hence S^w is usc at λ.

Theorem 3.1 yields that $S^w(\lambda)$ is a closed subset of X, and hence it is compact.

For given $(\lambda, \varepsilon) \in \Lambda \times \mathbb{R}_+$, it is clear that $g(x, y, \varepsilon, \lambda) = f(x, y, \lambda) + \varepsilon e(\lambda)$ is continuous as f and e are continuous. Arguing as above, we conclude that $\widetilde{S^w}$ is upper semicontinuous with compact values at (λ, ε).

(ii) Similarly to the first part, if S^w is not usc at λ, then there exist a superset U of $S^w(\lambda)$ and a sequence $\{\lambda_n\} \to \lambda$ such that there is $x_n \in S^w(\lambda_n) \setminus U$ for all n, $x_n \to \bar{x} \in K(\bar{x}, \lambda)$. For each $y \in K(\bar{x}, \lambda)$ there is $y_n \in K(x_n, \lambda_n)$, $y_n \to y$, and

$$f(x_n, y_n, \lambda_n) \in (Y \setminus -\operatorname{int} C(\lambda_n)).$$

For each neighborhood B of θ_Y, there is a balanced neighborhood B_1 of θ_Y, i.e., $-B_1 = B_1$, such that $B_1 + B_1 \subset B$. Since $Y \setminus -\operatorname{int} C(\cdot)$ is H-usc at λ and f is C-usc at $(\bar{x}, y, \lambda)$, there is $n_0 \in \mathbb{N}$ such that, for each $n \geqslant n_0$, we have

$$(Y \setminus -\operatorname{int} C(\lambda_n)) \subset (Y \setminus -\operatorname{int} C(\lambda)) + B_1, \text{ and}$$
$$f(x_n, y_n, \lambda_n) \in f(\bar{x}, y, \lambda) + B_1 - C(\lambda).$$

By the balance of B_1, we have

$$f(\bar{x}, y, \lambda) \in f(x_n, y_n, \lambda_n) - B_1 + C(\lambda) = f(x_n, y_n, \lambda_n) + B_1 + C(\lambda),$$

so,

$$f(\bar{x}, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda_n)) + B_1 + C(\lambda).$$

Since $C(\lambda)$ is a convex cone, $(Y \setminus -\operatorname{int} C(\lambda)) + C(\lambda) \subset (Y \setminus -\operatorname{int} C(\lambda))$. Therefore,

$$f(\bar{x}, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda)) + B_1 + B_1 + C(\lambda) \subset (Y \setminus -\operatorname{int} C(\lambda)) + B.$$

Since B is arbitrary and $Y \setminus -\operatorname{int} C(\lambda)$ is closed, we conclude that $f(\bar{x}, y, \lambda) \in (Y \setminus -\operatorname{int} C(\lambda))$, i.e., $\bar{x} \in S^w(\lambda)$, which is impossible as $x_n \notin U$ for all n. Hence S^w is usc at λ. Theorem 3.1 yields the closedness of $S^w(\lambda)$, and hence $S^w(\lambda)$ is compact as X is compact.

Using the same arguments as above, the upper semicontinuity with compact values of $\widetilde{S^w}$ is also established. □

Now let us present an important relationship between well-posedness and stability of the problems considered.

Theorem 3.4.

(i) *The problems* (WQEP) *and* (SQEP) *are well-posed under perturbations if and only if, for each* $\lambda \in \Lambda$, $\widetilde{S^w}$ *and* $\widetilde{S^s}$ *are upper semicontinuous with compact values at* $(\lambda, 0)$, *respectively.*

(ii) *The problems* (WQEP) *and* (SQEP) *are generalized Hadamard well-posed if and only if, for each* $\lambda \in \Lambda$, S^w *and* S^s *are upper semicontinuous with compact values at* λ, *respectively.*

Proof. We show only the well-posedness under perturbations for the problem (SQEP), since the other cases are similar. Suppose that for given $\lambda \in \Lambda$, $\widetilde{S^s}$ is usc at $(\lambda, 0)$ and $S^s(\lambda)$ is compact. Let $\{\lambda_n\} \subset \Lambda$ an arbitrary sequence converging to λ and $\{x_n\}$ an approximating sequence for (SQEP_λ) corresponding to $\{\lambda_n\}$. Then there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that, for each $n \in \mathbb{N}$, $x_n \in K(x_n, \lambda_n)$ and

$$f(x_n, y, \lambda_n) + \varepsilon_n e(\lambda_n) \in C(\lambda_n) \quad \forall\, y \in K(x_n, \lambda_n),$$

i.e., $x_n \in \widetilde{S^s}(\lambda_n, \varepsilon_n)$. Since $\widetilde{S^s}$ is upper semicontinuous with compact values at $(\lambda, 0)$, Lemma 2.2 implies that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $x_{n_k} \to \bar{x}$, for some $\bar{x} \in S^s(\lambda)$. Therefore, the problem (SQEP) is well-posed.

