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AUGMENTATION QUOTIENTS FOR BURNSIDE RINGS OF GENERALIZED DIHEDRAL GROUPS

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Abstract. Let H be a finite abelian group of odd order, $\mathcal{D}$ be its generalized dihedral group, i.e., the semidirect product of C_2 acting on H by inverting elements, where C_2 is the cyclic group of order two. Let $\Omega(\mathcal{D})$ be the Burnside ring of $\mathcal{D}$, $\Delta(\mathcal{D})$ be the augmentation ideal of $\Omega(\mathcal{D})$. Denote by $\Delta^n(\mathcal{D})$ and $Q_n(\mathcal{D})$ the nth power of $\Delta(\mathcal{D})$ and the nth consecutive quotient group $\Delta^n(\mathcal{D})/\Delta^{n+1}(\mathcal{D})$, respectively. This paper provides an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{D})$ and determines the isomorphism class of $Q_n(\mathcal{D})$ for each positive integer n.

Keywords: generalized dihedral group; Burnside ring; augmentation ideal; augmentation quotient

MSC 2010: 16S34, 20C05

1. Introduction

Let G be a finite group. A *G-set* is a finite set X together with an action of G on X:

$$G \times X \to X, \quad (g, x) \mapsto gx. \tag{1.1}$$

Two G-sets X and Y are said to be *isomorphic* (denoted by $X \cong Y$), if there exists a bijective map $f\colon X \to Y$ such that

$$f(gx) = gf(x), \quad g \in G,\ x \in X. \tag{1.2}$$

It is easy to verify that isomorphism of G-sets is an equivalence relation. The equivalence classes are called isomorphism classes. The isomorphism class of X is denoted

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by $[X]$. The sum $[X] + [Y]$ of two isomorphism classes $[X]$ and $[Y]$ is defined by

$$[X] + [Y] = [X \sqcup Y], \tag{1.3}$$

where $X \sqcup Y$ is the disjoint union of X and Y, which is also a G-set in the canonical way.

The *Burnside ring* $\Omega(G)$ is the group completion of the monoid (under addition) of isomorphism classes of G-sets. Its multiplication is induced by the Cartesian product of finite sets. Note that the Cartesian product $X \times Y$ is a G-set via coordinating action. By [6], $\Omega(G)$ is a commutative ring with an identity element. Its underlying group is a finitely generated free abelian group with the isomorphism classes of transitive G-sets as basis. Hence, by [6], its free rank is equal to the number of conjugacy classes of subgroups of G.

The number of fixed points of a G-set induces a ring homomorphism

$$\varphi\colon\ \Omega(G) \to \mathbb{Z}. \tag{1.4}$$

This homomorphism is called the *augmentation map*. Its kernel $\Delta(G)$ is called the *augmentation ideal* of $\Omega(G)$. Let $\Delta^n(G)$ and $Q_n(G)$ denote the nth power of $\Delta(G)$ and the nth consecutive quotient group $\Delta^n(G)/\Delta^{n+1}(G)$, respectively. Wu and Tang in [11] determined the isomorphism class of $Q_n(G)$ for all finite abelian groups and for any positive integer n. In particular, they showed that for any finite abelian group G, the isomorphism class of $Q_n(G)$ does not depend on n when n is large enough.

Despite these efforts, the isomorphism class of $Q_n(G)$ remained unclear for non-abelian groups. Let H be a finite abelian group of odd order, $\mathcal{D}$ be its generalized dihedral group. The goal of this article is to give an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{D})$ and determine the isomorphism class of $Q_n(\mathcal{D})$ for each positive integer n.

The result also yields $\mathrm{Tor}_1^{\Omega(\mathcal{D})}(\Omega(\mathcal{D})/\Delta^n(\mathcal{D}), \Omega(\mathcal{D})/\Delta(\mathcal{D}))$ because for any finite group G, $Q_n(G) \cong \mathrm{Tor}_1^{\Omega(G)}(\Omega(G)/\Delta^n(G), \Omega(G)/\Delta(G))$.

