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ON THE VISCOUS ALLEN-CAHN AND CAHN-HILLIARD SYSTEMS WITH WILLMORE REGULARIZATION

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Abstract. We consider the viscous Allen-Cahn and Cahn-Hilliard models with an additional term called the nonlinear Willmore regularization. First, we are interested in the well-posedness of these two models. Furthermore, we prove that both models possess a global attractor. In addition, as far as the viscous Allen-Cahn equation is concerned, we construct a robust family of exponential attractors, i.e. attractors which are continuous with respect to the perturbation parameter. Finally, we give some numerical simulations which show the effects of the viscosity term on the anisotropic and isotropic Cahn-Hilliard equation.

Keywords: viscous Cahn-Hilliard equation; viscous Allen-Cahn equation; Willmore regularization; well-posedness of models; global attractor; robust exponential attractors; anisotropy; simulations

MSC 2010: 35B40, 35B41, 35B45, 35K55

1. INTRODUCTION

The Allen-Cahn equation (see [1])

$$\frac{\partial u}{\partial t} = \Delta u - F'(u) \tag{1.1}$$

and the Cahn-Hilliard equation (see [6])

$$\frac{\partial u}{\partial t} = \Delta(F'(u) - \Delta u) \tag{1.2}$$

are central to material sciences, as they characterize important qualitative features of two-phase systems. Each of these equations governs the evolution of an order parameter $u = u(t, x)$: the Allen-Cahn equation describes the ordering of atoms within unit cells in a lattice, while the Cahn-Hilliard equation, a conservation law, describes the transport of atoms between unit cells.

Here, f is the derivative of a double-well potential F whose wells characterize the phases. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F_{\log}(s) = \frac{\lambda_1}{2}(1-s^2) + \frac{\lambda_2}{2}\Big((1-s)\ln\frac{1-s}{2} + (1+s)\ln\frac{1+s}{2}\Big), \quad s \in (-1,1),\ 0 < \lambda_2 \leqslant \lambda_1, \tag{1.3}$$

hence

$$f_{\log}(s) = -\lambda_1 s + \frac{\lambda_2}{2}\ln\frac{1+s}{1-s}, \tag{1.4}$$

although, as this will be the case in this article, such a function is very often approximated by regular ones, typically,

$$F(s) = \frac{1}{4}(s^2-1)^2, \tag{1.5}$$

hence

$$f(s) = s^3 - s \tag{1.6}$$

(see [5], [6], [9], [31], and [13]).

Both the Allen-Cahn and Cahn-Hilliard equations are based on the total free energy

$$\Psi(u) = \int_V \Big(F(u) + \frac{1}{2}\|\nabla u\|^2\Big)\,\mathrm{d}V, \tag{1.7}$$

with $F(u)$ the "coarse grain" free energy, a double-well potential whose wells define the phases. Each of these equations governs the evolution of an order parameter $u = u(t, x)$. The Allen-Cahn equation describes the evolution of a *non-conserved* order field during the anti-phase domain coarsening. It can be identified by the phase variable u appearing in the context of diffuse interface modelling. The Cahn-Hilliard approach, on the other hand, consists in assuming that the interface thickness between two phases in the system is small but greater than the real physical one. One phase is described geometrically by the smooth function u which is equal to 1 in one phase and -1 outside, and which varies continuously in the interfaces from one phase to the other (see Figure 1).

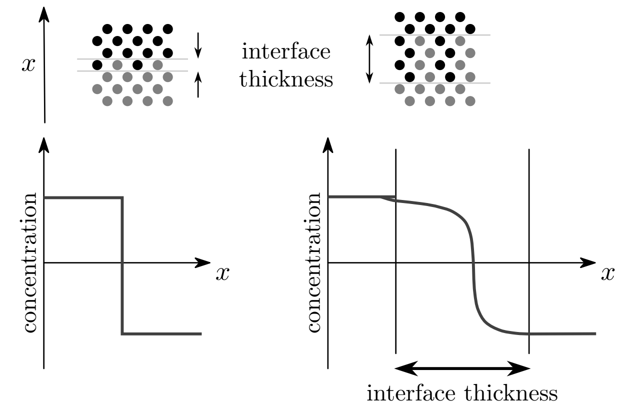


Figure 1. Sharp and diffuse interfaces

The standard derivation of the Allen-Cahn equation begins by assuming the relaxation dynamics (see [1])

$$\frac{\partial u}{\partial t} = -\frac{\delta \Psi}{\delta u},$$

where $\delta/\delta u$ is the variational derivative. To obtain (1.1), we write formally for a small variation and assuming proper boundary conditions,

$$\delta \Psi = \int_\Omega (\nabla u \cdot \nabla \delta u + f(u)\delta u)\,\mathrm{d}x = \int_\Omega (-\Delta u + f(u))\delta u\,\mathrm{d}x.$$

The Cahn-Hilliard equation is derived analogously. The starting point is to write the mass conservation and the relaxation dynamics (see [16]), i.e.

$$\frac{\partial u}{\partial t} = \Delta \mu, \quad \mu = \frac{\partial \Psi}{\partial u},$$

and this leads to equation (1.2).

A slightly more complicated model, which is based on a new balance law for microforces and which takes into account the working of internal microforces, was introduced in [19] (we can note that microforces describe forces which are associated with microscopic configurations of atoms, whereas standard forces are associated with macroscopic length scales, hence a reason to consider separate balance laws for microforces and standard forces). For an isotropic material, this leads to the following generalizations of equations (1.1) and (1.2):

$$\frac{\partial}{\partial t}(u + \xi(-\Delta)u) - \Delta u + f(u) = 0 \tag{1.8}$$

and

$$\frac{\partial}{\partial t}(u + \xi(-\Delta)u) = \Delta(-\Delta u + f(u)), \tag{1.9}$$

where ξ is a (small) positive parameter and where the term $\xi\partial_t u$ describes the influence of the internal microforces. These equations can also be viewed as viscous Allen-Cahn and Cahn-Hilliard equations, see e.g. [18], [3], [29], [19].

In particular, the viscous Cahn-Hilliard equation was proposed in [29] as a model of phase separation in mixtures of polymers, where the intermolecular friction forces can be significant. This model arises as a singular limit of the phase-field model in phase transition. It contains both the Cahn-Hilliard and Allen-Cahn equations as particular limits (see [3], [14]).

These equations have been studied intensively, see e.g. the review articles [14], [30] and, among many references, [29], [4]. In particular, in the case of regular potentials, the problem is well understood and one has existence and uniqueness of solutions and existence of the finite dimensional global attractor.

Furthermore, a robust family of exponential attractors with regular nonlinear terms for equation (1.9) has been studied in [32], while in [11] the authors study the one with singular nonlinear terms. In addition, in [27] the authors introduce the viscous Cahn-Hilliard equation but with dynamic boundary conditions.

In this paper we deal with the asymptotic behavior of the two equations with an additional term describing the influence of the internal microforces of order four and six in space, respectively.

These evolution equations arise for instance in anisotropic crystal models (see e.g. [35]), since the regularization term which contains a small parameter is added to the Ginzburg-Landau free energy. More precisely, we will consider the Willmore regularization of the viscous Allen-Cahn and the viscous Cahn-Hilliard equations. Indeed, denoting by $\beta > 0$ a small regularization parameter, the anisotropic energy functional reads

$$\mathcal{E}(u) = \int_\Omega \Big(\gamma(n)\Big(\frac{1}{2}|\nabla u|^2 + F(u)\Big) + \frac{\beta}{2}\omega^2\Big)\,\mathrm{d}x, \tag{1.10}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain containing the two-phase systems and $\omega = f(u) - \Delta u$. Moreover, $n = \nabla u/|\nabla u|$ is the unit normal and $\gamma(n)$ accounts for anisotropy effects. The Willmore regularization is relevant, e.g., in determining the equilibrium shape of a crystal in its own liquid matrix when the anisotropy effects are strong. Indeed, in that case the equilibrium interface may not be a smooth curve but may present facets and corners with slope of discontinuities (see e.g. [33]), which can lead to an ill-posed problem and requires regularization.

Although such models have already been studied from a numerical point of view [7], [36], to the best of our knowledge, their mathematical analysis has been performed only when γ is constant ($\gamma(n) = \pm 1$, see [25] and [23]) and very recently, in a slightly different anisotropic model [21], [22]. However, while the choice $\gamma(n) = -1$ does not lead to dissipativity, in the aforementioned anisotropic case it is difficult to carry the study beyond existence and uniqueness of solutions. Therefore, because in this paper we are concerned with attractors, we restrict ourselves to $\gamma(n) = 1$.

In that case, the relaxation dynamics leads to the (isotropic) Willmore regularization of the viscous Allen-Cahn equation

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} - \Delta u + f(u) + \beta\omega f'(u) - \beta\Delta\omega = 0, \tag{1.11}$$

$$\omega = f(u) - \Delta u, \tag{1.12}$$

whereas the mass conservation and the relaxation dynamics provide the (isotropic) Willmore regularization of the viscous Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \Delta K(u), \tag{1.13}$$

$$K(u) = f(u) - \Delta u + \xi\partial_t u + \beta\omega f'(u) - \beta\Delta\omega, \tag{1.14}$$

$$\omega = f(u) - \Delta u. \tag{1.15}$$

When ξ is equal to zero, we see that (1.11)–(1.12) formally becomes the regularized Allen-Cahn equation (see [25]) and (1.13)–(1.15) the regularized Cahn-Hilliard equation (see [23]). We also note that the authors in [8] study the Willmore regularization in terms of finite dimensional exponential attractors (depending on a small regularization parameter $\beta > 0$), for the isotropic Allen-Cahn and Cahn-Hilliard equations based on (1.10).

Our aim in this article is to study the asymptotic behavior of the viscous Allen-Cahn and Cahn-Hilliard equations, respectively, (1.11) and (1.13)–(1.15). First, we treat the viscous Allen-Cahn model with Willmore regularization. We prove the well-posedness, existence of global attractors and construct a family of robust exponential attractors. Then we study the viscous Cahn-Hilliard model and prove the existence, uniqueness of solutions and the existence of global attractors. Finally, we give some numerical results, illustrating the influence of the viscosity parameter ξ on the isotropic and strong anisotropic Cahn-Hilliard equations with willmore regularization.

We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$, where $S(t)$ denotes the solution operator mapping the initial datum onto the solution at time t) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. Furthermore, the finite-dimensionality means, roughly speaking, that even though the initial phase space is infinite-dimensional, the reduced dynamics is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. We refer the reader to [2], [28], and [34] for more details and discussions on this. Now, an exponential attractor $\mathcal{M}$ is only positively invariant (i.e. $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geqslant 0$), contains the global attractor, has by definition finite fractal dimension and attracts (uniformly) the bounded sets of initial data. Compared to the global attractor, an exponential attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations. We refer the reader to [10] and [28] for discussions on this subject.

Assumptions and notation. As far as the nonlinear term f is concerned, we assume more generally that f is of class $\mathcal{C}^4$ and that

$$f(0) = 0, \quad f'(s) \geqslant -c_0, \quad c_0 \geqslant 0, \quad s \in \mathbb{R}, \tag{1.16}$$

$$f(s)s \geqslant c_1 F(s) - c_2 \geqslant -c_2', \quad c_1 > 0, \quad c_2, \quad c_2' \geqslant 0, \quad s \in \mathbb{R}, \tag{1.17}$$

where $F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$,

$$sf(s)f'(s) - f(s)^2 \geqslant c_3 f(s)^2 - c_4, \quad c_3 > 0, \quad c_4 \geqslant 0, \quad s \in \mathbb{R}, \tag{1.18}$$

$$|f'(s)| \leqslant \varepsilon |f(s)| + c_5 \quad \forall \varepsilon > 0, \quad c_5(\varepsilon) \geqslant 0, \quad s \in \mathbb{R}, \tag{1.19}$$

$$sf''(s) \geqslant 0, \quad s \in \mathbb{R}. \tag{1.20}$$

We can note that (1.16)–(1.19) are satisfied by polynomials of the form $f(s) = \sum_{i=1}^{2p+1} a_i s^i$, $a_{2p+1} > 0$, and in particular by the usual cubic nonlinear term (1.6). Assumption (1.20), which allows to obtain dissipative estimates (see below), is more restrictive; it is however reasonable as it is satisfied by the cubic nonlinear term (1.6).

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product with associated norm $\|\cdot\|$ and we set $\|\cdot\|_{-1} = \|(-\Delta)^{-1/2}\cdot\|$, where $(-\Delta)^{-1}$ is the inverse minus Laplace operator associated with Neumann boundary conditions and acting on functions with null average. Furthermore, $\|\cdot\|_X$ denotes the norm in the Banach space X.