Conversely, suppose that the problem (SQEP) is well-posed. Let $\lambda \in \Lambda$, $\{(\lambda_n, \varepsilon_n)\} \subset \Lambda \times \mathbb{R}_+$, with $(\lambda_n, \varepsilon_n) \to (\lambda, 0)$ and $x_n \in \widetilde{S^s}(\lambda_n, \varepsilon_n)$. Then, for each $n \in \mathbb{N}$, $x_n \in K(x_n, \lambda_n)$, and

$$f(x_n, y, \lambda_n) + \varepsilon_n e(\lambda_n) \in C(\lambda_n) \quad \forall\, y \in K(x_n, \lambda_n),$$

i.e., $\{x_n\}$ is an approximating sequence of (SQEP$_\lambda$) corresponding to $\{\lambda_n\}$. Using the well-posedness of the problem (SQEP), we can find a subsequence of $\{x_n\}$ converging to some point of $S^s(\lambda) = \widetilde{S^s}(\lambda, 0)$. Using Lemma 2.2, we obtain that $\widetilde{S^s}$ is upper semicontinuous with compact values at $(\lambda, 0)$. □

Combining Theorems 3.2 and 3.3, we establish the results of well-posedness under perturbations and generalized Hadamard well-posedness for the problems (WQEP) and (SQEP).

Corollary 3.1.

(a) *If the conditions of Theorem* 3.2(a) *or those of Theorem* 3.3(a) *are satisfied, then the problem* (WQEP) *is well-posed under perturbations and generalized Hadamard well-posed.*

(b) *If the conditions of Theorem* 3.2(b) *or those of Theorem* 3.3(b) *are satisfied, then the problem* (SQEP) *is well-posed under perturbations and generalized Hadamard well-posed.*

Passing to uniquely well-posedness under perturbations and Hadamard well-posedness, we obtain results similar to those of Theorem 3.4 and Corollary 3.1.

Theorem 3.5. *Suppose that, for each $\lambda \in \Lambda$,* (WQEP$_\lambda$) *and* (SQEP$_\lambda$) *have a unique solution. Then*

(i) (WQEP) *and* (SQEP) *are uniquely well-posed under perturbations if and only if, for each $\lambda \in \Lambda$, $\widetilde{S^w}$ and $\widetilde{S^s}$ are upper semicontinuous with compact values at* $(\lambda, 0)$, *respectively;*

(ii) (WQEP) *and* (SQEP) *are Hadamard well-posed if and only if, for each $\lambda \in \Lambda$, S^w and S^s are upper semicontinuous with compact values at λ, respectively.*

Corollary 3.2.

(a) *If the conditions of Theorem* 3.2(a) *or those of Theorem* 3.3(a) *are satisfied and assuming further that, for each $\lambda \in \Lambda$, the problem* (WQEP$_\lambda$) *has a unique solution, then* (WQEP) *is uniquely well-posed under perturbations and Hadamard well-posed.*

(b) *If the conditions of Theorem* 3.2(b) *or those of Theorem* 3.3(b) *are satisfied and assuming further that, for each $\lambda \in \Lambda$, the problem* (SQEP$_\lambda$) *has a unique solution, then* (SQEP) *is uniquely well-posed under perturbations and Hadamard well-posed.*

4. Applications

4.1. Lower and upper bounded equilibrium problems. Let X and Λ be as in Section 2. Let $g\colon X \times X \times \Lambda \to \mathbb{R}$ be a vector valued mapping and let $\alpha, \beta \in \mathbb{R}_+$, $\alpha < \beta$, be fixed. For each $\lambda \in \Lambda$, we consider the lower and upper bounded equilibrium problem studied in [19], [45]:
(BEP$_\lambda$) Find $\bar{x} \in K(\lambda)$ such that, for each $y \in K(\lambda)$,

$$\alpha \leqslant g(\bar{x}, y, \lambda) \leqslant \beta.$$

Setting $Y = \mathbb{R}^2, C(\lambda) \equiv \mathbb{R}^2_+$ and $f(x, y, \lambda) = (g(x, y, \lambda) - \alpha; \beta - g(x, y, \lambda))$, (SQEP$_\lambda$) reduces to (BEP$_\lambda$).

Combining Theorems 3.2 and 3.4, we establish sufficient conditions for the well-posedness of the lower and upper bounded equilibrium problem as follows.