Two related problems of recent interest have been to investigate the augmentation ideals and their consecutive quotients for integral group rings and representation rings of finite groups. These problems have been well studied in [1]–[5], [7]–[10].

2. Preliminaries

In this section, we provide some useful results about $\Omega(G)$, $\Delta^n(G)$, $Q_n(G)$ and finitely generated free abelian groups.

Let X be a G-set. There is an equivalence relation on X given by saying that x is related to y if there exists $g \in G$ with $gx = y$. The equivalence classes are called *orbits*. It is easy to see that the orbit of x is

$$Gx = \{gx\colon\ g \in G\}. \tag{2.1}$$

A G-set is *transitive* if it has only one orbit. For instance, if $x \in X$, the orbit Gx is a transitive G-set. As another example, if K is a subgroup of G, the set G/K of left cosets of K is a transitive G-set under $g(hK) = (gh)K$.

Lemma 2.1. *Except for the order of terms, each isomorphism class of a G-set has a unique expression as a sum of isomorphic classes of transitive G-sets. Indeed, if X has orbits $X_1, \ldots, X_r$, then $[X] = [X_1] + \ldots + [X_r]$.*

There is a standard form for each transitive G-set. The *stabilizer* of an element $x \in X$ is the set

$$G_x = \{g \in G\colon\ gx = x\}. \tag{2.2}$$

It is easy to verify that the stabilizer G_x is a subgroup of G. The following lemma shows that every transitive G-set is isomorphic to G/K for some subgroup K of G.

Lemma 2.2. *For any G-set X and $x \in X$,*

$$Gx \cong G/G_x, \quad gx \mapsto gG_x. \tag{2.3}$$

For each $x \in X$ and $g \in G$, $G_{gx} = gG_xg^{-1}$. Any isomorphism of G-sets $f\colon\ X \to Y$ preserves stabilizers: $G_x = G_{f(x)}$. From these two facts we get the following lemma.

Lemma 2.3. *Let K and L be two subgroups of G. Then $G/K \cong G/L$ if and only if K and L are conjugate in G.*

Assembling the above facts, we have proved the following theorems.

Theorem 2.4. *Let $\mathcal{K}$ be a set of subgroups of G, one from each conjugacy class of subgroups of G. Each isomorphism class $[X]$ of a G-set has a unique expression*

$$\sum_{K \in \mathcal{K}} d_K [G/K], \tag{2.4}$$

where d_K is the number of orbits in X that are isomorphic to G/K.

Theorem 2.5. *For each finite group G, $\Omega(G)$ is, additively, the free abelian group based on the isomorphism classes $[G/K]$ as K ranges through the set $\mathcal{K}$ of representatives of the conjugacy classes of subgroups of G.*

Recall that $\Delta(G)$ is the kernel of $\varphi\colon \Omega(G) \to \mathbb{Z}$ which sends $[X]$ to the number $\#(X)$ of fixed points of X. Brief calculations show that

$$\#(G/K) = \begin{cases} 1 & \text{if } K = G, \\ 0 & \text{if } K < G. \end{cases} \tag{2.5}$$

From this we get the following corollary.

Corollary 2.6. *For each finite group G, the underlying group of $\Delta(G)$ is the free abelian group based on the isomorphism classes $[G/K]$ as K ranges through a set of representatives of the conjugacy classes of proper subgroups of G.*

The multiplication in $\Omega(G)$ is completely determined by the product

$$[G/K][G/L] = [(G/K) \times (G/L)], \tag{2.6}$$

where K, L are subgroups of G. The following lemma tackles this product.