We set, whenever it makes sense, $\langle\cdot\rangle = \mathrm{Vol}^{-1}(\Omega)\int_\Omega \cdot\,\mathrm{d}x$, being understood that for $\varphi \in H^{-1}(\Omega) = H^1(\Omega)'$, $\langle\varphi\rangle = \mathrm{Vol}^{-1}(\Omega)\langle\varphi, 1\rangle_{H^{-1}(\Omega),H^1(\Omega)}$, and we note that

$$\varphi \mapsto (\|\varphi - \langle\varphi\rangle^2\|_{-1}^2 + \langle\varphi\rangle^2)^{1/2}$$

is a norm on $H^{-1}(\Omega)$ which is equivalent to the usual one.

We define the following spaces:

$$V_\xi = \begin{cases} H^1(\Omega) & \text{for } \xi > 0, \\ L^2(\Omega) & \text{for } \xi = 0 \end{cases} \quad \text{and} \quad H_\xi = \begin{cases} L^2(\Omega) & \text{for } \xi > 0, \\ H^{-1}(\Omega) & \text{for } \xi = 0, \end{cases} \tag{1.21}$$

where the spaces V_ξ and H_ξ are equipped with the following norms, respectively:

$$\|v\|_{V_\xi}^2 = \|v\|^2 + \xi\|\nabla v\|^2 \quad \text{and} \quad \|v\|_{H_\xi}^2 = \|v\|_{-1}^2 + \xi\|v\|^2. \tag{1.22}$$

Throughout the paper, the same letter c, c_M (and sometimes c', c'' and c''') denotes constants which may vary from line to line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

2. Viscous Allen-Cahn system

Setting the problem: Let Ω be an open bounded domain of $\mathbb{R}^n$, $n = 1, 2$, or 3, with a smooth boundary Γ. The unknown function is a scalar $u = u(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$ and we consider the viscous Allen-Cahn system (for $0 \leqslant \xi < 1$ and taking $\beta = 1$)

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} - \Delta u + f(u) + \omega f'(u) - \Delta\omega = 0 \quad \text{on } \Omega, \tag{2.1}$$

$$\omega = f(u) - \Delta u \quad \text{on } \Omega \tag{2.2}$$

together with the Dirichlet boundary condition

$$u = \omega = 0 \quad \text{on } \Gamma, \tag{2.3}$$

and initial data

$$u|_{t=0} = u_0, \tag{2.4}$$

where f is the cubic function defined by (1.6) and F is the antiderivative of f defined by (1.5).

2.1. A priori estimates.

We multiply (2.1) by $\partial u/\partial t$ and have, integrating over Ω and by parts,

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \xi\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\,\mathrm{d}x + \left(\!\left(\omega f'(u) - \Delta\omega, \frac{\partial u}{\partial t}\right)\!\right) = 0,$$

which yields, noting that it follows from (2.2) that

$$\left(\!\!\left(\omega f'(u), \frac{\partial u}{\partial t}\right)\!\!\right) - \left(\!\!\left(\Delta\omega, \frac{\partial u}{\partial t}\right)\!\!\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2,$$

the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2\right) + 2\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \xi\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right) = 0. \tag{2.5}$$

In particular, it follows from (2.5) that the energy decreases along the trajectories as expected.

We then multiply (2.1) by u and obtain, owing to (2.2),

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \|\nabla u\|^2 + ((f(u),u)) + \int_\Omega uf(u)f'(u)\,\mathrm{d}x + \|\Delta u\|^2 \\ + 2((f'(u)\nabla u, \nabla u)) + ((uf''(u)\nabla u, \nabla u)) = 0,\end{aligned} \tag{2.6}$$

which yields, owing again to (2.2) and using integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V_\xi} + \|\nabla u\|^2 + ((f(u),u)) + \|\omega\|^2 \\ &\quad + \int_\Omega (uf(u)f'(u) - f(u)^2)\,\mathrm{d}x + ((uf''(u)\nabla u, \nabla u)) = 0,\end{aligned} \tag{2.7}$$

hence, in view of (1.17), (1.18), and (1.20),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V_\xi} + c\left(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2\right) \leqslant c', \quad c > 0. \tag{2.8}$$

Summing (2.5) and (2.8), we find a differential inequality of the form

$$\frac{\mathrm{d}E_{1,\xi}}{\mathrm{d}t} + c\left(E_{1,\xi} + \left\|\frac{\partial u}{\partial t}\right\|^2_{V_\xi}\right) \leqslant c', \quad c > 0, \tag{2.9}$$

where

$$E_{1,\xi} = \|u\|^2_{V_\xi} + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2. \tag{2.10}$$

In particular, it follows from (2.9) and Gronwall's lemma that

$$E_{1,\xi}(t) \leqslant E_{1,\xi}(0)\mathrm{e}^{-ct} + c', \quad c > 0, \tag{2.11}$$

hence, in view of (1.16) (which yields that $\|\omega\|^2 \geqslant \|\Delta u\|^2 + \|f(u)\|^2 - 2c_0\|\nabla u\|^2$), (2.11) and classical elliptic regularity results,

$$\|u(t)\|_{H^2(\Omega)} \leqslant Q(\|u_0\|_{H^2(\Omega)})e^{-ct} + c', \quad c > 0,\ t \geqslant 0. \tag{2.12}$$

Next, we multiply (2.1) by $-\Delta u$ and integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{\xi}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta u\|^2 + ((f'(u)\nabla u, \nabla u)) - ((\omega f'(u), \Delta u)) + ((\Delta f(u), \Delta u)) + \|\nabla\Delta u\|^2 = 0. \tag{2.13}$$

Noting that, owing to the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$ (here $n \leqslant 3$) and (2.2),

$$|((f'(u)\nabla u, \nabla u))| + |((\omega f'(u), \Delta u))| + |((\Delta f(u), \Delta u))| \leqslant Q(\|u\|_{H^2(\Omega)}) \tag{2.14}$$

(indeed, it follows from (2.2) that $\|\omega\| \leqslant Q(\|u\|_{H^2(\Omega)})$), we obtain

$$\frac{d}{dt}(\|\nabla u\|^2 + \xi\|\Delta u\|^2) + c\|u\|^2_{H^3(\Omega)} \leqslant Q(\|u\|_{H^2(\Omega)}), \quad c > 0. \tag{2.15}$$

We then multiply (2.1) by $-\Delta\partial u/\partial t$ and find, owing to (2.2),

$$\begin{aligned}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 &+ \xi\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 \\ &+ \Big(\Big(f'(u)\nabla u, \nabla\frac{\partial u}{\partial t}\Big)\Big) + \Big(\Big(\nabla(\omega f'(u)), \nabla\frac{\partial u}{\partial t}\Big)\Big) \\ &- \Big(\Big(\nabla\Delta f(u), \nabla\frac{\partial u}{\partial t}\Big)\Big) + \frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 = 0.\end{aligned} \tag{2.16}$$

We have

$$\begin{aligned}\Big|\Big(\Big(f'(u)\nabla u, \nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| &\leqslant \|f'(u)\|_{L^\infty(\Omega)}\|\nabla u\|\,\Big\|\nabla\frac{\partial u}{\partial t}\Big\| \\ &\leqslant \frac{1}{8}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + \|f'(u)\|^2_{L^\infty(\Omega)}\|\nabla u\|^2 \\ &\leqslant \frac{1}{8}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + Q(\|u\|_{H^2(\Omega)})\end{aligned} \tag{2.17}$$

(here we have used the fact that $H^2(\Omega)$ is continuously embedded in $C(\overline{\Omega})$, noting that

$$|f'(u)| \leqslant Q(|u|) \leqslant Q(\|u\|_{L^\infty(\Omega)})$$

for some monotone increasing and continuous function Q) and, proceeding similarly,

$$\left|\left(\left(\nabla(\omega f'(u)),\nabla\frac{\partial u}{\partial t}\right)\right)\right| \leqslant \left|\left(\left(\omega f''(u)\nabla u,\nabla\frac{\partial u}{\partial t}\right)\right)\right| + \left|\left(\left(f'(u)\nabla\omega,\nabla\frac{\partial u}{\partial t}\right)\right)\right| \leqslant \|\omega\|\,\|f''(u)\|_{L^\infty(\Omega)}\|\nabla u\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| + \|f'(u)\|_{L^\infty(\Omega)}\|\nabla\omega\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| \leqslant \frac{1}{8}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + Q(\|u\|_{H^2(\Omega)})(\|\nabla\Delta u\|^2+1), \tag{2.18}$$

noting that it follows from (2.2) that

$$\|\nabla\omega\|^2 \leqslant Q(\|u\|_{H^2(\Omega)}) + 2\|\nabla\Delta u\|^2.$$

Finally,

$$\nabla\Delta f(u) = f''(u)\Delta u\nabla u + f'(u)\nabla\Delta u + f'''(u)\nabla u|\nabla u|^2 + 2f''(u)\nabla\nabla u\cdot\nabla u,$$

so, owing to (2.12), the Hölder's inequality and proper Sobolev embeddings,

$$\left|\left(\left(\nabla\Delta f(u),\nabla\frac{\partial u}{\partial t}\right)\right)\right| \leqslant \left|\left(\left(f''(u)\Delta u\nabla u,\nabla\frac{\partial u}{\partial t}\right)\right)\right| + \left|\left(\left(f'(u)\nabla\Delta u,\nabla\frac{\partial u}{\partial t}\right)\right)\right| + \left|\left(\left(f'''(u)|\nabla u|^2\nabla u,\nabla\frac{\partial u}{\partial t}\right)\right)\right| + 2\left|\left(\left(f''(u)\nabla\nabla u\cdot\nabla u,\nabla\frac{\partial u}{\partial t}\right)\right)\right| \leqslant \|f''(u)\|_{L^\infty(\Omega)}\|\Delta u\|\,\|\nabla u\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| + \|f'(u)\|_{L^\infty(\Omega)}\|\nabla\Delta u\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| + \|f'''(u)\|_{L^\infty(\Omega)}\|\nabla u\|^2_{L^4(\Omega)}\|\nabla u\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| + 2\|f''(u)\|_{L^\infty(\Omega)}\|\Delta u\|\,\|\nabla u\|\,\left\|\nabla\frac{\partial u}{\partial t}\right\| \leqslant \frac{1}{4}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + Q(\|u\|_{H^2(\Omega)})\|\nabla\Delta u\|^2. \tag{2.19}$$

It thus follows from (2.16)–(2.19) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2+\|\nabla\Delta u\|^2) + \left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \xi\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 \leqslant Q(\|u\|_{H^2(\Omega)})(\|\nabla\Delta u\|^2+1). \tag{2.20}$$

We multiply (2.1) by $\Delta^2 u$ and integrate over Ω to obtain, owing to (2.2),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 + \|\nabla\Delta u\|^2 + ((f(u),\Delta^2u)) + ((\omega f'(u),\Delta^2u)) - ((\Delta f(u),\Delta^2u)) + \|\Delta^2u\|^2 = 0, \tag{2.21}$$

which yields, noting that

$$|((f(u),\Delta^2u))|+|((\omega f'(u),\Delta^2u))|+|((\Delta f(u),\Delta^2u))|\leqslant\frac{1}{2}\|\Delta^2u\|^2+Q(\|u\|_{H^2(\Omega)}),$$

the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2+\xi\|\nabla\Delta u\|^2)+\|u\|^2_{H^4(\Omega)}\leqslant Q(\|u\|_{H^2(\Omega)}). \tag{2.22}$$

We finally multiply (2.1) by $\Delta^2\frac{\partial u}{\partial t}$ and integrate over Ω to find, owing to (2.2),

$$\begin{aligned}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2+\xi\Big\|\nabla\Delta\frac{\partial u}{\partial t}\Big\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2&\\ +\Big(\Big(\Delta f(u),\Delta\frac{\partial u}{\partial t}\Big)\Big)+\Big(\Big(\Delta(\omega f'(u)),\Delta\frac{\partial u}{\partial t}\Big)\Big)&\\ -\Big(\Big(\Delta^2 f(u),\Delta\frac{\partial u}{\partial t}\Big)\Big)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta^2u\|^2=0.&\end{aligned} \tag{2.23}$$

We have

$$\Big|\Big(\Big(\Delta f(u),\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|\leqslant\frac{1}{8}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2+Q(\|u\|_{H^2(\Omega)}). \tag{2.24}$$