Corollary 4.1. *Assume that*
(i) *g is continuous*;
(ii) *K is continuous and with compact values.*

Then the family $\{(\mathrm{BEP}_\lambda)\colon\ \lambda \in \Lambda\}$ is well-posed under perturbations and generalized Hadamard well-posed. Moreover, if the solution set is a singleton, then the problem is uniquely and Hadamard well-posed.

Recently, there have been many works devoted to the lower and upper bounded equilibrium problems, such as existence conditions [19], [45], [21], stability conditions [46], and the references therein. To the best of our knowledge, there have not been any papers on the well-posedness for the lower and upper bounded equilibrium problems.

4.2. Traffic network problems. The Wardrop equilibrium flows for the transportation network problem was first introduced in 1952 by Wardrop [44] together with a basic traffic network principle. Since then, traffic network problems have been intensively studied from both the theory and methodology view points. Contributing to the development of traffic network problems, Smith [41] derived a spotlight point by claiming that the Wardrop equilibrium flows of a network are expressed as solutions of the corresponding variational inequality. Because of diverse practical situations, these demands may depend on the equilibrium vector flows which were considered in [20], [36]; the so-called elastic network problems, and then the Wardrop equilibrium flows of the elastic network problem are obtained as solutions of the corresponding quasivariational inequality.

Let us recall the mathematical model of the elastic traffic network problem studied in [2], [6], [20], [36]. Let N the set of nodes and L that of arcs (or links). We denote the collection of origin-destination pairs (O/D pairs for short) by $W = (W_1, \dots, W_l)$. Suppose that the pair W_j, $j = 1, \dots, l$, is connected by a set P_j of paths and P_j contains r_j paths, $r_j \geqslant 1$. Let us denote the path vector flow by $F = (F_1, \dots, F_m)$, where $m = r_1 + \ldots + r_l$. The capacity of these paths must be taken into account in practice (see, [26]), and hence we assume that the compact capacity restriction is defined by

$$F \in X := \{F \in \mathbb{R}^m \colon\ 0 \leqslant \gamma_s \leqslant F_s \leqslant \Gamma_s,\ s = 1, \dots, m\}.$$

Suppose that the travel cost on the path flow F_s, $s = 1, \dots, m$, depends on the whole path vector flow F and is $T_s(F) \geqslant 0$. Then we derive the path cost vector $T(F) = (T_1(F), \dots, T_m(F))$.

A path vector flow H is named an equilibrium vector flow as defined in [44], if for each W_j and $p, s \in P_j$,

$$[T_p(H) < T_s(H)] \Longrightarrow [H_s = \gamma_s \text{ or } H_p = \Gamma_p].$$

Now assume that the perturbation on the traffic is expressed by a parameter λ of a metric space Λ. Suppose that the travel demands g_j of the O/D pair W_j depend on $\lambda \in \Lambda$ and also on the equilibrium vector flow H as interpreted in [20], [36]. We denote the travel vector demand by $g = (g_1, \dots, g_l)$ and set

$$\varphi_{js} = \begin{cases} 1, & \text{if } s \in P_j, \\ 0, & \text{if } s \notin P_j, \end{cases}$$

$$\varphi = \{\varphi_{js}\},\ j = 1, \dots, l;\ s = 1, \dots, m.$$

Then the path vector flows meeting the travel demands are termed the feasible path vector flows and form the constraint set

$$K(H, \lambda) = \{F \in X \colon\ \varphi F = g(H, \lambda)\}.$$

The matrix φ is called the O/D pair—path incidence matrix. Assume further that the path costs also depend on the parameter λ, $T_s(F, \lambda)$, $s = 1, \dots, m$.

Our traffic network problem is equivalent to a quasivariational inequality as follows.

Lemma 4.1 ([41]). *A path vector flow $H \in K(H,\lambda)$ is an equilibrium flow if and only if it is a solution of the following quasivariational inequality:*
(QVI_λ) *find $H \in K(H,\lambda)$ such that, for each $F \in K(H,\lambda)$,*

$$\langle T(H,\lambda), F - H\rangle \geqslant 0.$$

Setting $f(H,F,\lambda) = \langle T(H,\lambda), F-H\rangle$, ($\mathrm{WQEP}_\lambda$) and ($\mathrm{SQEP}_\lambda$) become ($\mathrm{QVI}_\lambda$).

Lemma 4.2 ([6]). *If g is continuous at (H,λ), then K is continuous with convex and compact values at (H,λ).*

The following result is derived from Theorems 3.2, 3.4 and Lemmas 4.1, 4.2.

Corollary 4.2. *If g and T are continuous, then the family of parametric elastic network problems is well-posed under perturbations and generalized Hadamard well-posed. Furthermore, if* (QVI_λ) *has a unique solution, then this problem is uniquely well-posed under perturbations and Hadamard well-posed.*

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