Lemma 2.7. *Let K be a subgroup of G, L be a normal subgroup of G. Then*

$$[G/K][G/L] = \frac{|G|}{|KL|}[G/(K \cap L)]. \tag{2.7}$$

P r o o f. Let (gK, hL) be an element of the Cartesian product $(G/K) \times (G/L)$, where $g, h \in G$. A short calculation shows that

$$G_{(gK,hL)} = gKg^{-1} \cap hLh^{-1} = gKg^{-1} \cap L = g(K \cap L)g^{-1}. \tag{2.8}$$

Hence each orbit in $(G/K) \times (G/L)$ is isomorphic to $G/(K \cap L)$. Then the lemma follows from the cardinalities of $[G/K]$, $[G/L]$ and $[G/(K \cap L)]$. $\square$

Thanks to the above results, we get two useful properties of $Q_n(G)$ and $\Delta^n(G)$.

Theorem 2.8. *$Q_n(G)$ is a finite abelian group for any positive integer n.*

P r o o f. Note that $\Delta(G)$ is finitely generated as a free abelian group. Thus we only need to show $Q_n(G)$ is torsion, which is equivalent to showing that for each proper subgroup K of G, there is a positive integer m such that

$$m[G/K] \in \Delta^2(G). \tag{2.9}$$

We prove this by induction on $|K|$. If $|K| = 1$, then $K = \{1\}$. By Lemma 2.7, we get $[G/\{1\}]^2 = |G|[G/\{1\}]$, which implies $|G|[G/\{1\}] \in \Delta^2(G)$. Assume (2.9) holds for all proper subgroups of G whose orders are less than $|K|$.

To finish the proof, we need the following assertion.

Assertion 2.9. *Regarding the square of* $[G/K]$, *we have*

$$[G/K]^2 = d[G/K] + \sum_{i=1}^{r}[G/L_i], \tag{2.10}$$

where d is a positive integer, $L_1, \dots, L_r$ are proper subgroups of K.

P r o o f. Let $(gK, hK) \in (G/K) \times (G/K)$, where $g, h \in G$. Short calculations show

$$G_{(gK,hK)} = gKg^{-1} \cap hKh^{-1} = g(K \cap g^{-1}hKh^{-1}g)g^{-1}. \tag{2.11}$$

Thus each orbit in $(G/K) \times (G/K)$ is isomorphic to G/L for some subgroup L of K. To finish the proof, we just need to show that there is an orbit in $(G/K) \times (G/K)$ which is isomorphic to G/K itself. This is obviously true since $G_{(K,K)} = K$. □

We return now to the proof of Theorem 2.8. By Assertion 2.9, we get

$$d[G/K] = [G/K]^2 - \sum_{i=1}^{r}[G/L_i]. \tag{2.12}$$

Due to the induction assumption, there are r positive integers $m_1, \dots, m_r$ such that

$$m_i[G/L_i] \in \Delta^2(G), \quad i = 1, \dots, r. \tag{2.13}$$

From these it follows that

$$m_1 \dots m_r d[G/K] = m_1 \dots m_r [G/K]^2 - m_1 \dots m_r \sum_{i=1}^{r}[G/L_i] \in \Delta^2(G), \tag{2.14}$$

as required. □

Corollary 2.10. *For each positive integer n, $\Delta^n(G)$ has free rank $s(G)-1$, where $s(G)$ is the number of conjugacy classes of subgroups of G.*

P r o o f. It follows from the fact that $\Delta(G)$ has free rank $s(G)-1$ and the quotient $\Delta(G)/\Delta^n(G)$ is torsion. □

At last, we recall a classical result about finitely generated free abelian groups.

Lemma 2.11. *Let K be a finitely generated free abelian group of rank r. If the r elements $g_1, \ldots, g_r$ generate K, then they form a basis of K.*

3. Necessary tools

In this section, we construct a basis of $\Delta(\mathcal{D})$ as a finitely generated free abelian group. Then we determine the multiplication in $\Omega(\mathcal{D})$.

Let H be a finite abelian group of odd order. Recall that the generalized dihedral group $\mathcal{D}$ of H is the semidirect product of C_2 acting on H by inverting elements, where C_2 is the cyclic group of order two. Denote by σ the generate of C_2. Then $\mathcal{D}$ can be partitioned into H and σH. Its multiplication is determined by

$$\sigma^{-1}h\sigma = h^{-1}, \quad h \in H. \tag{3.1}$$

The following lemma provides a set of representatives of the conjugacy classes of subgroups of $\mathcal{D}$.