Furthermore,

$$\Delta(\omega f'(u))=f'(u)\Delta\omega+2f''(u)\nabla u\cdot\nabla\omega+\omega f''(u)\Delta u+\omega f'''(u)|\nabla u|^2,$$

which yields, owing to (2.2),

$$\begin{aligned}&\Big|\Big(\Big(\Delta(\omega f'(u)),\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|\\ &\leqslant\Big|\Big(\Big(f'(u)\Delta\omega,\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|+2\Big|\Big(\Big(f''(u)\nabla u\cdot\nabla\omega,\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|\\ &\quad+\Big|\Big(\Big(\omega f''(u)\Delta u,\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|+\Big|\Big(\Big(\omega f'''(u)|\nabla u|^2,\Delta\frac{\partial u}{\partial t}\Big)\Big)\Big|\\ &\leqslant\|f'(u)\|_{L^\infty(\Omega)}\|\Delta\omega\|\Big\|\Delta\frac{\partial u}{\partial t}\Big\|+2\|f''(u)\|_{L^\infty(\Omega)}\|\nabla u\|\|\nabla\omega\|\Big\|\Delta\frac{\partial u}{\partial t}\Big\|\\ &\quad+\|\omega\|\|f''(u)\|_{L^\infty(\Omega)}\|\Delta u\|\Big\|\Delta\frac{\partial u}{\partial t}\Big\|+\|\omega\|\|f'''(u)\|_{L^\infty(\Omega)}\|\nabla u\|^2_{L^4(\Omega)}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|\\ &\leqslant\frac{1}{8}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2+Q(\|u\|_{H^3(\Omega)})(\|\Delta^2u\|^2+1).\end{aligned} \tag{2.25}$$

Finally,

$$\Delta^2 f(u) = f'(u)\Delta^2 u + 2f''(u)\nabla\Delta u\cdot\nabla u + f''(u)|\Delta u|^2 + 2f''(u)\nabla\nabla u\cdot\nabla\nabla u \\ + 4f'''(u)\nabla\nabla u\cdot\nabla u\cdot\nabla u + f'''(u)|\nabla u|^2\Delta u + f^{(4)}(u)|\nabla u|^4$$

and, proceeding as above, we can prove that

$$\Big|\Big(\Big(\Delta^2 f(u), \Delta\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant \frac{1}{4}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2 + Q(\|u\|_{H^3(\Omega)})(\|\Delta^2 u\|^2 + 1). \tag{2.26}$$

It thus follows from (2.23)–(2.26) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\Delta u\|^2 + \|\Delta^2 u\|^2) + \Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2 + \xi\Big\|\nabla\Delta\frac{\partial u}{\partial t}\Big\|^2 \leqslant Q(\|u\|_{H^2(\Omega)})(\|\Delta^2 u\|^2 + 1). \tag{2.27}$$

2.2. Well-posedness and existence of the global attractor. We first state the following theorem.

Theorem 2.1. *Assume that* $u_0 \in H^2(\Omega)\cap H^1_0(\Omega)$. *Then* (2.1)–(2.4) *possesses a unique variational solution* u *such that*

$$u \in L^\infty(\mathbb{R}^+; H^2(\Omega)\cap H^1_0(\Omega)) \cap L^\infty(\tau,\infty; H^4(\Omega))$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))\cap L^2(\tau,T;H^2(\Omega)) \quad \forall\,\tau>0, \quad \forall\,0<\tau\leqslant T, \ \text{for } \xi=0,$$
$$\frac{\partial u}{\partial t} \in L^2(0,T;H^1(\Omega))\cap L^2(\tau,T;H^3(\Omega)) \quad \forall\,\tau>0, \quad \forall\,0<\tau\leqslant T, \ \text{for } \xi>0.$$

Furthermore,

$$\omega \in L^\infty(\mathbb{R}^+; L^2(\Omega))\cap L^\infty(\tau,\infty; H^2(\Omega)\cap H^1_0(\Omega)) \quad \forall\,\tau>0.$$

P r o o f. a) Uniqueness: Let (u_1,ω_1) and (u_2,ω_2) be two solutions of (2.1)–(2.4) with initial data $u_{0,1}$ and $u_{0,2}$ respectively, where ω_i, $i=1,2$, are defined by (2.2).

We set $u=u_1-u_2$, $\omega=\omega_1-\omega_2$, $u_0=u_{0,1}-u_{0,2}$ and have

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) + \omega_1 f'(u_1) - \omega_2 f(u_2) - \Delta\omega = 0, \tag{2.28}$$
$$\omega = f(u_1) - f(u_2) - \Delta u, \tag{2.29}$$
$$u = \omega = 0 \quad \text{on } \Gamma, \tag{2.30}$$
$$u|_{t=0} = u_0. \tag{2.31}$$

We multiply (2.28) by u and obtain, integrating over Ω,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2+\|\nabla u\|^2+((f(u_1)-f(u_2),u)) \\ +((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))-((\Delta(f(u_1)-f(u_2)),u))+\|\Delta u\|^2=0. \tag{2.32}$$

We note that, owing to (1.16),

$$((f(u_1)-f(u_2),u))\geqslant -c_0\|u\|^2 \tag{2.33}$$

and that, owing to (2.11), (2.29), the regularity of f and Poincaré's inequality,

$$\begin{aligned}|((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))| &\leqslant |((\omega f'(u_1),u))|+|((\omega_2(f'(u_1)-f'(u_2)),u))| \\ &\leqslant \|\omega\|\,\|u\|\,\|f'(u)\|_{L^\infty(\Omega)}+\|\omega_2\|\,\|u\|^2_{L^4(\Omega)}\|f''(u)\|_{L^\infty(\Omega)} \\ &\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})(\|\omega\|\|u\|+\|\omega_2\|\|u\|^2_{L^4(\Omega)}) \\ &\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})(\|\Delta u\|\|u\|+\|\nabla u\|^2) \\ &\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|u\|\|\Delta u\| \\ &\leqslant \frac{1}{4}\|\Delta u\|^2+Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|u\|^2;\end{aligned} \tag{2.34}$$

here we have used the interpolation inequality

$$\|\nabla u\|^2\leqslant c\|u\|\|\Delta u\|$$

and the fact that

$$\|\omega\|\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|u\|_{H^2(\Omega)}, \tag{2.35}$$

which follows from (2.29). Finally,

$$|((f(u_1)-f(u_2),\Delta u))|\leqslant \frac{1}{8}\|\Delta u\|^2+Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|u\|^2. \tag{2.36}$$

We finally deduce from (2.32)–(2.36) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2+\xi\|\nabla u\|^2)+\|\Delta u\|^2 &\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})\|u\|^2 \\ &\leqslant Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)})(\|u\|^2+\xi\|\nabla u\|^2).\end{aligned} \tag{2.37}$$

Gronwall's lemma then yields

$$\|u(t)\|^2_{V_\xi}\leqslant c\mathrm{e}^{c't}\|u_0\|^2_{V_\xi}, \tag{2.38}$$

where c and c' only depends on $\|u_{0,i}\|_{H^2(\Omega)}$, $i=1,2$, (and are in particular independent of ξ).

This gives the uniqueness as well as the continuous dependence with respect to the initial data in the V_ξ-norm.

b) Existence: The proof of existence is based on a classical Galerkin scheme and the a priori estimates derived in the previous subsection.

A weak (variational) formulation for (2.1)–(2.4) reads

$$\Big(\Big(\frac{\partial u}{\partial t},v\Big)\Big)-\xi\Big(\Big(\Delta\frac{\partial u}{\partial t},v\Big)\Big)-((\Delta u,v))+((f(u),v))+((\omega f'(u),v)) \\ -((\Delta\omega,v))=0 \quad \forall\, v\in H^1(\Omega), \tag{2.39}$$

$$((\omega,v))=((f(u),v))-((\Delta u,v)) \quad \forall\, v\in H^1(\Omega), \tag{2.40}$$

$$u|_{t=0}=u_0. \tag{2.41}$$

We can note that all estimates in Subsection 3.1 follow (formally) from this variational formulation.

Let $v_0,v_1,\ldots$ be an orthonormal (in $L^2(\Omega)$) and orthogonal (in $H^1(\Omega)$) family associated with the eigenvalues $0=\lambda_0<\lambda_1\leqslant\ldots$ of the self-adoint, bounded and strictly positive operator $-\Delta$ (note that v_0 is a constant). We set

$$V_m=\mathrm{Span}\{v_0,v_1,\ldots,v_m\}$$

and consider the approximated problem:

Find $(u_m,\omega_m)\colon\ [0,T]\to V_m\times V_m$ such that

$$\Big(\Big(\frac{\partial u_m}{\partial t},v\Big)\Big)-\xi\Big(\Big(\Delta\frac{\partial u_m}{\partial t},v\Big)\Big)-((\Delta u_m,v))+((f(u_m),v))+((\omega_m f'(u_m),v)) \\ -((\Delta\omega_m,v))=0 \quad \forall\, v\in V_m, \tag{2.42}$$

$$((\omega_m,v))=((f(u_m),v))-((\Delta u_m,v)) \quad \forall\, v\in V_m, \tag{2.43}$$

$$u_m|_{t=0}=u_{0,m}, \tag{2.44}$$

where $u_{0,m}=P_m u_0$, P_m is the orthogonal projector from $L^2(\Omega)$ onto V_m.

The existence of a local (in time) solution to (2.42)–(2.44) is standard. Indeed, we have to solve a Lipschitz continuous finite-dimensional system of ODE's to find u_m, which yields ω_m over an interval $(0,T_m)$ for certain $T_m>0$.

We will consider now a maximal solution defined over $[0,T_m]$ and we will prove that $T_m=T$. In other words, we will prove that the local (in time) solution obtained is a global (in time) solution. We have, owing to (2.12),

$$\|u_m(t)\|^2_{H^2(\Omega)}\leqslant Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-ct}+c'.$$

Moreover,

$$\sup_{t\in[0,T]} \|u_m(t)\|^2_{H^2(\Omega)} \leqslant c', \tag{2.45}$$

where, c' is independent of m. We thus conclude that the solution is global in time and $T_m = T$.

The passage to the limit: In this step we get a limit boundary problem by letting m tend to infinity. We then have, owing to (2.12), up to a subsequence, which will not be relabeled, that

$$u_m \to u \quad \text{weak star in } L^\infty(0,T;H^2(\Omega)). \tag{2.46}$$

Next we have by (2.9) that

$$\frac{\mathrm{d}E^m_{1,\xi}}{\mathrm{d}t} + c\Big(E^m_{1,\xi} + \Big\|\frac{\partial u_m}{\partial t}\Big\|^2_{V_\xi}\Big) \leqslant c', \quad c>0. \tag{2.47}$$

We then integrate (2.47) from 0 to t and obtain

$$E^m_{1,\xi}(T) + \int_0^t \Big\|\frac{\partial u_m(s)}{\partial t}\Big\|^2_{V_\xi} \mathrm{d}s \leqslant c + E^m_{1,\xi}(0) \quad \forall\, t\in[0,T],\ c\geqslant 0, \tag{2.48}$$

where the constant c is independent of m. It then follows that

$$\frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(0,T;V_\xi). \tag{2.49}$$

Considering now the set

$$W = \Big\{u\in L^2(0,T;H^2(\Omega));\ \frac{\partial u}{\partial t}\in L^2(0,T;H^1(\Omega))\Big\},$$

we obtain, owing to the classical Aubin-Lions compacteness lemma, that

$$W \hookrightarrow L^2(0,T;H^1(\Omega)), \quad \text{with compact inection}$$

and consequently,

$$u_m \to u \quad \text{strongly in } C([0,T];H^{2-\varepsilon}(\Omega))\ \forall\,\varepsilon>0 \text{ and a.e.} \tag{2.50}$$

Moreover, since f is a continuous polynomial,

$$f(u_m(t,x)) \to f(u(t,x)) \quad \text{a.e.}$$

and since $f(u_m)$ is bounded in $L^2((0,T)\times\Omega)$,

$$f(u_m)\to f(u)\quad \text{weakly in } L^2((0,T)\times\Omega),$$

owing to the weak dominated convergence theorem. We also have, owing to (2.46),

$$\Delta u_m \to \Delta u \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)). \tag{2.51}$$

Therefore, owing to (2.2), we have

$$\omega_m \to \omega \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)). \tag{2.52}$$

Noting that, owing to (2.2) $\|\omega\| \leqslant Q(\|u_m\|_{H^2(\Omega)})$, we finally obtain

$$\omega_m \to \omega \quad \text{weakly in } L^2(0,T;H^2(\Omega)). \tag{2.53}$$

As far as the passage to the limit is concerned, the most delicate part is to prove that

$$\int_0^T\int_\Omega \omega_m f'(u_m)\varphi\,\mathrm{d}x\,\mathrm{d}t \underset{m\to\infty}{\longrightarrow} \int_0^T\int_\Omega \omega f'(u)\varphi\,\mathrm{d}x\,\mathrm{d}t$$

for φ regular enough.