Lemma 3.1. *Let K be a subgroup of $\mathcal{D}$.*

(i) *If $K \subset H$, then K is normal in $\mathcal{D}$.*
(ii) *If $K \not\subset H$, then K is conjugate to $N \cup \sigma N$ in $\mathcal{D}$, where $N = K \cap H$.*

P r o o f. It is easy to see that (i) is a direct corollary of (3.1). For (ii), since $K \not\subset H$, there is $g \in H$ such that $\sigma g \in K$. We claim that

$$K = N \cup \sigma g N. \tag{3.2}$$

Suppose (3.2) has been proved. Since H has odd order, there exists an integer k such that $g^{2k+1} = 1$. Then the lemma follows from

$$g^{-k}(N \cup \sigma g N)g^k = N \cup \sigma g^{2k+1} N = N \cup \sigma N. \tag{3.3}$$

To show (3.2), we just need to prove that $K \cap \sigma H$ is contained in $\sigma g N$. Suppose $\sigma h \in K$, where $h \in H$. Short calculations show that

$$(\sigma g)^{-1}(\sigma h) = g^{-1}h \in K \cap H = N. \tag{3.4}$$

From this it follows that

$$\sigma h = \sigma g(g^{-1}h) \in \sigma g N, \tag{3.5}$$

as required. □

Thanks to Lemma 3.1, we get a basis of $\Omega(\mathcal{D})$, hence a basis of $\Delta(\mathcal{D})$. For convenience, we fix the following notation.

▷ For any subgroup N of H, denote $[\mathcal{D}/N]$ and $[\mathcal{D}/(N \cup \sigma N)]$ by α_N and β_N, respectively.

▷ For any subset $\Gamma \subset \Omega(\mathcal{D})$, denote by $\mathbb{Z}\,\Gamma$ the set of all $\mathbb{Z}$-linear combinations of elements of Γ.

Theorem 3.2. *The underlying group of $\Omega(\mathcal{D})$ is the free abelian group with basis*

$$\{\alpha_N\colon\ N \leqslant H\} \cup \{\beta_N\colon\ N \leqslant H\}. \tag{3.6}$$

Proof. Note that for each subgroup N of H, $N \cup \sigma N$ is a subgroup of $\mathcal{D}$. Then the theorem follows from Lemma 3.1 and the fact that all subgroups appearing in (3.6) are pairwise non-conjugate. □

Corollary 3.3. *$\Delta(\mathcal{D})$ is, additively, the free abelian group based on*

$$\{\alpha_N\colon\ N \leqslant H\} \cup \{\beta_N\colon\ N < H\}. \tag{3.7}$$

Now we determine the multiplication in $\Omega(\mathcal{D})$.

Lemma 3.4. *Let M, N be two subgroups of H. Then*

$$\alpha_M\alpha_N = \frac{2|H|}{|MN|}\alpha_{M\cap N}, \tag{3.8}$$

$$\alpha_M\beta_N = \frac{|H|}{|MN|}\alpha_{M\cap N}, \tag{3.9}$$

$$\beta_M\beta_N \in \mathbb{Z}\{\alpha_{M\cap N}, \beta_{M\cap N}\}. \tag{3.10}$$

P r o o f. The first two identities are direct corollaries of Lemma 2.7 since each subgroup of H is normal in $\mathcal{D}$. Now consider the last identity. Note that for any $x \in \big(\mathcal{D}/(M \cup \sigma M)\big) \times \big(\mathcal{D}/(N \cup \sigma N)\big)$, there exist $g, h \in H$ such that

$$x = \big(g(M \cup \sigma M), h(N \cup \sigma N)\big). \tag{3.11}$$

Short calculations show

$$\begin{aligned} \mathcal{D}_x \cap H &= g(M \cup \sigma M)g^{-1} \cap h(N \cup \sigma N)h^{-1} \cap H \\ &= (M \cup \sigma g^{-2} M) \cap (N \cup \sigma h^{-2} N) \cap H \\ &= M \cap N, \end{aligned} \tag{3.12}$$

where $\mathcal{D}_x$ is the stabilizer of x. So by Lemma 3.1, we get

▷ $\mathcal{D}_x = M \cap N$, if $\mathcal{D}_x \subset H$,

▷ $\mathcal{D}_x$ is conjugate to $(M \cap N) \cup \sigma(M \cap N)$ in $\mathcal{D}$, if $\mathcal{D}_x \not\subset H$.

From this it follows that

$$[\mathcal{D}x] = \begin{cases} \alpha_{M \cap N} & \text{if } \mathcal{D}_x \subset H, \\ \beta_{M \cap N} & \text{if } \mathcal{D}_x \not\subset H, \end{cases} \tag{3.13}$$

where $\mathcal{D}x$ is the orbit of x. Then the last identity is clear. □

Lemma 3.5. *For any subgroup N of H, we have*

$$\beta_N^2 = \beta_N + d_N \alpha_N, \tag{3.14}$$

where d_N is a natural number.

P r o o f. Let $H/N = \{h_1 N, \ldots, h_r N\}$, where $h_1, \ldots, h_r \in H$. It is easy to verify that

$$\mathcal{D}/(N \cup \sigma N) = \big\{h_1(N \cup \sigma N), \ldots, h_r(N \cup \sigma N)\big\}. \tag{3.15}$$

Denote $\big(h_i(N \cup \sigma N), h_j(N \cup \sigma N)\big)$ by x_{ij}. Then brief calculations show

$$\begin{aligned} \mathcal{D}_{x_{ij}} &= h_i(N \cup \sigma N)h_i^{-1} \cap h_j(N \cup \sigma N)h_j^{-1} \\ &= (N \cup \sigma h_i^{-2} N) \cap (N \cup \sigma h_j^{-2} N) \\ &= N \cup \sigma(h_i^{-2} N \cap h_j^{-2} N). \end{aligned} \tag{3.16}$$

Note that

$$h_i^{-2}N \cap h_j^{-2}N = \begin{cases} h_i^{-2}N & \text{if } h_i^2 h_j^{-2} \in N, \\ \emptyset & \text{if } h_i^2 h_j^{-2} \notin N. \end{cases} \tag{3.17}$$

So by the proof of Lemma 3.4, we get

$$[\mathcal{D}x_{ij}] = \begin{cases} \beta_N, & h_i^2 h_j^{-2} \in N, \\ \alpha_N, & h_i^2 h_j^{-2} \notin N. \end{cases} \tag{3.18}$$

Regarding $h_i^2 h_j^{-2}$, we have the following assertion.

Assertion 3.6. *$h_i^2 h_j^{-2} \in N$ if and only if $i = j$.*

P r o o f. We just need to show that $h_i^2 h_j^{-2} \in N$ implies $i = j$. Recall that H has odd order, so there exists an integer l such that $(h_i h_j^{-1})^{2l-1} = 1$. A short calculation shows

$$h_i h_j^{-1} = (h_i h_j^{-1})^{2l} = (h_i^2 h_j^{-2})^l \in N. \tag{3.19}$$

This implies $h_i N = h_j N$, hence $i = j$. □

We return now to the proof of Lemma 3.5. Due to Assertion 3.6, there are exactly r elements $x_{11}, \ldots, x_{rr}$ whose orbits are isomorphic to $\mathcal{D}/(N \cup \sigma N)$. Note that $\mathcal{D}/(N \cup \sigma N)$ has r elements. Thus there is exactly one orbit in $\big(\mathcal{D}/(N \cup \sigma N)\big) \times \big(\mathcal{D}/(N \cup \sigma N)\big)$ which is isomorphic to $\mathcal{D}/(N \cup \sigma N)$. From this it follows that

$$\beta_N^2 = \beta_N + d_N \alpha_N, \tag{3.20}$$

where d_N is the number of orbits in $\big(\mathcal{D}/(N \cup \sigma N)\big) \times \big(\mathcal{D}/(N \cup \sigma N)\big)$ that are isomorphic to $\mathcal{D}/N$, as required. □

4. Main results

In this section, we give an explicit $\mathbb{Z}$-basis for $\Delta^n(\mathcal{D})$ and determine the isomorphism class of $Q_n(\mathcal{D})$ for each positive integer n.