We have, say, for $\varphi\in C^2([0,T]\times\overline{\Omega})$ such that $\varphi(T)=\varphi(0)=0$,

$$\begin{aligned}\int_0^T\int_\Omega (\omega_m f'(u_m)-\omega f'(u))\varphi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega (\omega_m-\omega)f'(u)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega \omega_m\left(f'(u_m)-f'(u)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.54}$$

The passage to the limit in the first integral on the right-hand side of (2.54) is straightforward, while the passage to the limit in the second one follows from the above convergences which yield in particular the inequality

$$\left|\int_0^T\int_\Omega \omega_m(f'(u_m)-f'(u))\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant c\|u_m-u\|_{L^2((0,T)\times\Omega)}.$$

Finally, it follows from (2.9)–(2.10) and (2.12) that $u\in L^\infty(\mathbb{R}^+;H^2(\Omega))$ and consequently, $\omega\in L^\infty(\mathbb{R}^+;L^2(\Omega))$. □

It follows from Theorem 2.1 that we can define the family of solution operators

$$S_\xi(t)\colon\ \Phi\to\Phi,\quad u_0\to u(t),\quad t\geqslant 0,\ \xi>0,$$

which maps the initial datum onto the solution at time t with $\Phi = H^2(\Omega) \cap H_0^1(\Omega)$. This family of operators forms a semigroup, i.e.

$$x \mapsto S_\xi(t)x \quad \text{is continuous}, \ t \geqslant 0,$$
$$S_\xi(0) = \mathrm{Id}, \quad S_\xi(t+s) = S_\xi(t) \circ S_\xi(s), \quad t, s \geqslant 0,$$

where Id denotes the identity operator.

We now have the following theorem:

Theorem 2.2. *The semigroup $S_\xi(t)$ is dissipative in Φ in the sense that it possesses a bounded absorbing set $\mathcal{B}_1 \subset \Phi$, i.e., for all $B \subset \Phi$ bounded there exists $t_0 = t_0(B)$ such that $t \geqslant t_0$ implies $S_\xi(t)B \subset \mathcal{B}_1$. Furthermore, we can choose $\mathcal{B}_1$ such that $\mathcal{B}_1 \subset H^4(\Omega)$.*

Proof. The dissipativity in Φ immediately follows from (2.12).

Let now $\mathcal{B}_0$ be a bounded absorbing set in Φ. Let $B \subset \Phi$ be bounded and $t_0 = t_0(B)$ be such that $t \geqslant t_0$ implies $S_\xi(t)B \subset \mathcal{B}_0$.

It follows from (2.9) and (2.12) that

$$\int_t^{t+r} \Big\| \frac{\partial u}{\partial t} \Big\|_{V_\xi}^2 \,\mathrm{d}s \leqslant c(r), \quad t \geqslant t_0, \ r > 0, \tag{2.55}$$

and from (2.15) we have

$$\int_t^{t+r} \|u\|_{H^3(\Omega)}^2 \,\mathrm{d}s \leqslant c(r), \quad t \geqslant t_0, \ r > 0; \tag{2.56}$$

again, all constants are independent of ξ.

We thus deduce from (2.20), (2.56), and the uniform Gronwall's lemma that (assuming, as above, that $\|u_0\|_{H^2(\Omega)} \leqslant R$)

$$\|u(t)\|_{H^3(\Omega)} \leqslant c, \quad t \geqslant t_1 \ (\geqslant t_0), \tag{2.57}$$

$$\int_t^{t+r} \Big(\Big\| \nabla \frac{\partial u}{\partial t} \Big\|^2 + \xi \Big\| \Delta \frac{\partial u}{\partial t} \Big\|^2 \Big) \,\mathrm{d}s \leqslant c_r, \quad t \geqslant t_1. \tag{2.58}$$

Note that this also yields the existence of a bounded absorbing set for the associated dynamical system on $H^3(\Omega)$.

Integrating (2.22) with respect to time, we have

$$\int_t^{t+r} \|u\|_{H^4(\Omega)}^2 \,\mathrm{d}s \leqslant c_r, \quad t \geqslant t_1. \tag{2.59}$$

We deduce again from above and the uniform Gronwall's lemma applied to (2.27) that

$$\|u(t)\|_{H^4(\Omega)} \leqslant c, \quad t \geqslant t_2 \ (\geqslant t_1) \tag{2.60}$$

and

$$\int_t^{t+r} \Big(\Big\|\Delta \frac{\partial u}{\partial t}\Big\|^2 + \xi \Big\|\nabla\Delta \frac{\partial u}{\partial t}\Big\|^2 \Big)\, \mathrm{d}s \leqslant c_r, \quad t \geqslant t_2, \ r > 0. \tag{2.61}$$

Consequently, this yields the existence of a bounded absorbing set for the associated dynamical system on $H^4(\Omega)$. □

Note that from (2.60) we deduce that $u \in L^\infty(\tau, \infty; H^4(\Omega))$ for all $\tau > 0$, and it follows from (2.27) that

$$\frac{\partial u}{\partial t} \in L^2(\tau, T; H^2(\Omega)) \quad \forall\, \tau > 0, \quad \forall\, 0 < \tau \leqslant T, \ \text{for } \xi = 0,$$
$$\frac{\partial u}{\partial t} \in L^2(\tau, T; H^3(\Omega)) \quad \forall\, \tau > 0, \quad \forall\, 0 < \tau \leqslant T, \ \text{for } \xi > 0.$$

As a consequence of Theorem 2.2, we have the following result.

Theorem 2.3. *The semigroup $S_\xi(t)$ possesses the global attractor $\mathcal{A}_\xi$ on the phase space Φ, which is compact in $H^2(\Omega)$ and bounded in $H^4(\Omega)$.*

Remark 2.1. Replacing, if necessary, $\mathcal{B}_1$ by (the closure of) $\bigcup_{t \geqslant t_1} S_\xi(t)\mathcal{B}_1$, where t_1 is such that $t \geqslant t_1$, implies $S_\xi(t)\mathcal{B}_1 \subset \mathcal{B}_1$, we can assume, without loss of generality, that $\mathcal{B}_1$ is (closed and) positively invariant by $S_\xi(t)$, i.e., $S_\xi(t)\mathcal{B}_1 \subset \mathcal{B}_1$ for all $t \geqslant 0$.

2.3. Robust exponential attractors.

Estimates on the difference of two solutions: Let (u_1, ω_1) and (u_2, ω_2) be two solutions of (2.1)–(2.4) with initial data $u_{0,1}$ and $u_{0,2}$, respectively, where ω_i, $i = 1, 2$, are defined in (2.2).

We set $u = u_1 - u_2$, $\omega = \omega_1 - \omega_2$, $u_0 = u_{0,1} - u_{0,2}$ and have

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) + \omega_1 f'(u_1) - \omega_2 f(u_2) - \Delta\omega = 0, \tag{2.62}$$

$$\omega = f(u_1) - f(u_2) - \Delta u, \tag{2.63}$$

$$u = \omega = 0 \quad \text{on } \Gamma, \tag{2.64}$$

$$u|_{t=0} = u_0. \tag{2.65}$$

Now, we derive a smoothing property on the difference of two solutions which is the key estimate for proving the existence of exponential attractors.

We multiply (2.62) by $t\frac{\partial u}{\partial t}$ and integrate over Ω and by parts to get

$$t\Big\|\frac{\partial u}{\partial t}\Big\|^2+\xi t\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\|\nabla u\|^2+\|\Delta u\|^2)\big)-\frac{1}{2}(\|\nabla u\|^2+\|\Delta u\|^2)$$
$$+t\Big(\Big(f(u_1)-f(u_2),\frac{\partial u}{\partial t}\Big)\Big)+t\Big(\Big(\omega_1 f'(u_1)-\omega_2 f'(u_2),\frac{\partial u}{\partial t}\Big)\Big)$$
$$-t\Big(\Big(\Delta(f(u_1)-f(u_2)),\frac{\partial u}{\partial t}\Big)\Big)=0. \tag{2.66}$$

We note that, owing to (2.12),

$$\Big|\Big(\Big(f(u_1)-f(u_2),\frac{\partial u}{\partial t}\Big)\Big)\Big|\leqslant\frac{1}{8}\Big\|\frac{\partial u}{\partial t}\Big\|^2+c\|u\|^2, \tag{2.67}$$

where here and below all the constants only depend on the absorbing set $\mathcal{B}_1$ constructed above. Furthermore,

$$\Big|\Big(\Big(\omega_1 f'(u_1)-\omega_2 f'(u_2),\frac{\partial u}{\partial t}\Big)\Big)\Big|\leqslant\Big|\Big(\Big(\omega f'(u_1),\frac{\partial u}{\partial t}\Big)\Big)\Big|$$
$$+\Big|\Big(\Big(\omega_2(f'(u_1)-f'(u_2)),\frac{\partial u}{\partial t}\Big)\Big)\Big|\leqslant\frac{1}{8}\Big\|\frac{\partial u}{\partial t}\Big\|^2+c\|u\|^2_{H^2(\Omega)}; \tag{2.68}$$

here we have used the fact that

$$\|\omega\|\leqslant c\|u\|_{H^2(\Omega)}.$$

Finally,

$$\Delta(f(u_1)-f(u_2))=f'(u_1)\Delta u_1-f'(u_2)\Delta u_2+f''(u_1)|\nabla u_1|^2-f''(u_2)|\nabla u_2|^2,$$

so, owing once more to (2.12),

$$\Big|\Big(\Big(\Delta(f(u_1)-f(u_2)),\frac{\partial u}{\partial t}\Big)\Big)\Big|\leqslant\frac{1}{4}\Big\|\frac{\partial u}{\partial t}\Big\|^2+c\|u\|^2_{H^2(\Omega)}. \tag{2.69}$$

It finally follows from (2.66)–(2.69) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\|\nabla u\|^2+\|\Delta u\|^2)\big)+t\Big\|\frac{\partial u}{\partial t}\Big\|^2+\xi t\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2$$
$$\leqslant ct\|u\|^2_{H^2(\Omega)}+(\|\nabla u\|^2+\|\Delta u\|^2). \tag{2.70}$$

Integrating (2.70) from 0 to t, we have

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant c\frac{1+t}{t}\int_0^t \|u\|^2_{H^2(\Omega)}\,\mathrm{d}s \tag{2.71}$$

and noting that it follows from (2.37) that

$$\int_0^t \|u\|^2_{H^2(\Omega)}\,\mathrm{d}s \leqslant c\mathrm{e}^{c't}\|u_{0,1}-u_{0,2}\|^2_{V_\xi}, \tag{2.72}$$

we deduce from (2.71) and (2.72) that

$$\|u(t)\|^2_{H^2(\Omega)} \leqslant c\frac{1+t}{t}\mathrm{e}^{c't}\|u_0\|^2_{V_\xi}, \tag{2.73}$$

where all constants are independent of ξ.

Let finally (u^ξ,ω^ξ) and (u^0,ω^0) be two solutions to (2.1)–(2.2) for $\xi>0$ and $\xi=0$, respectively, with the same initial datum u_0. We set $u=u^\xi-u^0$ and $\omega=\omega^\xi-\omega^0$. We then have

$$\frac{\partial u}{\partial t}+\xi(-\Delta)\frac{\partial u}{\partial t}-\Delta u+f(u^\xi)-f(u^0)+\omega^\xi f'(u^\xi) -\omega^0 f'(u^0)-\Delta\omega=\xi\Delta\frac{\partial u^0}{\partial t}, \tag{2.74}$$

$$\omega=f(u^\xi)-f(u^0)-\Delta u, \tag{2.75}$$

$$u|_{t=0}=u_0. \tag{2.76}$$

Multiplying (2.74) by u and integrating over Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2+\|\nabla u\|^2 + ((f(u^\xi)-f(u^0),u))+((\omega^\xi f'(u^\xi)-\omega^0 f'(u^0),u)) - ((\Delta(f(u^\xi)-f(u^0)),u))+\|\Delta u\|^2=\xi\Big(\Big(\frac{\partial u^0}{\partial t},\Delta u\Big)\Big). \tag{2.77}$$

We have, owing to (1.16),

$$((f(u^\xi)-f(u^0),u))\geqslant -c_0\|u\|^2, \tag{2.78}$$

$$|((f(u^\xi)-f(u^0),\Delta u))|\leqslant \frac{1}{8}\|\Delta u\|^2+c\|u\|^2, \tag{2.79}$$

and owing to (2.75), we have

$$|((\omega^\xi f'(u^\xi) - \omega^0 f'(u^0), u))| \leqslant \frac{1}{4}\|\Delta u\|^2 + c\|\nabla u\|^2, \tag{2.80}$$

and

$$\Big|\xi\Big(\Big(\frac{\partial u^0}{\partial t}, \Delta u\Big)\Big)\Big| \leqslant \frac{1}{8}\|\Delta u\|^2 + c\xi^2\Big\|\frac{\partial u^0}{\partial t}\Big\|^2. \tag{2.81}$$

We finally deduce from (2.77)–(2.81) and a proper interpolation inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + \xi\|\nabla u\|^2) + \|\Delta u\|^2 \leqslant c\xi^2\Big\|\frac{\partial u^0}{\partial t}\Big\|^2 + c'\|u\|^2. \tag{2.82}$$

Now, to find $\|\partial u^0/\partial t\|^2$, we multiply the equation

$$\frac{\partial u^0}{\partial t} - \Delta u^0 + f(u^0) + \omega^0 f'(u^0) - \Delta\omega^0 = 0,$$

by $\partial u^0/\partial t$ and have, in view of (2.2),

$$\Big\|\frac{\partial u^0}{\partial t}\Big\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u^0\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}F(u^0)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 = 0. \tag{2.83}$$

We integrate over $[0,t]$ and over $[t,t+1]$ to have

$$\begin{aligned}\|\nabla u^0(t)\|^2 &+ 2\int_0^t \Big\|\frac{\partial u^0}{\partial t}\Big\|^2\,\mathrm{d}s + 2\int_\Omega F(u^0(t))\,\mathrm{d}x + \|\omega^0(t)\|^2\\ &\leqslant \|\nabla u^0(0)\|^2 + \int_\Omega F(u^0(0))\,\mathrm{d}x + \|\omega^0(0)\|^2\\ &\leqslant \|u^0(0)\|^2_{H^2(\Omega)} + c\|u^0(0)\|^4_{L^4(\Omega)} + c'\end{aligned}$$

and

$$\begin{aligned}\|\nabla u^0(t+1)\|^2 + 2\int_t^{t+1} \Big\|\frac{\partial u^0}{\partial t}\Big\|^2\,\mathrm{d}s + 2\int_\Omega F(u^0(t+1))\,\mathrm{d}x + \|\omega^0(t+1)\|^2\\ \leqslant \|\nabla u^0(0)\|^2 + \int_\Omega F(u^0(0))\,\mathrm{d}x + \|\omega^0(0)\|^2,\end{aligned}$$

respectively. Now, by the above two relations we have

$$\int_t^{t+1} \Big\|\frac{\partial u^0}{\partial t}\Big\|^2\,\mathrm{d}s \leqslant \text{constant}. \tag{2.84}$$

By estimate (2.82) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2+\xi\|\nabla u\|^2)+\|\Delta u\|^2 \leqslant c\xi^2\left\|\frac{\partial u^0}{\partial t}\right\|^2 + c'(\|u\|^2+\xi\|\nabla u\|^2).$$

Integrating from 0 to t, $t>0$ we obtain, owing to Gronwall's lemma and in view of (2.84),

$$\|u(t)\|^2+\xi\|\nabla u(t)\|^2 \leqslant c\xi^2 \mathrm{e}^{c''t}, \quad \|u(t)\|^2_{V_\xi} \leqslant c\xi^2\mathrm{e}^{c''t}, \tag{2.85}$$

where the constants c and c'' only depend on $\|u_0\|_{H^2(\Omega)}$ and are independent of ξ.

Next we recall the following result concerning the construction of a robust family of exponential attractors for a discrete dynamical system (see [11]; see also [12], [15], [17], [24] and [26] for generalizations):

Proposition 2.1. *Let H and H_1 be two Banach spaces such that the injection $H_1 \subset H$ is compact, let B be a bounded subset of H and $L_\xi\colon B \to B$, $\xi \in [0,\xi_0]$, $\xi_0 > 0$ be a family of operators such that*

a) *For every $x_1, x_2 \in B$ and every $\xi \in [0,\xi_0]$,*

$$\|L_\xi x_1 - L_\xi x_2\|_{H_1} \leqslant c\|x_1-x_2\|_H,$$

where the constant c is independent of ξ.

b) *For every $\xi \in [0,\xi_0]$, every $i \in \mathbb{N}$ and every $x \in B$,*

$$\|L^i_\xi x - L^i_0 x\|_H \leqslant c^i\xi,$$

where the constant c is independent of ξ.

Then there exists a family $\mathcal{M}_\xi \subset B$, $\xi \in [0,\xi_0]$, such that $\mathcal{M}_\xi$ is an exponential attractor for the discrete dynamical system generated by L_ξ, i.e.:

(i) *The set $\mathcal{M}_\xi$ is compact in H and has a finite fractal dimension in H,*

$$\dim_F \mathcal{M}_\xi \leqslant c.$$

(ii) *The set $\mathcal{M}_\xi$ is positively invariant,*

$$L_\xi\mathcal{M}_\xi \subset \mathcal{M}_\xi.$$

(iii) *The set $\mathcal{M}_\xi$ attracts B exponentially fast,*

$$\mathrm{dist}_H(L^iB,\mathcal{M}_\xi) \leqslant c\mathrm{e}^{-c'i}, \quad i\in\mathbb{N},\ c'>0,$$

where dist_H *denotes the Hausdorff semidistance between sets defined by*

$$\operatorname{dist}_H(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_H.$$

(iv) *Furthermore, the family* $\mathcal{M}_\xi$ *is Hölder continuous at* $\xi=0$,

$$\operatorname{dist}_{\rm sym}(\mathcal{M}_\xi,\mathcal{M}_0)\leqslant c\mathrm{e}^{c'},\quad c'\in(0,1),$$

where $\operatorname{dist}_{\rm sym}$ *denotes the Hausdorff symmetric distance between sets defined by*

$$\operatorname{dist}_{\rm sym}(A,B)=\max(\operatorname{dist}_H(A,B),\operatorname{dist}_H(B,A)).$$

Finally, all constants are independent of ξ *and can be computed explicitly.*

Based on Proposition 4.1, we can prove the following theorem.

Theorem 2.4. *For every* $\xi\in[0,\xi_0]$, $\xi_0>0$, *the semigroup* $S_\xi(t)$ *acting on* Φ *possesses an exponential attractor* $\mathcal{M}_\xi$ *on* Φ *such that*

1. *The set* $\mathcal{M}_\xi$ *has finite fractal dimension in* V_ξ,

$$\dim_F\mathcal{M}_\xi\leqslant c.$$

2. *The set* $\mathcal{M}_\xi$ *is positively invariant by* $S_\xi(t)$,

$$S_\xi(t)\mathcal{M}_\xi\subset\mathcal{M}_\xi,\ t\geqslant 0.$$

3. *The set* $\mathcal{M}_\xi$ *attracts all bounded subsets of* Φ *exponentially fast, i.e. for every bounded subset* B *of* Φ *there exists a constant* $c=c(B)$ *such that*

$$\operatorname{dist}_{V_\xi}(S_\xi(t)B,\mathcal{M}_\xi)\leqslant c\mathrm{e}^{-c't},\quad t\geqslant 0,\ c'>0.$$

4. *The family of sets* $\mathcal{M}_\xi$ *is Hölder continuous at* 0,

$$\operatorname{dist}_{\rm sym}(\mathcal{M}_\xi,M_0)\leqslant c\xi^{c'},\quad c'\in(0,1).$$

Furthermore, all constants are independent of ξ *and can be computed explicitly.*

Proof. We first note that, owing to the uniform estimates obtained in the previous section, we have the existence of a uniform (with respect to ξ) absorbing set $\mathcal{B}_1\subset\Phi$, i.e., for all $B\subset\Phi$ bounded there exists $t_0=t_0(B)$ independent of $\xi\in[0,\xi_0]$ such that

$$S_\xi(t)B\subset\mathcal{B}_1,\quad t\geqslant t_0,\ \xi\in[0,\xi_0].$$

It is thus sufficient to construct the exponential attractor $\mathcal{M}_\xi$ on $\mathcal{B}_1$.

To do so, we first construct exponential attractors for a proper family of discrete semigroups and then pass to the continuous case.

From the above there exists $t_1 > 0$ independent of $\xi \in [0, \xi_0]$ such that

$$S_\xi(t)\mathcal{B}_1 \subset \mathcal{B}_1, \quad t \geqslant t_1,\ \xi \in [0, \xi_0].$$

We then set

$$L_\xi := S_\xi(t_1)$$

and consider the sets $H = V_\xi$ and $H_1 = H^2(\Omega)$. It follows from (2.73) and (2.85) that the assumptions of Proposition 2.1 are satisfied, hence the existence of a robust family of exponential attractors $\mathcal{M}_\xi^d$ for the discrete dynamical systems generated by the operators L_ξ. We finally set

$$\mathcal{M}_\xi = \bigcup_{t \in [0, t_1]} S_\xi(t)\mathcal{M}_\xi^d.$$

To finish the proof, it suffices to prove that the mapping $(t, x) \mapsto S_\xi(t)x$ is Hölder continuous on $[0, t_1] \times B$, uniformly with respect to $\xi \in [0, \xi_0]$. The Hölder continuity with respect to x follows from (2.12). $\square$

We then have:

Proposition 2.2. *For any solutions to* (2.1)–(2.4) *with initial datum belonging to* $\mathcal{B}_0$ *and for any* $T > 0$,

$$\|u(t_1) - u(t_2)\|_{V_\xi} \leqslant c(T, \mathcal{B}_0)|t_1 - t_2|^{1/2} \quad \forall\, t_1,\ t_2 \in [0, T]. \tag{2.86}$$

Proof. We have

$$u(t_1) - u(t_2) = \int_{t_1}^{t_2} \frac{\partial u}{\partial t}\,\mathrm{d}\tau,$$

which yields

$$\begin{aligned} \|u(t_2) - u(t_2)\|_{V_\xi} &\leqslant \left\| \int_{t_1}^{t_2} \frac{\partial u}{\partial t}\,\mathrm{d}\tau \right\|_{V_\xi} \leqslant \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{V_\xi} \mathrm{d}\tau \right| \\ &\leqslant |t_1 - t_2|^{1/2} \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{V_\xi}^2 \right|^{1/2}, \end{aligned} \tag{2.87}$$

where u is a solution of (2.1)–(2.4).

We note that, owing to (2.9),

$$\left|\int_{t_1}^{t_2} \left\|\frac{\partial u}{\partial t}\right\|_{V_\xi}^2 \,\mathrm{d}\tau\right| \leqslant c, \tag{2.88}$$

where the constant c depends only on $\mathcal{B}_1$ and T such that $t_1, t_2 \in [0, T]$, so

$$\|u(t_1) - u(t_2)\|_{V_\xi} \leqslant c|t_1 - t_2|^{1/2},$$

where the constant c depends only on $\mathcal{B}_1$ and T such that $t_1, t_2 \in [0, T]$. □

Since an exponential attractor yields the existence of the global attractor $\mathcal{A}_\xi \subset \mathcal{M}_\xi$, we deduce from Theorem 2.4 the following result.

Corollary 2.1. *The global attractor $\mathcal{A}_\xi$ has a finite fractal dimension in V_ξ.*

3. Viscous Cahn-Hilliard system

Setting the problem: It will be more convenient for us to rewrite equations (1.13)–(1.15) in the following form (taking $\beta = 1$):

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} = \Delta\mu, \tag{3.1}$$

$$\mu = f(u) - \Delta u + \omega f'(u) - \Delta\omega, \tag{3.2}$$

$$\omega = f(u) - \Delta u, \tag{3.3}$$

together with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = \frac{\partial \omega}{\partial \nu} = 0 \quad \text{on } \Gamma \tag{3.4}$$

and the initial condition

$$u(x, 0) = u_0(x),\ x \in \Omega. \tag{3.5}$$

Remark 3.1. We note that the above viscous Cahn-Hilliard system associated with periodic boundary conditions can also be treated in a similar way as below.

3.1. A priori estimates. We first note that integrating (3.1) over Ω we obtain the conservation of mass, namely

$$\langle u(t)\rangle = \langle u_0\rangle \quad \forall\, t \geqslant 0. \tag{3.6}$$

Multiplying (3.1) by $(-\Delta)^{-1}\partial u/\partial t$ and integrating over Ω and by parts, we have

$$\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \xi\Big\|\frac{\partial u}{\partial t}\Big\|^2 = -\Big(\Big(\mu, \frac{\partial u}{\partial t}\Big)\Big). \tag{3.7}$$

We then multiply (3.2) by $\frac{\partial u}{\partial t}$ and integrate over Ω to obtain

$$\Big(\Big(\mu, \frac{\partial u}{\partial t}\Big)\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \Big(\Big(\omega f'(u), \frac{\partial u}{\partial t}\Big)\Big) - \Big(\Big(\Delta\omega, \frac{\partial u}{\partial t}\Big)\Big). \tag{3.8}$$

Noting that it follows from (3.3) that

$$\Big(\Big(\omega f'(u), \frac{\partial u}{\partial t}\Big)\Big) - \Big(\Big(\Delta\omega, \frac{\partial u}{\partial t}\Big)\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega^2\,\mathrm{d}x, \tag{3.9}$$

we finally deduce from (3.7)–(3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2\Big) + 2\Big\|\frac{\partial u}{\partial t}\Big\|_{H_\xi}^2 = 0. \tag{3.10}$$

In particular, (3.10) yields that the free energy decreases along the trajectories as expected.