Theorem 4.1. *For each positive integer n, $\Delta^n(\mathcal{D})$ is, additively, the free abelian group based on*

$$\{2^{n-1}\alpha_H\} \cup \{\alpha_N \colon\ N < H\} \cup \{\beta_N \colon\ N < H\}. \tag{4.1}$$

P r o o f. We prove the theorem by induction on n. The case $n = 1$ is clear. Assume the theorem holds for $n-1$. Note that (4.1) has the same cardinality as (3.7), so due to Corollary 2.10 and Lemma 2.11, we only need to show it generates $\Delta^n(\mathcal{D})$. Recall that $\Delta(\mathcal{D})$ has basis

$$\{\alpha_H\} \cup \{\alpha_M\colon\ M < H\} \cup \{\beta_M\colon\ M < H\}. \tag{4.2}$$

So by the induction assumption, $\Delta^n(\mathcal{D}) = \Delta^{n-1}(\mathcal{D})\Delta(\mathcal{D})$ is generated by

$$\begin{aligned}\{2^{n-2}\alpha_H^2\} &\cup \{\alpha_H\alpha_N\colon\ N < H\} \cup \{\alpha_H\beta_N\colon\ N < H\}\\ &\cup \{\alpha_M\alpha_N\colon\ M < H,\ N < H\} \cup \{\alpha_M\beta_N\colon\ M < H,\ N < H\}\\ &\cup \{\beta_M\beta_N\colon\ M < H,\ N < H\}.\end{aligned} \tag{4.3}$$

Due to the first two identities in Lemma 3.4, we get for any two subgroups M, N of H,

$$\alpha_M\alpha_N = 2\alpha_M\beta_N, \tag{4.4}$$

which implies that $\{\alpha_H\alpha_N\colon\ N < H\}$ and $\{\alpha_M\alpha_N\colon\ M,N < H\}$ are redundant in (4.3). Hence, by a short calculation, $\Delta^n(\mathcal{D})$ is generated by

$$\{2^{n-1}\alpha_H\} \cup \{\alpha_N\colon\ N < H\} \cup \{\alpha_M\beta_N\colon\ M,N < H\} \cup \{\beta_M\beta_N\colon\ M,N < H\}. \tag{4.5}$$

Moreover, it is easy to see that $\{\alpha_M\beta_N\colon\ M,N < H\}$ is redundant too since, by the second identity of Lemma 3.4, it is a subset of $\mathbb{Z}\{\alpha_N\colon\ N < H\}$. To finish the proof, we need to show that the last subset of (4.5) can be replaced by $\{\beta_N\colon\ N < H\}$. Note that the last formula in Lemma 3.4 implies

$$\{\beta_M\beta_N\colon\ M < H,\ N < H\} \subset \mathbb{Z}\{\alpha_N\colon\ N < H\} + \mathbb{Z}\{\beta_N\colon\ N < H\}. \tag{4.6}$$

Thus we just need to show that $\{\beta_N\colon\ N < H\}$ is contained in $\Delta^n(\mathcal{D})$. By Lemma 3.5, we have for any subgroup N of H,

$$\beta_N = \beta_N^2 - d_N\alpha_N. \tag{4.7}$$

Then the theorem follows from the fact that either β_N^2 or α_N belongs to $\Delta^n(\mathcal{D})$. $\square$

Theorem 4.2. *For any positive integer* n,

$$Q_n(\mathcal{D}) \cong C_2. \tag{4.8}$$

P r o o f. This is a direct corollary of Theorem 4.1. □

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