We now multiply (3.1) by $(-\Delta)^{-1}\bar{u}$, $\bar{u} = u - \langle u\rangle$, and find, owing to (3.6) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}\|_{-1}^2 + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}\|^2 = -((\mu, u)) + \mathrm{Vol}(\Omega)\langle\mu\rangle\langle u_0\rangle, \tag{3.11}$$

where, owing to (3.2),

$$\langle\mu\rangle = \langle f(u)\rangle + \langle\omega f'(u)\rangle. \tag{3.12}$$

Multiplying then (3.2) by u, we have, owing to (3.3),

$$\begin{aligned}((\mu, u)) = {} & ((f(u), u)) + \|\nabla u\|^2 + ((f(u)f'(u), u)) + \|\Delta u\|^2 \\ & - ((\Delta f(u), u)) - ((f'(u)\Delta u, u)).\end{aligned} \tag{3.13}$$

Noting that

$$((f'(u)\Delta u, u)) = -((f'(u)\nabla u, \nabla u)) - ((uf''(u)\nabla u, \nabla u))$$

and

$$((\Delta f(u), u)) = -((f'(u)\nabla u, \nabla u)),$$

we obtain

$$((\mu, u)) = \|\nabla u\|^2 + ((f(u), u)) + \|\omega\|^2 + ((uf''(u)\nabla u, \nabla u)) + \int_\Omega (uf(u)f'(u) - f(u)^2)\,\mathrm{d}x$$

and we finally find, owing to (1.17), (1.18), (1.20), and (3.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}\|^2_{H_\xi} + c\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2 + \|f(u)\|^2\Big) \leqslant 2\mathrm{Vol}(\Omega)\langle\mu\rangle\langle u_0\rangle + c', \quad c > 0. \tag{3.14}$$

We now assume that

$$|\langle u_0\rangle| \leqslant M \quad (\text{hence, } |\langle u(t)\rangle| \leqslant M,\ t \geqslant 0),\ M \geqslant 0. \tag{3.15}$$

Therefore, owing to (1.19) and (3.12),

$$\begin{aligned} |2\mathrm{Vol}(\Omega)\langle\mu\rangle\langle u_0\rangle| &\leqslant c_M(|\langle f(u)\rangle| + |\langle\omega f'(u)\rangle|) \\ &\leqslant \frac{c}{2}\Big(\int_\Omega \omega^2\,\mathrm{d}x + \int_\Omega f(u)^2\,\mathrm{d}x\Big) + c'_M, \end{aligned} \tag{3.16}$$

where c is the constant appearing in (3.14), and we deduce from (3.14) and (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}\|^2_{H_\xi} + c\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2\Big) \leqslant c'_M, \quad c > 0. \tag{3.17}$$

Combining (3.10) and (3.17), we have an inequality of the form

$$\frac{\mathrm{d}E_{2,\xi}}{\mathrm{d}t} + c\left(E_{2,\xi} + \|\frac{\partial u}{\partial t}\|^2_{H_\xi}\right) \leqslant c'_M, \quad c > 0, \tag{3.18}$$

where

$$E_{2,\xi} = \|\bar{u}\|^2_{H_\xi} + \langle u\rangle^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\omega\|^2. \tag{3.19}$$

In particular, we deduce from (3.18) and Gronwall's lemma that

$$E_{2,\xi}(t) \leqslant E_{2,\xi}(0)\mathrm{e}^{-ct} + c'_M, \quad c > 0,\ t \geqslant 0. \tag{3.20}$$

Noting that, owing to (1.16),

$$\|\omega\|^2 \geqslant \|f(u)\|^2 + \|\Delta u\|^2 - 2c_0\|\nabla u\|^2, \tag{3.21}$$

we finally deduce from (3.19)–(3.21) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|f(u)\|^2 \leqslant Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-ct} + c'_M, \quad c > 0,\ t \geqslant 0. \tag{3.22}$$

We now multiply (3.1) by u, and integrating over Ω we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 = -((\nabla\mu, \nabla u)). \tag{3.23}$$

Multiplying then (3.2) by $-\Delta u$, we obtain in view of (3.3)

$$\begin{aligned}((\nabla\mu, \nabla u)) = \|\Delta u\|^2 + ((f'(u)\nabla u, \nabla u)) - ((\omega f'(u), \Delta u))\\ + ((\Delta f(u), \Delta u)) + \|\nabla\Delta u\|^2.\end{aligned} \tag{3.24}$$

Noting that, owing to the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$ (here, $n \leqslant 3$) and (3.3),

$$|((f'(u)\nabla u, \nabla u))| + |((\omega f'(u), \Delta u))| + |((\Delta f(u), \Delta u))| \leqslant Q(\|u_0\|_{H^2(\Omega)})$$

(indeed, it follows from (3.3) that $\|\omega\| \leqslant Q(\|u_0\|_{H^2(\Omega)})$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{V_\xi} + c\|u\|^2_{H^3(\Omega)} \leqslant Q(\|u\|_{H^2(\Omega)}), \quad c > 0. \tag{3.25}$$

We now multiply (3.1) by $\partial u/\partial t$ and integrate over Ω to get

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \xi\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 = -\left(\left(\nabla\mu, \nabla\frac{\partial u}{\partial t}\right)\right). \tag{3.26}$$

Multiplying then (3.2) by $-\Delta\partial u/\partial t$, we obtain, in view of (3.3),

$$\begin{aligned}\left(\left(\nabla\mu, \nabla\frac{\partial u}{\partial t}\right)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \left(\left(f'(u)\nabla u, \nabla\frac{\partial u}{\partial t}\right)\right) + \left(\left(\nabla(\omega f'(u)), \nabla\frac{\partial u}{\partial t}\right)\right)\\ - \left(\left(\nabla\Delta f(u), \nabla\frac{\partial u}{\partial t}\right)\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2.\end{aligned} \tag{3.27}$$

Substituting (3.27) in (3.26), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2 + \|\nabla\Delta u\|^2) + \left\|\frac{\partial u}{\partial t}\right\|^2_{V_\xi} + \left(\left(f'(u)\nabla u, \nabla\frac{\partial u}{\partial t}\right)\right)\\ + \left(\left(\nabla(\omega f'(u)), \nabla\frac{\partial u}{\partial t}\right)\right) - \left(\left(\nabla\Delta f(u), \nabla\frac{\partial u}{\partial t}\right)\right) = 0.\end{aligned} \tag{3.28}$$

We have

$$\Big|\Big(\Big(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant \|f'(u)\|_{L^\infty(\Omega)}\|\nabla u\|\Big\|\nabla\frac{\partial u}{\partial t}\Big\| \leqslant \frac14\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + Q(\|u\|_{H^2(\Omega)})\|\nabla u\|^2, \tag{3.29}$$

$$\Big|\Big(\Big(\nabla(\omega f'(u)),\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant \Big|\Big(\Big(f'(u)\nabla\omega,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| + \Big|\Big(\Big(\omega f''(u)\nabla u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant \frac14\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + Q(\|u\|_{H^2(\Omega)})(\|\nabla\Delta u\|^2+1), \tag{3.30}$$

noting that it follows from (3.3)

$$\|\nabla\omega\| \leqslant Q(\|u\|_{H^2(\Omega)}) + 2\|\nabla\Delta u\|^2$$

and that, owing to Hölder's inequality and proper Sobolev embeddings,

$$\Big|\Big(\Big(\nabla\Delta f(u),\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \leqslant \Big|\Big(\Big(f''(u)\Delta u\nabla u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| + \Big|\Big(\Big(f'(u)\nabla\Delta u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big|, \\ \Big|\Big(\Big(f'''(u)|\nabla u|^2\nabla u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| + 2\Big|\Big(\Big(f''(u)\nabla\nabla u\cdot\nabla u,\nabla\frac{\partial u}{\partial t}\Big)\Big)\Big| \\ \leqslant \frac14\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + Q(\|u\|_{H^2(\Omega)})\|\nabla\Delta u\|^2. \tag{3.31}$$

It thus follows from (3.28)–(3.31) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2+\|\nabla\Delta u\|^2) + \Big\|\frac{\partial u}{\partial t}\Big\|^2_{V_\xi} \leqslant Q(\|u\|_{H^2(\Omega)})(\|\nabla\Delta u\|^2+1). \tag{3.32}$$

Rewriting (3.1) in the equivalent form

$$\mu = \langle\mu\rangle - \xi\frac{\partial u}{\partial t} - (-\Delta)^{-1}\frac{\partial u}{\partial t}, \tag{3.33}$$

we obtain

$$\|\nabla\mu\| \leqslant c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1} + \xi\Big\|\nabla\frac{\partial u}{\partial t}\Big\|\Big). \tag{3.34}$$

Noting that, proceeding as in (3.16),

$$|\langle\mu\rangle| \leqslant c(\|u\|^2_{H^2(\Omega)} + \|f(u)\|^2+1),$$

we finally find

$$\|\mu\|_{H^1(\Omega)} \leqslant c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1} + \Big\|\frac{\partial u}{\partial t}\Big\|_{V_\xi} + \|u\|^2_{H^2(\Omega)} + \|f(u)\|^2 + 1\Big). \tag{3.35}$$

Having this, (1.19), (3.2), (3.3), and (3.35) yield

$$\|\omega\|_{H^2(\Omega)} \leqslant c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1} + \Big\|\frac{\partial u}{\partial t}\Big\|_{V_\xi} + \|u\|^2_{H^2(\Omega)} + \|f(u)\|^2 + 1\Big). \tag{3.36}$$

3.2. The dissipative semigroup.

Theorem 3.1. *We assume that* (3.15) *holds and that* $u_0 \in H^2(\Omega)$ *with* $\partial u_0/\partial\nu = 0$ *on* Γ. *Then* (3.1)–(3.5) *possesses a unique (weak) solution such that for all* T,

$$u \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T; H_\xi),$$
$$\mu \in L^2(0,T; H^1(\Omega)), \quad \text{and} \quad \omega \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

Proof. a) Existence: The proof of existence of solutions is based on the a priori estimates derived in the previous subsection and, e.g., a standard Galerkin scheme.

In particular, it follows from (3.18)–(3.19) and (3.22) that we can construct a sequence of solutions u_m to a proper approximated problem such that

$$\begin{gathered} u_m \to u \quad \text{weak star in } L^\infty(0,T; H^2(\Omega)), \\ \text{strongly in } C([0,T]; H^{2-\varepsilon}(\Omega)) \quad \forall\, \varepsilon > 0, \text{ and a.e.}, \\ \frac{\partial u_m}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(0,T; H_\xi), \\ \mu_m \to \mu \quad \text{weakly in } L^2(0,T; H^1(\Omega)), \\ \omega_m \to \omega \quad \text{weak star in } L^\infty(0,T; L^2(\Omega)) \text{ and weakly in } L^2(0,T; H^2(\Omega)), \end{gathered}$$

as $m \to \infty$ for all $T > 0$.

The passage to the limit is then standard and can be done as in the previous section.

Finally, it follows from (3.18)–(3.19) and (3.22) that $u \in L^\infty(\mathbb{R}^+; H^2(\Omega))$ and consequently, $\omega \in L^\infty(\mathbb{R}^+; L^2(\Omega))$.

b) Uniqueness: Let (u_1, μ_1, ω_1) and (u_2, μ_2, ω_2) be two solutions of (3.1)–(3.5) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, such that

$$|\langle u_{i,0}\rangle| \leqslant M, \quad i = 1, 2. \tag{3.37}$$

We set $(u,\mu,\omega)=(u_1,\mu_1,\omega_1)-(u_2,\mu_2,\omega_2)$ and $u_0=u_{1,0}-u_{2,0}$. Then we have

$$\frac{\partial}{\partial t}(u+\xi(-\Delta)u)=\Delta\mu, \tag{3.38}$$

$$\mu=f(u_1)-f(u_2)-\Delta u+\omega_1 f'(u_1)-\omega_2 f'(u_2)-\Delta\omega, \tag{3.39}$$

$$\omega=f(u_1)-f(u_2)-\Delta u, \tag{3.40}$$

$$\frac{\partial u}{\partial\nu}=\frac{\partial\mu}{\partial\nu}=\frac{\partial\omega}{\partial\nu}=0\quad\text{on }\Gamma, \tag{3.41}$$

$$u|_{t=0}=u_0. \tag{3.42}$$

We can rewrite (3.38) in the equivalent form

$$\frac{\partial}{\partial t}(\bar u+\xi(-\Delta)\bar u)=\Delta\mu \tag{3.43}$$

and then multiply (3.43) by $(-\Delta)^{-1}\bar u$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar u\|_{-1}^2+\xi\|\bar u\|^2)=-((\mu,u))+\mathrm{Vol}(\Omega)\langle\mu\rangle\langle u\rangle. \tag{3.44}$$

Multiplying now (3.39) by u and integrating over Ω, we have

$$\begin{aligned}((\mu,u))=\|\nabla u\|^2+((f(u_1)-f(u_2),u))+((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))\\ -((f(u_1)-f(u_2),\Delta u))+\|\Delta u\|^2.\end{aligned} \tag{3.45}$$

We have

$$((f(u_1)-f(u_2),u))\geqslant -c_0\|u\|^2. \tag{3.46}$$

Furthermore,

$$\begin{aligned}|((f(u_1)-f(u_2),\Delta u))|&\leqslant \|f(u_1)-f(u_2)\|_{L^\infty(\Omega)}\|\Delta u\|\\ &\leqslant \frac{1}{8}\|\Delta u\|^2+Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|u\|^2\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\big|((\omega_1 f'(u_1)-\omega_2 f'(u_2),u))\big|&\leqslant |((\omega_1(f'(u_1)-f'(u_2)),u))|+|((\omega f'(u_2),u))|\\ &\leqslant |((\omega_1(f'(u_1)-f'(u_2)),u))|+|((f'(u_2)\Delta u,u))|\\ &\quad+|((f'(u_2)(f(u_1)-f(u_2)),u))|\\ &\leqslant Q(\|u_{1,0}\|_{H^2_{\mathrm{per}}(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|\omega_1\|_{H^2(\Omega)}\|u\|^2\\ &\quad+\frac{1}{8}\|\Delta u\|^2+Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})\|u\|^2\\ &\leqslant \frac{1}{8}\|\Delta u\|^2+Q(\|u_{1,0}\|_{H^2(\Omega)},\|u_{2,0}\|_{H^2(\Omega)})(\|\omega_1\|_{H^2(\Omega)}+1)\|u\|^2.\end{aligned} \tag{3.48}$$

Similarly,

$$(3.49)\qquad |\mathrm{Vol}(\Omega)\langle\mu\rangle\langle u\rangle| \leqslant \frac{1}{4}\|\Delta u\|^2 + Q(\|u_{0,1}\|_{H^2(\Omega)}, \|u_{0,2}\|_{H^2(\Omega)})(\|\omega_1\|+1)(\|u\|^2+|\langle u\rangle|^2).$$

We finally deduce from (3.44)–(3.49) and the interpolation inequality

$$(3.50)\qquad \|\bar u\| \leqslant c\|\bar u\|_{-1}^{1/2}\|\nabla\bar u\|^{1/2} \leqslant c\|\bar u\|_{-1}^{1/2}\|\Delta\bar u\|^{1/2}$$

that

$$(3.51)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\big(\|\bar u\|_{-1}^2 + |\langle u\rangle|^2 + \xi(\|\bar u\|^2+\langle u\rangle^2)\big) + \|\Delta u\|^2 \leqslant Q(\|u_{0,1}\|_{H^2(\Omega)}, \|u_{0,2}\|_{H^2(\Omega)})(\|\omega_1\|^2+\|\omega_1\|^2_{H^2(\Omega)}+1)(\|\bar u\|_{-1}^2+|\langle u\rangle|^2 + \xi(\|\bar u\|^2+\langle u\rangle^2)).$$

Gronwall's lemma then yields, owing to (3.18), (3.22), and (3.36) (written for (u_1,μ_1,ω_1)),

$$(3.52)\qquad \|u(t)\|^2_{H_\xi} \leqslant c\mathrm{e}^{c't}\|u_0\|^2_{H_\xi},$$

where c and c' only depend on $\|u_{0,i}\|_{H^2(\Omega)}$, $i=1,2$, and M (and are, in particular, independent of ξ).

This gives the uniqueness as well as the continuous dependence with respect to the initial data in the H_ξ-norm. □

It follows from the above results that we can define the semigroup

$$S(t)\colon\ \Phi_M \to \Phi_M, \quad u_0 \mapsto u(t), \quad t \geqslant 0$$

(i.e., $S(0)=\mathrm{Id}$ and $S(t+s)=S(t)\circ S(s)$, $t,s\geqslant 0$), where

$$\Phi_M = \{v\in H^2(\Omega)\colon\ |\langle v\rangle| \leqslant M\}, \quad M\geqslant 0.$$

Theorem 3.2. *The semigroup $S(t)$ is dissipative in Φ_M in the sense that it possesses a bounded absorbing set $\mathcal{B}_2\subset\Phi_M$, i.e., for all $B\subset\Phi_M$ bounded there exists $t_0=t_0(B)$ such that $t\geqslant t_0$ implies $S(t)B\subset\mathcal{B}_2$. Furthermore, we can choose $\mathcal{B}_2$ such that $\mathcal{B}_2\subset H^3(\Omega)$.*

P r o o f. The dissipativity in Φ_M immediately follows from (3.22).

We now (formally) differentiate (3.1)–(3.3) with respect to time and have, setting $(q, v, r) = (\partial u/\partial t, \partial \mu/\partial t, \partial \omega/\partial t)$,

$$\frac{\partial q}{\partial t} + \xi(-\Delta)\frac{\partial q}{\partial t} = \Delta v, \tag{3.53}$$

$$v = f'(u)q - \Delta q + \omega f''(u)q + rf'(u) - \Delta r, \tag{3.54}$$

$$r = f'(u)q - \Delta q. \tag{3.55}$$

We multiply (3.53) by $(-\Delta)^{-1}q$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{-1}^2 + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 = -((v, q)). \tag{3.56}$$

Multiplying then (3.54) by q, we find, owing to (3.55),

$$\begin{aligned}((v, q)) = ((f'(u)q, q)) + \|\nabla q\|^2 + ((\omega f''(u)q, q)) + ((f'(u)^2 q, q)) \\ - 2((f'(u)q, \Delta q)) + \|\Delta q\|^2.\end{aligned} \tag{3.57}$$

We thus deduce from (3.56)–(3.57) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|_{-1}^2 + \xi\|q\|^2) + \|\nabla q\|^2 + \|\Delta q\|^2 + ((f'(u)q, q)) + ((\omega f''(u)q, q)) \\ + ((f'(u)^2 q, q)) - 2((f'(u)q, \Delta q)) = 0.\end{aligned} \tag{3.58}$$

Now, we have

$$((f'(u)q, q)) \geqslant -c_0\|q\|^2, \tag{3.59}$$

$$|((\omega f''(u)q, q))| \leqslant Q(\|u\|_{H^2(\Omega)})\|\omega\|_{H^2(\Omega)}\|q\|^2, \tag{3.60}$$

$$|((f'(u)^2 q, q))| \leqslant Q(\|u\|_{H^2(\Omega)})\|q\|^2, \tag{3.61}$$

and

$$|((f'(u)q, \Delta q))| \leqslant \|f'(u)\|_{L^\infty(\Omega)}\|q\|\,\|\Delta q\| \leqslant Q(\|u\|_{H^2(\Omega)})\|q\|\|\Delta q\|. \tag{3.62}$$

It finally follows from (3.58)–(3.62) and the interpolation inequality (3.50) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|q\|_{-1}^2 + \xi\|q\|^2) + \|\nabla q\|^2 + \|\Delta q\|^2 &\leqslant Q(\|u\|_{H^2(\Omega)})(\|\omega\|_{H^2\Omega}^2 + 1)\|q\|_{-1}^2 \\ &\leqslant Q(\|u\|_{H^2(\Omega)})(\|\omega\|_{H^2\Omega}^2 + 1)(\|q\|_{-1}^2 + \xi\|q\|^2).\end{aligned} \tag{3.63}$$

In the second step, we multiply (3.1) by $q = \partial u/\partial t$ and have

$$\|q\|^2 + \xi\|\nabla q\|^2 = -((\nabla\mu, \nabla q)). \tag{3.64}$$

Multiplying then (3.2) by $-\Delta q = -\Delta \partial u/\partial t$, we obtain, owing to (3.3),

$$((\nabla \mu, \nabla q)) = -((f(u), \Delta q)) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 - ((\omega f'(u), \Delta q)) + ((\Delta f(u), \Delta q)) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \Delta u\|^2. \tag{3.65}$$

We have

$$|((f(u), \Delta q))| \leqslant Q(\|u\|_{H^2(\Omega)})\|\Delta q\|, \tag{3.66}$$

$$|((\omega f'(u), \Delta q))| \leqslant Q(\|u\|_{H^2(\Omega)})\|\omega\|\|\Delta q\|, \tag{3.67}$$

and

$$|((\Delta f(u), \Delta q))| \leqslant Q(\|u\|_{H^2(\Omega)})\|\Delta q\|. \tag{3.68}$$

We thus deduce from (3.64)–(3.68) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|^2 + \|\nabla \Delta u\|^2) \leqslant Q(\|u\|_{H^2(\Omega)})(\|\omega\|^2 + 1)(\|\Delta q\|^2 + 1). \tag{3.69}$$

Let now $\mathcal{B}_2$ be a bounded absorbing set in Φ_M. Let also $B \subset \Phi_M$ be bounded and $t_0 = t_0(B)$ be such that $t \geqslant t_0$ implies $S(t)B \subset \mathcal{B}_2$.

It follows from (3.18) and (3.22) that

$$\int_t^{t+r} \left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \xi\left\|\frac{\partial u}{\partial t}\right\|^2\right) \mathrm{d}s \leqslant C(r), \quad t \geqslant 0,\ r > 0, \tag{3.70}$$

and from (3.25) we obtain

$$\int_t^{t+r} \|u\|_{H^3(\Omega)}^2 \,\mathrm{d}s \leqslant C(r), \quad t \geqslant t_0,\ r > 0. \tag{3.71}$$

Then we get from (3.32) that

$$\int_t^{t+r} \left\|\frac{\partial u}{\partial t}\right\|_{V_\xi}^2 \mathrm{d}s \leqslant C(r), \quad t \geqslant t_0,\ r > 0. \tag{3.72}$$

Consequently, we deduce from (3.36) that

$$\int_t^{t+r} \|\omega\|_{H^2(\Omega)}^2 \,\mathrm{d}s \leqslant C(r), \quad t \geqslant t_0,\ r > 0. \tag{3.73}$$

Applying the uniform Gronwall's lemma to (3.63), we have

$$\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \xi\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 \leqslant C(r), \quad t \geqslant t_0 + r,\ r > 0, \tag{3.74}$$

and

$$\int_t^{t+r} \|q\|_{H^2(\Omega)}^2 \,\mathrm{d}s \leqslant C(r), \quad t \geqslant t_0 + r,\ r > 0. \tag{3.75}$$

Applying the uniform Gronwall lemma the second time, now to (3.69) (note that $\|\omega\|^2 \leqslant Q(\|u\|_{H^2(\Omega)})$), we finally deduce that

$$\|u(t)\|_{H^3(\Omega)} \leqslant C(r), \quad t \geqslant t_0 + 2r,\ r > 0, \tag{3.76}$$

which finishes the proof of the theorem. □

It follows from (3.76) that the semigroup $S(t)$ possesses a bounded absorbing set which is compact in $H^2(\Omega)$ and bounded in $H^3(\Omega)$. We thus deduce from standard results the following theorem.

Theorem 3.3. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}_M$ which is compact in $H^2(\Omega)$ and bounded in $H^3(\Omega)$.*

4. Numerical Simulations

We split the sixth-order (in space) equation into a system of three second-order ones. Consequently, we use a P1-finite element for the space discretization together with a semi-implicit Euler time discretization (i.e. implicit for the linear terms and explicit for the nonlinear ones). The numerical simulations are performed with the software FreeFem++ [20].

In the numerical results presented below, the domain Ω is the square $(0,1)\times(0,1)$. The triangulation is obtained by dividing Ω into 130×130 rectangles and by dividing every rectangle along the same diagonal. We take here $f(s) = 4s^3 - 6s^2 + 2s$.

In the figures below, the values of solutions between 0 and $\frac{1}{2}$ are represented in (light) yellow, while the values of solutions between $\frac{1}{2}$ and 1 are represented in (dark) red.

4.1. Isotropic case with Willmore regularization. We consider the following isotropic viscous Cahn-Hilliard equation defined by

$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} = \frac{1}{\varepsilon}\Delta\mu, \tag{4.1}$$

$$\mu = -\varepsilon\Delta u + \frac{1}{\varepsilon}f(u) + \frac{\beta}{\varepsilon^2}\omega f'(u) - \beta\Delta\omega, \tag{4.2}$$

$$\omega = \frac{1}{\varepsilon}f(u) - \varepsilon\Delta u, \tag{4.3}$$

$$u,\ \mu,\ \omega \text{ are } \Omega\text{-periodic}, \tag{4.4}$$

$$u|_{t=0} = u_0, \tag{4.5}$$

where ε here defines the interface thickness and β defines a small regularization parameter $0 \leqslant \beta \leqslant 1$.

In Figure 2, we present numerical solutions corresponding to the initial datum u_0 randomly distributed between 0 and 1, as shown in Figure 2(a), but with different parameters ξ. Figure 2(b) corresponds to the case when $\xi = 0$, i.e. the non-viscous isotropic one, while Figures 2(c), 2(d), and 2(e) correspond to the viscous isotropic case with $\xi = 0.01$, $\xi = 0.1$, and $\xi = 1$, respectively. In these four cases, the step size is 10^{-8}, and we show the solution after 5 iterations ($t = 5 \times 10^{-8}$). We can see that when ξ is close to zero, the solutions evolve more rapidly. Figure 2(f) corresponds to the case when $\xi = 0$, i.e. the non-viscous isotropic case, while Figures 2(g), 2(h), and 2(i) correspond to the viscous isotropic case with $\xi = 0.01$, $\xi = 0.1$, and $\xi = 1$, respectively. In these four cases, the step size is 10^{-8}, and we show the solution after 200 iterations ($t = 2 \times 10^{-6}$). We can see that when ξ is close to zero, the solutions evolve more rapidly. In this test, $\varepsilon = 0.05$ and $\beta = 0.001$.

4.2. Anisotropic case with Willmore regularization. We consider in this case the following anisotropic viscous Cahn-Hilliard problem

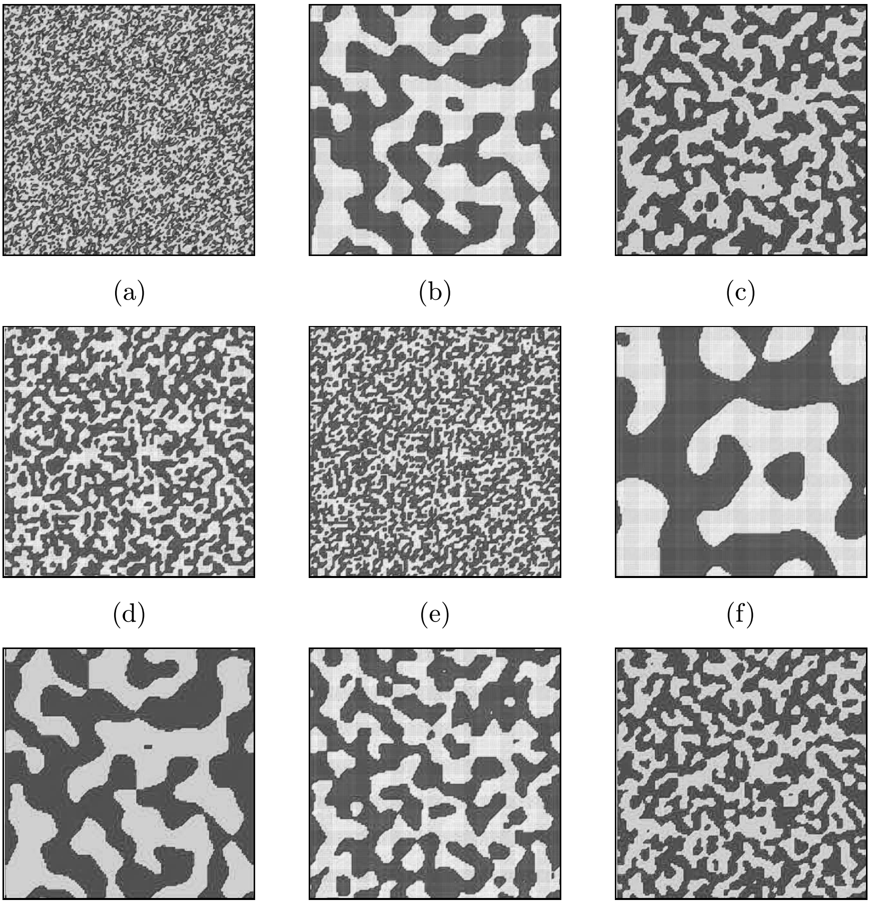
$$\frac{\partial u}{\partial t} + \xi(-\Delta)\frac{\partial u}{\partial t} = \frac{1}{\varepsilon}\Delta\mu, \tag{4.6}$$

$$\mu = -\varepsilon\,\mathrm{div}(g'(\nabla u)) + \frac{1}{\varepsilon}f(u) + \frac{\beta}{\varepsilon^2}\omega f'(u) - \beta\Delta\omega, \tag{4.7}$$

$$\omega = \frac{1}{\varepsilon}f(u) - \varepsilon\Delta u, \tag{4.8}$$

$$u,\ \mu,\ \omega \text{ are } \Omega\text{-periodic}, \tag{4.9}$$

$$u|_{t=0} = u_0, \tag{4.10}$$



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 2. Isotropic model: (a) Initial condition u_0. (b) Solution at $t = 5 \times 10^{-8}$ with $\xi = 0$. (c) Solution at $t = 5 \times 10^{-8}$ with $\xi = 0.01$. (d) Solution at $t = 5 \times 10^{-8}$ with $\xi = 0.1$. (e) Solution at $t = 5 \times 10^{-8}$ with $\xi = 1$. (f) Solution at $t = 2 \times 10^{-6}$ with $\xi = 0$. (g) Solution at $t = 2 \times 10^{-6}$ with $\xi = 0.01$. (h) Solution at $t = 2 \times 10^{-6}$ with $\xi = 0.1$. (i) Solution at $t = 2 \times 10^{-6}$ with $\xi = 1$.

where the anisotropic term g is defined by

$$g(s_1, s_2) = \begin{cases} \frac{1}{2}\gamma^2\Big(\frac{s_1}{|s|}, \frac{s_2}{|s|}\Big)|s|^2 & \text{for } (s_1, s_2) \neq (0,0), \\ 0 & \text{for } (s_1, s_2) = (0,0), \end{cases}$$

where $\gamma(n)$ describes the anisotropic function and $n = \nabla u/|\nabla u|$ is the outer normal unit vector.

The well-posedness of (4.6)–(4.10) in the two dimensional case for $\xi = 0$ has been studied in [22], while in [21], the authors study the non-viscous anisotropic Cahn-Hilliard model (i.e. for $\xi = 0$) but in the one-dimensional case and they prove the existence and uniqueness of the solution.

We consider here the four-fold symmetric anisotropic function

$$\gamma(n) = \gamma(n_1, n_2) = 1 + \alpha \cos(4\theta) = 1 + \alpha\Big(4\sum_{i=1}^{2} n_i^4 - 3\Big). \tag{4.11}$$

In this case, we have for $\nabla u \neq (0,0)$

$$g(\nabla u) = \frac{|\nabla u|^2}{2}\Big[1 + \alpha\Big(\frac{4}{|\nabla u|^4}\Big[\Big(\frac{\partial u}{\partial x}\Big)^4 + \Big(\frac{\partial u}{\partial y}\Big)^4\Big] - 3\Big)\Big]^2. \tag{4.12}$$

To avoid the problem with $|\nabla u| = 0$, we use the square regularization, i.e. $|\nabla u|$ is replaced by $\sqrt{|\nabla u|^2 + \delta^2}$.

In Figure 3, we present numerical solutions corresponding to the initial datum u_0 randomly distributed between 0 and 1, as shown in Figure 3(a), but with different parameters ξ. Figure 3(b) corresponds to the case when $\xi = 0$, i.e. the anisotropic non-viscous case. Figures 3(c), 3(d), and 3(e) correspond to the anisotropic viscous case with $\xi = 0.01$, $\xi = 0.1$, and $\xi = 1$, respectively. In these four cases, the step size is 10^{-8}, and we show the solution after 5 iterations ($t = 5 \times 10^{-8}$). We can see that when ξ is close to zero, the solutions evolve more rapidly. Figure 3(f) corresponds to the case when $\xi = 0$, i.e. the anisotropic non-viscous case. Figures 3(g), 3(h), and 3(i) correspond to the anisotropic viscous case with $\xi = 0.01$, $\xi = 0.1$, and $\xi = 1$, respectively. In these four cases, the step size is 10^{-8}, and we show the solution after 200 iterations ($t = 2 \times 10^{-6}$). We can see that when ξ is close to zero, the solutions evolve more rapidly. In this test, $\varepsilon = 0.05$, $\alpha = 0.9$, $\delta = 0.0001$, and $\beta = 0.001$.

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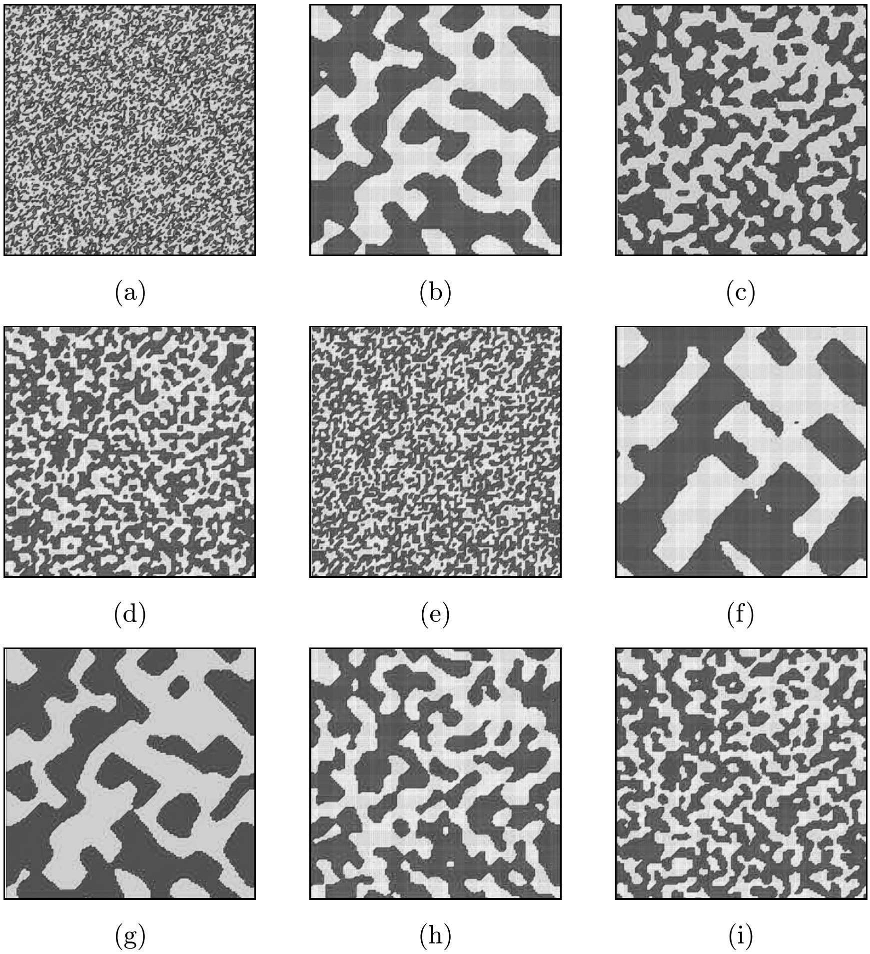

Figure 3. Anisotropic model: (a) Initial condition u_0. (b) Solution at $t = 5 \times 10^{-8}$ with $\xi = 0$. (c) Solution at $t = 5\times10^{-8}$ with $\xi = 0.01$. (d) Solution at $t = 5\times10^{-8}$ with $\xi = 0.1$. (e) Solution at $t = 5\times10^{-8}$ with $\xi = 1$. (f) Solution at $t = 2\times10^{-6}$ with $\xi = 0$. (g) Solution at $t = 2 \times 10^{-6}$ with $\xi = 0.01$. (h) Solution at $t = 2 \times 10^{-6}$ with $\xi = 0.1$. (i) Solution at $t = 2 \times 10^{-6}$ with $\xi = 1$.

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